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Z. POLNIAKOWSKI (Poznań)

On some properties of Riesz means

1. In this paper we prove several theorems concerning Riesz mean of the sequence $\{s_n\}$:

$$t_n = \sum_{\nu=0}^{n} p_\nu s_\nu \Big/ \sum_{\nu=0}^{n} p_\nu$$

for complex p_n. We prove a Mercerian theorem (Theorem 1), a theorem concerning the inclusion of two Riesz means (Theorem 2) and two Tauberian theorems (Theorems 3 and 4). In the proofs we make use of the method of difference equations.

We assume, as usual, that

$$\Delta^k a_n = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} a_{n+\nu}, \qquad \Delta a_n = \Delta^1 a_n.$$

We shall say that the sequence $\{a_n\}$ *has the property* C with the constant K if there exist N and $K > 1$ such that

$$(1) \qquad |a_n| \to \infty \quad \text{and} \quad \sum_{\nu=0}^{n-1} |\Delta a_\nu| \leqslant K|a_n| \quad \text{for } n \geqslant N$$

or

$$(2) \qquad a_n \to 0, \quad a_n \neq 0 \quad \text{and} \quad \sum_{\nu=n}^{\infty} |\Delta a_\nu| \leqslant K|a_n| \quad \text{for } n \geqslant N.$$

2. LEMMA. *Suppose that*

$$(3a) \qquad \operatorname{re} h_n < 0$$

or

$$(3b) \qquad \operatorname{re} h_n > 0 \quad (n = 0, 1, 2, \ldots),$$

$$(4) \qquad |h_n| \leqslant A|\operatorname{re} h_n| \quad (n = 0, 1, 2, \ldots),$$

$$(5) \qquad \sum_{n=0}^{\infty} |h_n| = \infty,$$

(6) $(\Delta b_n)/b_n \sim h_n$ *in case* (3a), $(\Delta b_{n-1})/b_n \sim h_n$ *in case* (3b) $(n \to \infty)$.

Then the sequence $\{b_n\}$ has the property C *with every constant $K > A$. Moreover, $|b_n| \to \infty$ in case* (3a) *and $b_n \to 0$ in case* (3b).

Proof. Let us observe that for $b \neq 0$ we have

$$|a/b| \geqslant \operatorname{re}(a/b), \qquad (|a|-|b|)/|b| \geqslant \operatorname{re}[(a-b)/b].$$

For arbitrary sequence $\{b_n\}$ ($b_n \neq 0$) it follows that

$$(7) \quad (\Delta|b_n|)/|b_n| \leqslant \operatorname{re}[(\Delta b_n)/b_n] \quad \text{and} \quad (\Delta|b_{n-1}|)/|b_n| \geqslant \operatorname{re}[(\Delta b_{n-1})/b_n].$$

a) In the case of $\operatorname{re} h_n < 0$ we infer that $\operatorname{re}[(\Delta b_n)/b_n] \sim \operatorname{re} h_n \ (n \to \infty)$, since

$$\operatorname{re}[(\Delta b_n)/b_n] = \operatorname{re}[(\Delta b_n)/b_n h_n]\operatorname{re} h_n - \operatorname{im}[(\Delta b_n)/b_n h_n]\operatorname{im} h_n.$$

From this we obtain for $n \geqslant N$ (with some N), by (7):

$$(\Delta|b_n|)/|b_n| \leqslant \operatorname{re}[(\Delta b_n)/b_n] \leqslant \tfrac{1}{2}\operatorname{re} h_n < 0$$

and

$$|b_n| \geqslant |b_0| \prod_{\nu=0}^{n-1} (1-\tfrac{1}{2}\operatorname{re} h_\nu) \geqslant -\tfrac{1}{2}|b_0| \operatorname{re} \sum_{\nu=0}^{n-1} h_\nu \to \infty$$

by (3a), (4) and (5). Next, we have by (7) for $n \geqslant N$

$$\frac{|\Delta b_n|}{|\Delta|b_n||} \leqslant \frac{|\Delta b_n|}{|b_n|} \cdot \frac{1}{-\operatorname{re}[(\Delta b_n)/b_n]} \sim \frac{|h_n|}{-\operatorname{re} h_n}.$$

Since $\Delta|b_n| < 0$ for $n \geqslant N$, then the sequence $\{|b_n|\}$ has the property C with every constant greater than 1. We have $|\Delta b_n|/|\Delta|b_n|| \leqslant A+\delta$ for sufficiently large n by (4) ($\delta > 0$), and we obtain the desired result in case (3a).

b) In the case of $\operatorname{re} h_n > 0$ we infer, as in a), that for a given ε ($0 < \varepsilon < 1$) there exists N_1 such that for $n \geqslant N_1$

$$\Delta|b_{n-1}| \geqslant (1-\varepsilon)|b_n| \operatorname{re} h_n > 0 \quad \text{and} \quad |\Delta b_{n-1}| \leqslant (1+\varepsilon)|b_n h_n|.$$

Next, we prove that $\lim_n b_n = 0$, applying the inequality

$$|b_n| \leqslant |b_0| / \prod_{\nu=1}^{n} [1+(1-\varepsilon)\operatorname{re} h_\nu] \qquad (n \geqslant N_1)$$

and the hypotheses (4) and (5). We obtain for $n \geqslant N_1$

$$\sum_{\nu=n}^{\infty} |\Delta b_\nu| \leqslant (1+\varepsilon) \sum_{\nu=n+1}^{\infty} |b_\nu h_\nu| \leqslant (1+\varepsilon) A |b_n| \sum_{\nu=n+1}^{\infty} \operatorname{re} h_\nu / \prod_{j=n+1}^{\nu} [1+(1-\varepsilon)\operatorname{re} h_j]$$

$$\leqslant \frac{1+\varepsilon}{1-\varepsilon} A |b_n|$$

in virtue of the equality

$$\sum_{\nu=n+1}^{\infty} v_\nu \Big/ \prod_{j=n+1}^{\nu} (1+v_j) = 1,$$

where $v_\nu = (1-\varepsilon)\operatorname{re} h_\nu$ (compare Sierpiński [5], p. 131).

COROLLARY. *Suppose that for* $n = 0, 1, 2, \ldots$

(8) $$|\tau_n| < A_1(\sigma_n - \sigma_n^2 - \tau_n^2) \quad \textit{with some } A_1 > 0,$$

(9) $$\sum_{n=0}^{\infty} |H_n| = \infty \quad \textit{with } H_n = \sigma_n + i\tau_n,$$

(10) $$b_n \neq 0, \quad (\Delta b_{n-1})/b_{n-1} = H_n.$$

Then the sequence $\{b_n\}$ *has the property* C *with every constant* $K > \sqrt{1+A_1^2}$ *and* $b_n \to 0$.

Proof. Let us observe that $|H_n| < 1$ by (8) since $\operatorname{re}(1/H_n) > 1$. Setting $h_n = (\Delta b_{n-1})/b_n$ we obtain $h_n = H_n/(1-H_n)$.

We shall prove that the sequence $\{h_n\}$ satisfies the hypotheses of Lemma 2 in the case of $\operatorname{re} h_n > 0$. We have

$$h_n = \frac{\sigma_n + i\tau_n}{1-\sigma_n - i\tau_n} = \frac{\sigma_n - \sigma_n^2 - \tau_n^2 + i\tau_n}{(1-\sigma_n)^2 + \tau_n^2}$$

and $\operatorname{re} h_n > 0$ by (8). Furthermore $|\operatorname{im} h_n| < A_1 \operatorname{re} h_n$ by (8), and $|h_n| < \sqrt{1+A_1^2} \operatorname{re} h_n$.

The equality $\sum_{n=0}^{\infty} |h_n| = \infty$ is also satisfied by (9). Now we apply Lemma 2 in case (3b).

3.1. LEMMA. *Suppose that*

(11) $$0 < |a_n| \leqslant -K \operatorname{re} a_n \quad \textit{for some } K > 1 \textit{ and } n = 0, 1, 2, \ldots$$

(12) $$\sum_{n=0}^{\infty} \frac{1}{|a_n|} = \infty,$$

(13a) $$\overline{\lim_n} |\varphi_n| = M < \infty$$

or

(13b) $$\lim_n \varphi_n = s.$$

Then every solution of the difference equation

(14) $$L(x_n) = \varphi_n \quad (n = 1, 2, \ldots),$$

where $L(x_n) = x_n + a_n \Delta x_{n-1}$, *satisfies the relation* $\overline{\lim_n} |x_n| \leqslant MK$ *in case* (13a) *or* $\lim_n x_n = s$ *in case* (13b).

Proof. Assuming $1/a_n = (\Delta b_{n-1})/b_{n-1} = h_n$, $n = 1, 2, \ldots$, we see that the sequence $\{b_n\}$ has the property C with every constant $K^* > K$ and $|b_n| \to \infty$ by Lemma 2. Now it suffices to apply 2.1.1, [3], p. 3 and 2.1.2, [4], p. 113.

3.2. LEMMA. *Suppose that*

(15) $|\mathrm{im}\, a_n| < K_1(\mathrm{re}\, a_n - 1)$ *for some* $K_1 > 0$ *and* $n = 0, 1, 2, \ldots$,

(16) $$\sum_{n=0}^{\infty} \frac{1}{|a_n|} = \infty,$$

(17a) $$\overline{\lim_n} |\varphi_n| = M < \infty$$

or

(17b) $$\lim_n \varphi_n = s.$$

Then the difference equation (14) *has a solution* $\{\bar{x}_n\}$ *such that* $\overline{\lim_n} |\bar{x}_n| \leqslant M\sqrt{1+K_1^2}$ *in case* (17a) *or* $\lim_n \bar{x}_n = s$ *in case* (17b).

Proof. Setting $1/a_n = (\Delta b_{n-1})/b_{n-1} = H_n$, $n = 1, 2, \ldots$, we obtain that the sequence $\{b_n\}$ has the property C with every constant $K_1^* > \sqrt{1+K_1^2}$ by Lemma 2 and Corollary. Now it suffices to apply 2.1.1, [3], p. 3 and 2.1.2, [4], p. 113.

THEOREM 1. *If*

(18) $\lambda_n \neq 0$, $p_n \neq 0$, $P_n = \sum_{\nu=0}^{n} p_\nu \neq 0$ $(n = 0, 1, 2, \ldots)$,

(19) $|p_n/\lambda_n P_{n-1}| \leqslant K \mathrm{re}(p_n/\lambda_n P_{n-1})$ *with some* $K > 1$ $(n = 1, 2 \ldots)$,

(20) $$\sum_{n=1}^{\infty} |p_n/\lambda_n P_{n-1}| = \infty,$$

then the hypothesis

$$\lim_n \varphi_n = s,$$

where $\varphi_n = \lambda_n s_n + (1-\lambda_n) t_n$, $t_n = \sum_{\nu=0}^{n} p_\nu s_\nu / P_n$, *implies*

$$\lim_n t_n = s.$$

Moreover, if

(21) $$\underline{\lim_n} |\lambda_n| > 0,$$

then

$$\lim_n s_n = s.$$

Compare Karamata [2]. Let us observe that the transformation $t_n = \sum_{\nu=0}^{n} p_\nu s_\nu / P_n$ may be not regular.

Proof. We have $s_n = t_n - (P_{n-1}/p_n)\Delta t_{n-1}$ and

$$\varphi_n = t_n - \lambda_n (P_{n-1}/p_n)\Delta t_{n-1}.$$

Applying Lemma 3.1 for $a_n = -\lambda_n P_{n-1}/p_n$ we obtain $\lim_n t_n = s$. If (21) is satisfied then $\lim_n s_n = s$ since

$$s_n = (\varphi_n - t_n)/\lambda_n + t_n.$$

Let us observe that if λ_n does not satisfy hypothesis (20), the conclusion of Theorem 1 may be not true. For example, if $p_n/\lambda_n P_{n-1} = a_n > 0$, $n = 1, 2, \ldots, \sum_{n=1}^{\infty} a_n < \infty$, we obtain

$$t_n - (1/a_n)\Delta t_{n-1} = \varphi_n \quad (n = 1, 2, \ldots), \tag{22}$$

and $t_n = c_n/x_n$, where $x_n = \prod_{\nu=1}^{n} (1+a_\nu)$. We calculate the unknown sequence $\{c_n\}$ substituting the value for t_n into (22). We obtain

$$c_n = \sum_{\nu=2}^{n} a_\nu \varphi_\nu x_{\nu-1} + \text{const}, \qquad n = 2, 3, \ldots$$

It follows that there exists a sequence $\{t_n\}$ satisfying (22) and such that $\lim_n c_n = \sigma$ and $\lim_n t_n = \sigma/\sigma_1 \neq s$, where $\sigma_1 = \lim_n x_n$.

4. LEMMA. *Suppose that*

$$0 < |a_n| \leqslant -K \operatorname{re} a_n \quad \textit{with some } K > 1 \ (n = 0, 1, 2, \ldots), \tag{23}$$

$$\sum_{n=0}^{\infty} 1/|a_n| = \infty, \tag{24}$$

$$|\beta_n - a_n \Delta(\gamma_n/a_n)| \leqslant K_1, \quad |\gamma_n/a_n| \leqslant K_2 \quad (n = 0, 1, 2, \ldots), \tag{25}$$

$$\overline{\lim_n} |\varphi_n| = M < \infty \tag{26a}$$

or

$$\lim_n \varphi_n = s \quad \textit{and} \quad \beta_n = B. \tag{26b}$$

Then the difference equation

$$L(x_n) = \beta_n \varphi_n + \gamma_n \Delta \varphi_{n-1} \quad (n = 1, 2, \ldots), \tag{27}$$

where $L(x_n) = x_n + a_n \Delta x_{n-1}$, *has a solution* $\{\bar{x}_n\}$ *bounded in case* (26a) *or such that* $\lim_n \bar{x}_n = Bs$ *in case* (26b).

Proof. We set

$$x_n = \gamma_{n+1}\varphi_n/a_{n+1} + x_n^{(1)} \quad (n = 0, 1, 2, \ldots),$$

and we obtain

$$\begin{aligned} L(x_n^{(1)}) &= L(x_n) - L(\gamma_{n+1}\varphi_n/a_{n+1}) \\ &= \beta_n\varphi_n + \gamma_n \Delta\varphi_{n-1} - L(\gamma_{n+1}\varphi_n/a_{n+1}), \end{aligned} \tag{28}$$

$$L(x_n^{(1)}) = A_n\varphi_n \quad (n = 1, 2, \ldots),$$

where $A_n = \beta_n - a_n \Delta(\gamma_n/a_n) - \gamma_{n+1}/a_{n+1}$.

In case (26a) it follows by (25) and Lemma 3.1 that (28) has a solution $\{\bar{x}_n^{(1)}\}$ such that $\overline{\lim_n}|\bar{x}_n^{(1)}| \leqslant M(K_1+K_2)K$. We set $\bar{x}_n = \gamma_{n+1}\varphi_n/a_{n+1} + \bar{x}_n^{(1)}$, $n = 0, 1, 2, \ldots$, and we obtain $\overline{\lim_n}|\bar{x}_n| \leqslant M(K_2+KK_1+KK_2)$. If $\lim_n \varphi_n = 0$ we infer from this that $\lim_n \bar{x}_n = 0$.

In case (26b) we set $x_n = x_n^* + Bs$, $\varphi_n = \varphi_n^* + s$ and we obtain from (27) the difference equation

$$L(x_n^*) = B\varphi_n^* + \gamma_n \Delta\varphi_{n-1}^* \qquad (n = 1, 2 \ldots),$$

which has a solution $\{\bar{x}_n^*\}$ such that $\lim_n \bar{x}_n^* = 0$, since $\lim_n \varphi_n^* = 0$. Now we set $\bar{x}_n = \bar{x}_n^* + Bs$.

Remark. It follows from the proof of Lemma 4 that the conclusion of this Lemma remains true if we replace the hypotheses (23) and (24) by the (weaker) hypothesis

(29) $\{b_n\}$ has the property C with every constant $> K$ and $|b_n| \to \infty$, where $b_n = \prod_{\nu=0}^{n}(1-1/a_\nu)$ (compare proof of Lemma 3.1, it is $a_n = b_{n-1}/\Delta b_{n-1}$).

THEOREM 2. *Suppose that*

$$p_n \neq 0,\ q_n \neq 0,\ |P_n| \to \infty,\ \sum_{\nu=0}^{n} |p_\nu| \leqslant K|P_n|, \quad \textit{where } P_n = \sum_{\nu=0}^{n} p_\nu, \tag{30}$$

$$Q_{n-1}/q_n = O(P_{n-1}/p_n), \quad \textit{where} \quad Q_n = \sum_{\nu=0}^{n} q_\nu, \tag{31}$$

$$\Delta\{p_n Q_{n-1}/P_{n-1}q_n\} = O(p_n/P_{n-1}). \tag{32}$$

Then the relation

$$\lim_n \sum_{\nu=0}^{n} q_\nu s_\nu / Q_n = s$$

implies

$$\lim_n \sum_{\nu=0}^{n} p_\nu s_\nu / P_n = s.$$

Proof. If

$$t_n^{(1)} = \sum_{\nu=0}^{n} p_\nu s_\nu / P_n, \qquad t_n^{(2)} = \sum_{\nu=0}^{n} q_\nu s_\nu / Q_n,$$

then

$$s_n = t_n^{(1)} - (P_{n-1}/p_n)\, \Delta t_{n-1}^{(1)} = t_n^{(2)} - (Q_{n-1}/q_n)\, \Delta t_{n-1}^{(2)}.$$

Now we apply Lemma 4 (see Remark) with $\alpha_n = -P_{n-1}/p_n$, $\beta_n = 1$, $\gamma_n = -Q_{n-1}/q_n$.

An analogous theorem for real p_n and q_n is proved in Hardy [1] (Theorem 14).

5. Lemma. *Suppose that*

$$|\operatorname{im} \alpha_n| < K(|\operatorname{re} \alpha_n| - 1), \quad \operatorname{re} \alpha_n \cdot \operatorname{re} \alpha_{n+1} > 0, \quad K > 0 \quad (n = 0, 1, 2, \ldots), \tag{33}$$

$$\sum_{n=0}^{\infty} 1/|\alpha_n| = \infty, \tag{34}$$

$$\lim_n |\alpha_n| = \infty, \tag{35}$$

$$\overline{\lim_n} |\varphi_n| = M < \infty \tag{36a}$$

or

$$\lim_n \varphi_n = s. \tag{36b}$$

Then for every real $h \neq 0$ the difference equation

$$x_n + h\alpha_n \Delta x_{n-1} = \varphi_n \quad (n = 1, 2, \ldots)$$

has a solution $\{\bar{x}_n\}$ satisfying the relation $\overline{\lim_n} |\bar{x}_n| \leqslant M\sqrt{1+4K^2}$ in case (36a) *or $\lim_n \bar{x}_n = s$ in case* (36b).

Proof. We shall prove that the sequence $\{h\alpha_n\}$ satisfies the hypothesis (11) or (15). If $h \operatorname{re} \alpha_n < 0$ then we obtain from (33)

$$|h \operatorname{im} \alpha_n| \leqslant K(|h \operatorname{re} \alpha_n| - |h|) < -Kh \operatorname{re} \alpha_n$$

and the hypothesis (11) of Lemma 3.1 is satisfied with $\sqrt{1+K^2}$ instead of K. If $h \operatorname{re} \alpha_n > 0$ and $|h| > 1$ then

$$|h \operatorname{im} \alpha_n| \leqslant K(|h \operatorname{re} \alpha_n| - |h|) < K(h \operatorname{re} \alpha_n - 1).$$

If $h\,\mathrm{re}\,a_n > 0$ and $|h| \leqslant 1$ we choose N such that $|\mathrm{re}\,a_n| > 2/|h|-1$ for $n \geqslant N$. We obtain for $n \geqslant N$

$$|\mathrm{im}\,a_n| \leqslant K(|\mathrm{re}\,a_n|-1) < \{2/|h|\}K(|h\,\mathrm{re}\,a_n|-1).$$

In the two last cases the hypothesis (15) of Lemma 3.2 is satisfied by the sequence $\{ha_n\}$ except for $n = 0, 1, \ldots, N-1$ in the second case. Now it suffices to apply Lemma 3.1 or 3.2 for a_{N+n} instead of a_n.

6. LEMMA. *If the sequences* $\{a_n\}$ *and* $\{\varphi_n\}$ *satisfy the hypotheses* (33)-(36), *furthermore*

(37) $$C \text{ and } C_1 \text{ are real, } C_1 \neq 0,\ C_1 < C^2/4,$$

then the difference equation

(38) $$L(x_n) = \varphi_n + C_2 a_n \Delta\varphi_{n-1} \quad (n = 1, 2, \ldots),$$

where $L(x_n) = x_n + (C + C_1 \Delta a_{n-1}) a_n \Delta x_{n-1} + C_1 a_n a_{n-1} \Delta^2 x_{n-2}$, *has a solution* $\{\bar{x}_n\}$ *such that* $\overline{\lim\limits_n}|\bar{x}_n| \leqslant M\sqrt{1+4K^2}K_1$ *in case* (36a), (K_1 depends on C, C_1 and C_2 only, more exactly, $K_1 = |\lambda_1|+|\lambda_2|$, where λ_1 and λ_2 are defined in the proof) *and* $\lim\limits_n \bar{x}_n = s$ *in case* (36b).

Proof. We set $L_i(x_n) = x_n + h_i a_n \Delta x_{n-1}$, $i = 1, 2$, $h_1+h_2 = C$, $h_1h_2 = C_1$. By Lemma 5 the difference equation $L_i(x_n) = \varphi_n$ has a solution $\{\bar{x}_n^{(i)}\}$ such that $\overline{\lim\limits_n}|\bar{x}_n^{(i)}| \leqslant M\sqrt{1+4K^2}$ in case (36a) or $\lim\limits_n \bar{x}_n^{(i)} = s$ in case (36b). Now we set $\bar{x}_n = \lambda_1\bar{x}_n^{(1)} + \lambda_2\bar{x}_n^{(2)}$, where λ_1 and λ_2 satisfy the relations $\lambda_1+\lambda_2 = 1$, $\lambda_1h_2+\lambda_2h_1 = C_2$.

It may be computed that $L_2L_1(x_n) = L_1L_2(x_n) = L(x_n)$. We obtain

$$\begin{aligned} L(\bar{x}_n) &= L(\lambda_1\bar{x}_n^{(1)} + \lambda_2\bar{x}_n^{(2)}) = \lambda_1 L_2(\varphi_n) + \lambda_2 L_1(\varphi_n) \\ &= (\lambda_1+\lambda_2)\varphi_n + (\lambda_1h_2+\lambda_2h_1)a_n\Delta\varphi_{n-1}, \end{aligned}$$

so that the sequence $\{\bar{x}_n\}$ satisfies (38). Moreover, we have $\overline{\lim\limits_n}|\bar{x}_n| \leqslant M\sqrt{1+4K^2}(|\lambda_1|+|\lambda_2|)$ in case (36a) and $\lim\limits_n \bar{x}_n = s$ in case (36b).

7. LEMMA. *If*

$$P_n = \sum_{\nu=0}^{n} p_\nu, \quad t_n = \sum_{\nu=0}^{n} p_\nu s_\nu / P_n \quad \textit{for } n = 0, 1, 2, \ldots,$$

$$s_0 = 1, \quad s_n = 1 \Big/ \prod_{\nu=1}^{n}\left(1 - r\frac{p_\nu}{P_{\nu-1}}\right) \quad \textit{for } n = 1, 2, \ldots, \quad r \neq -1,$$

then

$$t_n = [1/(r+1)]s_n \quad \textit{for } n = 1, 2, \ldots$$

Proof. We seek the solution of the equation

$$\sum_{\nu=0}^{n} p_\nu s_\nu / P_n = \lambda s_n, \qquad \lambda \neq 0 \qquad (n = 1, 2, \ldots).$$

We obtain

$$p_n s_n = \lambda (P_n s_n - P_{n-1} s_{n-1}), \frac{s_n}{s_{n-1}} = 1 \Big/ \left(1 - r \frac{p_n}{P_{n-1}}\right), \qquad r = \frac{1}{\lambda} - 1 \neq -1.$$

THEOREM 3. *Suppose that*

$$\lim_n \sum_{\nu=0}^{n} p_\nu s_\nu / P_n = s, \qquad \textit{where} \qquad P_n = \sum_{\nu=0}^{n} p \neq 0 \qquad \textit{for } n = 0, 1, 2, \ldots$$

The relation

$$\lim_n s_n = s$$

holds if and only if

$$\lim_n \sum_{\nu=1}^{n} P_{\nu-1} a_\nu / P_n = 0, \qquad \textit{where} \qquad s_n = \sum_{\nu=0}^{n} a_\nu.$$

The proof follows immediately from the identity

$$\sum_{\nu=1}^{n} P_{\nu-1} a_\nu / P_n = s_n - \sum_{\nu=0}^{n} p_\nu s_\nu / P_n,$$

which may be obtained by Abel transformation.

Let us remark that if $p_n \neq 0$ then the sequence with $t_n = \sum_{\nu=1}^{n} P_{\nu-1} a_\nu / P_n$ is the Riesz mean of the sequence $\{P_{n-1} a_n / p_n\}_{n=1,2,\ldots}$, $a_0 = 0$.

THEOREM 4. *Suppose that*

(39) $$p_n \neq 0, \quad P_n = \sum_{\nu=0}^{n} p_\nu \neq 0, \quad |\mathrm{im}(P_{n-1}/p_n)| < K[\mathrm{re}(P_{n-1}/p_n) - 1], \quad K > 0,\ n = 1, 2, \ldots,$$

(40) $$\lim_n |P_{n-1}/p_n| = \infty,$$

(41) $$\sum_{n=1}^{n} |p_n / P_{n-1}| = \infty,$$

(42) $$|P_{n-1} a_n / p_n| \leqslant M \qquad (n = 1, 2, \ldots).$$

Then the hypothesis

$$\lim_n \sum_{\nu=0}^{n} p_\nu s_\nu / P_n = s, \qquad \textit{where} \qquad s_n = \sum_{\nu=0}^{n} a_\nu,$$

implies

$$\lim_n s_n = s.$$

An analogous theorem for real p_n is proved in [1] (Theorem 67).

Proof. We may assume $M>1$. Under the hypothesis $s=0$ we set for a given ε $(0<\varepsilon<1)$:

$$\sum_{\nu=0}^{n} p_\nu s_\nu / P_n + A(P_{n-1}/p_n)\Delta s_{n-1} = \varphi_n \qquad (n=1,2,\ldots)$$

with $A = \dfrac{\varepsilon^2}{4M^2(1+4K^2)} < \dfrac{1}{4}$.

Multiplying by P_n we obtain

$$p_n s_n + A\frac{P_n P_{n-1}}{p_n}\Delta s_{n-1} - A\frac{P_{n-1}P_{n-2}}{p_{n-1}}\Delta s_{n-2} = P_n\varphi_n - P_{n-1}\varphi_{n-1},$$

$$s_n + A\frac{P_{n-1}}{p_n}\left(1-\frac{P_{n-2}}{p_{n-1}}+\frac{P_{n-1}}{p_n}\right)\Delta s_{n-1} - A\frac{P_{n-1}P_{n-2}}{p_n p_{n-1}}\Delta^2 s_{n-2} = \varphi_n - \frac{P_{n-1}}{p_n}\Delta\varphi_{n-1},$$

$$L(s_n) = \varphi_n - \frac{P_{n-1}}{p_n}\Delta\varphi_{n-1}, \tag{43}$$

where $L(s_n)$ is defined as in Lemma 6 with $a_n = -P_{n-1}/p_n$, $C = C_1 = -A$.

We have $\overline{\lim\limits_n}|\varphi_n| \leqslant AM$ by (42). By Lemma 6 the difference equation (43) has a solution $\{\bar{x}_n\}$ such that $\overline{\lim\limits_n}|\bar{x}_n| \leqslant AM\sqrt{1+4K^2}K_1$, where $K_1 = |\lambda_1|+|\lambda_2|$ and λ_1, λ_2 satisfy the relations $\lambda_1+\lambda_2=1$, $\lambda_1 h_2+\lambda_2 h_1 = 1$; furthermore $h_i^2+Ah_i-A=0$, $i=1,2$, $h_1 = -\frac{1}{2}A+\sqrt{\frac{1}{4}A^2+A}>0$, $h_1<1$ and $h_2 = -\frac{1}{2}A-\sqrt{\frac{1}{4}A^2+A}<0$, $h_2>-1$.

It follows

$$\lambda_1 = \frac{1-h_1}{h_2-h_1}, \qquad \lambda_2 = \frac{1-h_2}{h_1-h_2}, \qquad |\lambda_1|+|\lambda_2| = \frac{1+\frac{1}{2}A}{\sqrt{\frac{1}{4}A^2+A}} < \frac{2}{\sqrt{A}}$$

and

$$\overline{\lim_n}|\bar{x}_n| \leqslant 2\sqrt{A}M\sqrt{1+4K^2} = \varepsilon.$$

We obtain

$$s_n = \bar{x}_n + \mu_1 x_n^{(1)} + \mu_2 x_n^{(2)}, \tag{44}$$

where

$$x_n^{(i)} = \frac{1}{\prod\limits_{\nu=1}^{n}\left(1-\dfrac{p_\nu}{h_i P_{\nu-1}}\right)} \qquad (i=1,2)$$

is a solution of the difference equation $L_i(x_n)=0$ (see Lemma 6) and consequently $L(x_n^{(i)})=0$ since $L_1L_2 = L_2L_1 = L$. We have $\Delta x_{n-1}^{(2)}/x_n^{(2)} = -p_n/h_2P_{n-1}$ and $\lim\limits_n x_n^{(2)} = 0$ by Lemma 2.

We shall prove indirectly that $\mu_1 = 0$. Suppose that $\mu_1 \neq 0$. By 3.1, we have

$$\overline{\lim_n}\left|\frac{\sum_{\nu=0}^{n} p_\nu \bar{x}_\nu}{P_n}\right| \leqslant \varepsilon\sqrt{1+K^2}$$

since the sequence $t_n = \sum_{\nu=0}^{n} p_\nu \bar{x}_\nu / P_n$ satisfies the difference equation

$$t_n - \frac{P_{n-1}}{p_n}\Delta t_{n-1} = \bar{x}_n.$$

We infer from (44) that

$$\lim_n\left(\frac{\sum_{\nu=0}^{n} p_\nu \bar{x}_\nu}{P_n} + \mu_1 \frac{\sum_{\nu=0}^{n} p_\nu x_\nu^{(1)}}{P_n}\right) = \lim_n \frac{\sum_{\nu=0}^{n} p_\nu s_\nu}{P_n} = 0,$$

since the transformation $\sum_{\nu=0}^{n} p_\nu x_\nu^{(2)}/P_n$ is regular by (39), (41) and Lemma 2 for $b_n = P_{n-1}$. Moreover

$$|\mu_1|\,\overline{\lim_n}\left|\frac{\sum_{\nu=0}^{n} p_\nu x_\nu^{(1)}}{P_n}\right| \leqslant \varepsilon\sqrt{1+K^2}.$$

Now, we obtain by (40)

$$\frac{\Delta x_{n-1}^{(1)}}{x_{n-1}^{(1)}} = \frac{-p_n/h_1 P_{n-1}}{1-p_n/h_1 P_{n-1}} \sim -\frac{p_n}{h_1 P_{n-1}}.$$

We infer from this that $|b_n| = |x_{n-1}^{(1)}| \to \infty$, by Lemma 2. It follows $|\sum_{\nu=0}^{n} p_\nu x_\nu^{(1)}/P_n| \to \infty$ by Lemma 7. The obtained contradiction proves that $\mu_1 = 0$ and $\overline{\lim_n}|s_n| = \overline{\lim_n}|\bar{x}_n| \leqslant \varepsilon$, by (44). Since ε may be arbitrarily small we obtain $\lim_n s_n = 0$.

Under the hypothesis $\lim_n \sum_{\nu=0}^{n} p_\nu s_\nu / P_n = s \neq 0$ we set $s_n^* = s_n - s$ and infer that $\lim_n s_n^* = 0$. Then $\lim_n s_n = s$.

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES