ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XI (1967)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XI (1967)

C. E. Aull (Blacksburg, Va)

A certain class of topological spaces

Introduction. This paper is concerned primarily with topological spaces such that every point is the intersection of a countable number of closed neighborhoods. These spaces will be designated as E_1 spaces. Spaces such that each point is a G_δ have been studied by several authors. For a summary of results see Anderson [3]. Regular spaces that satisfy the latter property are E_1 spaces. As a consequence most of the results in Anderson [3] apply to E_1 spaces. Perfectly normal T_1 spaces and hereditary Lindelöf T_2 spaces are also E_1 spaces.

The following are proved. Every T_2, C_1 (first countable) topological space is an E_1 space. Locally countably paracompact (see Definitions 3 and 4) E_1 spaces are T_3 from which it follows that a T_2 space is metrizable iff it is locally countably paracompact and has a σ-locally finite base. Countably compact E_1 spaces are maximally countably compact and minimally E_1.

In general the notation of Kelley [7] will be used. However M^* will indicate the complement of M. $\overline{M}^*$ will indicate the complement of M closure and $\overline{M^*}$ will indicate the closure of M complement. If "a" is used as a subscript, it is understood that "a" is a member of an index set A.

E_0 and E_1 spaces.

Definition 1. A topological space is an E_0 *space* if every point is a G_δ.

Definition 2. A topological space is an E_1 *space* if every point is the intersection of a countable number of closed neighborhoods.

Clearly every E_1 space is E_0 and T_2, and every E_0 space is T_1. But an E_0 space may not be T_2. As an example, consider a countable space such that the only non-trivial closed sets are finite. Also see Anderson [3]. However, since in a regular space every open neighborhood of a point contains a closed neighborhood, the following relation is satisfied.

THEOREM 1. *Every regular E_0 space is E_1.*

In a T_2 space it can be easily shown that every paracompact subset is closed. We will show that in an E_1 space, every countably paracompact subset is closed. Countable paracompactness was introduced by Dowker [5].

DEFINITION 3. A subset M of a topological space $(X, \mathscr{T})$ is *countably paracompact* if every countable open cover of M has an open locally finite refinement.

Remark. In the above definition, we are using the topology for the original space rather than the relative topology for M. The refinement is locally finite with respect to every point in the space. While all subsets of the real line with the usual topology are countably paracompact subspaces, only the closed subsets are countably paracompact subsets as given in Definition 3.

THEOREM 2. *Every countably paracompact subset of an E_1 space is closed.*

Proof. Let M be a countably paracompact subset and let $x \notin M$. Let $\{C_n\}$ be a countable family of closed neighborhoods of x such that $[x] = \bigcap C_n$. $\{C_n^*\}$ is a countable cover of M and $x \notin \overline{C_n^*}$ for any n. Hence there is a locally finite refinement $\{U_a\}$ such that $x \notin \overline{U}_a$. $(\bigcup \overline{U}_a)^*$ is an open set containing x and not intersecting M. Since x is arbitrary, M is closed.

COROLLARY 2. *Every countably compact E_1 space is maximally countably compact and minimally E_1.*

Proof. It follows from Theorem 2 that every countably compact subset of an E_1 space is closed and it is clear that with a stronger topology (one with additional open sets) the space will also be E_1. Then using arguments similar to those used in the theorem that a T_2 compact space is maximally compact and minimally T_2, one can prove the corollary. See for instance Vaidnanthaswamy [8], p. 104.

With this background, we introduce an example that is E_0 and T_2 but not E_1.

EXAMPLE 1. Let X be the ordinals up to and including the first uncountable ordinal. Let a subbase for the topology $(X, \mathscr{T})$ consist of the open sets of the order topology and the family of sets $\{U_n\}$ containing Ω defined as follows. Let an isolated point be of class 1 if it is preceded by a non-isolated point under the order topology or if it has no predecessors. Inductively, let an isolated point be of class $n+1$ if it is preceded by an isolated point of class n. Let U_n consist of Ω and all isolated points of class greater than or equal to n. $(X, \mathscr{T})$ is E_0 and T_2 and not E_1 or T_3.

Proof. All points of $X \sim \Omega$ are G_δ's under the order topology. Furthermore $\bigcap U_n = [\Omega]$. Since $\mathscr{T}$ is stronger than the order topology, $(X, \mathscr{T})$

is an E_0 space. Since X is T_2 under the order topology, $(X, \mathcal{T})$ is also T_2. $\mathcal{T}$ is weaker than the topology for X suing the subbase consisting of the sets of the order topology and Ω itself. This latter space is a countably compact E_1 space and is hence minimally E_1, so $(X, \mathcal{T})$ is not E_1. From Theorem 1, it follows that the above example is another example of a T_2 space that is not T_3.

Locally countably paracompact E_1-spaces.

DEFINITION 4. A topological space $(X, \mathcal{T})$ is *locally countably paracompact* if it has a countably paracompact subset for every x which is a neighborhood of x.

THEOREM 3. *Every locally countably paracompact E_1 space is T_3.*

Proof. Let F be closed and let $x \notin F$. Let P_x be the countably paracompact neighborhood of x and let $\{C_n\}$ be a countable family of closed neighborhoods of x such that $[x] = \bigcap C_n$. $\{C_n^*\}$ is a countable cover of $P_x \cap F$ having an open locally finite refinement $\{U_a\}$, since $P_x \cap F$ is closed by Theorem 2. $V = (\bigcup U_a) \cup P_x^*$ is an open set containing F such that $x \notin \overline{V}$. Hence $(X, \mathcal{T})$ is regular.

The above theorem is a generalization of a theorem of Alexandroff and Urysohn [2] that a countably compact C_1, T_2 space is T_3. In a previous paper [4], the author showed that a T_2 space is metrizable iff it is countably paracompact and has a σ-locally finite base. As a result of Theorem 3 we have the following modification.

COROLLARY 3. *A T_2 space is metrizable iff it is locally countably paracompact and has a σ-locally finite base.*

Relation to C_1 spaces. Clearly every C_1, T_3 space is E_1. However a stronger theorem can be proved.

THEOREM 4. *Every T_2, C_1 space is E_1. Every T_1, C_1 space is E_0.*

Proof. Let $\{U_n\}$ be the countable base for a point x. For each $y \in X$, $y \neq x$, there exists $U(y)$, $x \in U(y)$ and $y \notin \overline{U(y)}$. For each $U(y)$ there exists n such that $U_n \subset U(y)$. So $\bigcap \overline{U}_n \subset \bigcap \overline{U(y)} = [x]$, and $\{\overline{U}_n\}$ is the desired countable family of closed neighborhoods. The proof of the second statement is similar.

We now turn to a restricted converse of Theorem 4 due to Alexandroff and Urysohn [2], p. 66. A proof is included only for sake of completeness.

THEOREM 5. *A locally countably compact E_1 space is C_1. A regular E_0 locally countably compact space is C_1.*

Let $\{C_n\}$ be a countable family of closed neighborhoods with intersection $[x]$ and let U_n be the corresponding open neighborhoods such that $x \in U_n \subset C_n$. Let $V_n = \bigcap_{k=1}^{n} U_k$. Let F_x be the countably compact neighborhood of x. F_x is closed by Theorem 2. Let W_x be an open set such that $x \in W_x \subset F_x$. We wish to show that $\{W_n\}$ is a countable base at x where $W_n = W_x \cap V_n$. Let T be an open set containing x. Consider the set $T^* \cap F_x$. $\{\overline{W}_n^*\}$ is a countable open cover of $T^* \cap F_x$. There is a finite subcover. Let p be the largest subscript of this subcover. Clearly $\overline{W}_p^*$ covers $F_x \cap T^*$ so that $W_p \subset F_x^* \cup T$. Since $W_p \cap F_n^* = \emptyset$, $W_p \subset T$. This proves the first assertion. The second follows from Theorem 1.

Relation to Lindelöf spaces. There are Lindelöf, even compact T_2 spaces that do not satisfy E_0, such as the one point compactification of an uncountable discrete space. But the hereditary Lindelöf spaces are closely related to E_1.

THEOREM 6. *Every E_0, T_2 Lindelöf space is E_1. Every hereditary Lindelöf T_2 space is E_1.*

Proof. Let $x \in X$. For $y \neq x$, there exist U_y such that $x \notin \overline{U_y}$ by the T_2 property. Let $\{G_n\}$ be a countable open family with intersection x. The family consisting of $\{U_y\}$ and a member of $\{G_n\}$ is a cover of X and may be replaced by a countable subcover. For each member of $\{G_n\}$, there is such a countable family. Let $\{V_{mn}\}$ be the countable family associated with G_n, but with G_n excluded. $\{V_{mn}\}$ is a countable subfamily of $\{U_y\}$ which covers $[x]^*$. And $x \notin \overline{V}_{mn}$ for any V_{mn}. $\bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{V_{mn}^*} = [x]$ and the first assertion is proved. Consider the above family $\{U_y\}$ which in a hereditary Lindelöf space may be replaced by a countable family $\{U_n\}$. $\bigcap \overline{U}_n^* = [x]$, so $(X, \mathscr{T})$ is E_0 and by the first assertion is E_1.

One may also prove that in a T_2 hereditary Lindelöf space, every compact subset is the intersection of a countable number of closed neighborhoods.

Some examples concerning locally countably paracompact spaces. There are topological spaces that are not locally countably paracompact. The example below of Alexandroff and Urysohn [2] is an example of such a space.

EXAMPLE 2. Let the Hausdorff neighborhoods of a point p in the Euclidean plane consist of open circles with p at the center excluding the points on the vertical diameters except p itself.

Since the resulting topology is a strengthening of the usual topology of the Euclidean plane it is an E_1 topology. Since this is a T_2 space which is not T_3, it can not be locally countably paracompact. Clearly by the homogeneity of the space, no point would have a countably paracompact neighborhood.

The well known example of Niemytski of a non normal completely regular space is locally countably paracompact in fact locally compact without being countably paracompact. See Vaidnanthaswamy [8], p. 155, for a general discussion and Gal [6], p. 157, for a discussion of the local compactness of Niemytski's example.

EXAMPLE 3. Let X be the upper half of the Euclidean plane bounded by the x-axis. Alter the topological structure of X by prescribing as new neighborhoods of each point p at the x-axis, the unions of p and open circles touching the x-axis at p.

We will show that this example of Niemytski's is not countably paracompact.

It is known that the two closed sets of the rational and irrational points of the x-axis do not possess disjoint open neighborhoods. For each rational, r, there is an open neighborhood U_r with closure not containing any irrational point. If X was countably paracompact, the rationals would be a countably paracompact subset and there would be a locally finite refinement of $\{U_r\}$ and hence an open set containing the rationals on the x-axis with closure disjoint from the irrationals on the x-axis, which is a contradiction. So $(X, \mathscr{T})$ is not countably paracompact. Since Example 1 is countably compact it is countably paracompact. Since it is not regular it is not locally compact or even locally paracompact.

Any countable T_3 space is hereditary Lindelöf and hence paracompact and hence locally countably paracompact. If the space is not C_1, the space will not be locally countably compact by Theorem 5 since such a space is E_1 by Theorem 6. For example of such a space due to Arens, see Kelley [7], p. 77.

References

[1] P. S. Alexandroff, *Bikompakte topologische Räume*, Math. Ann. 92 (1924), pp. 267-274.

[2] P. S. Alexandroff and P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wetensch. Amsterdam, 14 (1929), pp. 1-96.

[3] F. W. Anderson, *A lattice characterization of completely regular G_δ spaces*, Proc. Amer. Math. Soc. 6 (1955), pp. 757-765.

[4] C. E. Aull, *A note on countably paracompact spaces and metrization*, Proc. Amer. Math. Soc. 16 (1965), pp. 1316-1317.

[5] C. H. Dowker, *On countably paracompact spaces*, Canadian J. Math. 3 (1951), pp. 219-229.

[6] I. Gal, *Point set topology*, New York 1964.

[7] J. L. Kelley, *General Topology*, New York 1955.

[8] R. Vaidnanthaswamy, *Set topology*, 2nd ed., New York 1960.

VIRGINIA POLYTECHNIC INSTITUTE
KENT STATE UNIVERSITY