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Some properties of oscillatory solutions of certain differential equations of second order

We shall consider the second-order ordinary differential equation

\[(0.1) \quad [p(t)x']' + q(t, x) = 0,\]

where \(x = x(t)\), \(p\) is a positive continuous function on the interval \(I = \{t: a \leq t < \infty\}\), and \(q\) is a real-valued continuous function on the strip \(D = \{(t, x): t \in I\}\).

A solution \(x(t)\) of (0.1) is called oscillatory if for every \(t\) in \(I\) there are \(a, \beta\) such that \(\beta > a > t\), \(x(a) = 0\) and \(x(\beta) \neq 0\). A solution \(x(t)\) of (0.1) is called \(S\)-oscillatory if (i) it is oscillatory, (ii) the zeros of \(x\) have no accumulation point in \(I\), (iii) for every two points \(a, \beta\) such that \(a \leq a < \beta\) and \(x'(a) = x'(\beta) = 0\) there exists a \(\gamma\) such that \(a < \gamma < \beta\) and \(x(\gamma) = 0\), i.e., the zeros of \(x\) and \(x'\) follow alternately.

Equation (0.1) is called oscillatory \([S\text{-oscillatory}]\) if every non-zero solution of it is oscillatory \([S\text{-oscillatory, respectively}]\). It is well known that the equation

\[(0.2) \quad x'' + q(t)x = 0,\]

where \(q(t) \geq c > 0\), is \(S\)-oscillatory; on the other hand, the solution \(x(t) = 3\sin t + \sin 3t\) of the equation

\[x'' + \frac{15 - 18\sin^2 t}{1 + 2\cos^2 t} x = 0\]

is oscillatory and is not \(S\)-oscillatory.

We shall show (in Lemma 2) that under certain assumptions any oscillatory solution of (0.1) is \(S\)-oscillatory. Comparing (0.1) with another differential equations \([P(t)x']' + Q(t)x = 0\) we get certain conditions sufficient in order that (0.1) be \(S\)-oscillatory. The results obtained here overlap some results of R. V. Petropavlovskaja [13].

In the second part of the paper we obtain a condition sufficient in order that the series \(\sum_{n=0}^{\infty} (t_{2n+2} - t_{2n})\) be convergent, where \(t_0, t_2, \ldots\) denote
the successive zeros of an oscillatory solution $x(t)$. This condition is a generalization of one given by E. Gagliardo \[10\].

The last part of the paper concerns the values of an $S$-oscillatory solution $x(t)$ at the successive zeros of its derivative and the values of $x'(t)$ at the successive zeros of $x(t)$. Such investigation were initiated by G. Floquet \[9\] in 1883 who dealt with the solutions of (0.2) in the case where $q$ is periodic. In 1919 W. F. Osgood \[12\] investigated this problem for nonperiodic $q$. Further papers have been published by M. Biernecki \[1\], Z. Butlewski \[2\], \[4\], \[5\], \[6\], C. Taam \[15\], and others (cf. \[14\] and \[17\]). The results obtained in § 3 are more general than those in \[1\], \[2\], \[4\], and are somewhat different from those in \[6\] and \[15\].

§ 1. Let $\mathcal{P} = \{p : p \in C(\mathcal{I}), p(t) > 0 \text{ for } t \in \mathcal{I}\}$ and let $\mathcal{Q}_1$ be the set of all functions $q$ in $C(\mathcal{D})$ such that

$$
|q(t, x_1) - q(t, x_2)| \leq m(t)|x_1 - x_2|
$$

for some function $m$ in $C(\mathcal{I})$ depending on $q$. Here $C(\mathcal{I})$ denotes the set of all real-valued continuous functions on $\mathcal{I}$.

We begin by noticing that if $p \in \mathcal{P}$ and $q \in \mathcal{Q}_1$, then for any initial conditions $x(a) = c_0, x'(a) = c_1$ there exists a unique solution of (0.1) satisfying these conditions and defined for all $t \in \mathcal{I}$ (cf. L. Tonelli \[16\], R. Conti \[7\]).

**Lemma 1.** Suppose that $p \in \mathcal{P}, q \in \mathcal{Q}_1, P \in \mathcal{P}, Q \in C(\mathcal{I})$, the inequalities

$$
(1.2) \quad xq(t, x) \geq Q(t)x^2, \quad P(t) \leq P(t),
$$

$$
(1.3) \quad [P(t) - P(t)]^2 + [xq(t, x) - Q(t)x^2]^2 \neq 0
$$

are satisfied in $\mathcal{D}$ and $u(t)$ is a solution of the differential equation

$$
(1.4) \quad [P(t)u']' + Qu = 0
$$

such that $u(a) = u(\beta) = 0$, where $a \leq a < \beta$. Furthermore, suppose that $u(t) \neq 0$ for some $t$ in $(a, \beta)$. Then for every non-zero constant $c$ the function $x(t) = cu(t)$ is not a solution of the differential equation (0.1).

**Proof.** Suppose the contrary. Then $[P(t)u'(t)]' = -Q(t)u(t)$ and $[p(t)u'(t)]' = -c^{-1}q(t, cu(t))$, and the derivative of the function

$$
V(t) = [P(t) - P(t)]u(t)u'(t)
$$

would be nonnegative and not identically zero, because

$$
V'(t) = [P(t) - P(t)][u'(t)]^2 + c^{-2}\{cu(t)q[t, cu(t)] - Q(t)[cu(t)]^2\}.
$$

Consequently, $V(\beta) - V(a)$ would be positive, contradicting $V(a) = V(\beta) = 0$. 
LEMMa 2. Let \( x(t) \) be an oscillatory solution of (0.1) and let \( p \in \mathcal{P} \) and \( q \in \mathcal{Q}_2 \), where

\[
\mathcal{Q}_2 = \{ q: q \in \mathcal{Q}_1, q(t, 0) = 0, q(t, x) \not= 0 \text{ for } t \text{ in } \mathcal{I} \text{ and } x \not= 0 \}.
\]

Then \( x(t) \) is \( S \)-oscillatory.

Proof. Since \( q(t, 0) = 0 \) for \( t \in \mathcal{I} \), the function \( 0 \) is a solution of (0.1); consequently, in virtue of the uniqueness of the solution, \( x(t) \) does not vanish identically on any interval. Similarly, since \( q(t, x) \not= 0 \) for \( x \not= 0 \), \( x(t) \) does not vanish on any interval. An argument used by Z. Butlewski [3] shows that the zeros of \( x(t) \) must not have any accumulation point in \( \mathcal{I} \). Indeed, if there existed a monotone sequence \( \tau_1, \tau_2, \ldots \) of zeros of \( x(t) \) convergent to some \( \tau \) in \( \mathcal{I} \), then \( x(\tau) \) would be equal to 0,

\[
\frac{x'(\tau)}{x' (\tau_n)} = \lim_{n \to \infty} \frac{x(\tau) - x(\tau_n)}{\tau - \tau_n}
\]

would also be equal to zero, and \( x(t) \) would be identically 0. Thus, the zeros of \( x(t) \) can be arranged into an increasing sequence tending to \( \infty \). Now, if \( \alpha, \beta \) are two successive zeros of \( x'(t) \), then

\[
\int_{\alpha}^{\beta} q(t, x(t)) dt = p(\alpha)x'(\alpha) - p(\beta)x'(\beta) = 0
\]

and \( q(t, x(t)) \) must vanish at a point \( \gamma \) between \( \alpha \) and \( \beta \); since \( q(\gamma, x) \not= 0 \) if \( x \not= 0 \), \( x(\gamma) \) must vanish as well. Thus \( x(t) \) is \( S \)-oscillatory.

THEOREM 1. Suppose that \( p \in \mathcal{P} \), \( P \in \mathcal{P} \), \( q \in \mathcal{Q}_2 \), \( Q \in \mathcal{C}(\mathcal{I}) \), the equation (1.4) is oscillatory and (1.2) holds. Then the equation (0.1) is \( S \)-oscillatory.

Proof. If (1.3) is not satisfied, then (0.1) is \( S \)-oscillatory by Lemma 2. Now, if (1.3) does not hold, we argue as follows. Suppose that \( \alpha, \beta \) are two successive zeros of a non-zero solution \( u(t) \) of the equation (1.4). We shall show that any solution \( x(t) \) of (0.1) has at least one zero in \( \langle \alpha, \beta \rangle \). Indeed, if \( x(t) \not= 0 \) for \( t \) in \( \langle \alpha, \beta \rangle \), then the function

\[
S(t) = u(t) \left[ P(t)u'(t) - p(t)x'(t) \frac{u(t)}{x(t)} \right]
\]

is differentiable on \( \langle \alpha, \beta \rangle \). Taking into account (1.2) and (0.1), we infer that

\[
S'(t) = p(t) \left[ \frac{x'(t)u(t) - x(t)u'(t)}{x(t)} \right]^2 + [P(t) - p(t)][u'(t)]^2 + \frac{x(t)q[t, x(t)] - Q(t)[x(t)]^2}{x(t)} \left[ \frac{u(t)}{x(t)} \right]^2.
\]
Consequently, by (1.2) and (1.3), \( S'(t) \) is nonnegative and is not identically 0 because, by Lemma 1, \( x(t) \) is not identically equal to \( cu(t) \) for any \( c \neq 0 \), and this contradicts \( S(a) = S(\beta) = 0 \). We have thus shown that \( x(t) \) vanishes somewhere in \( (a, \beta) \); therefore \( x(t) \) is oscillatory and hence \( S \)-oscillatory (by Lemma 2).

**Corollary 1.** Suppose that \( p \in \mathcal{P} \), \( q \in \mathcal{L}_2 \), \( \lim_{t \to \infty} p(t) < \infty \), and there exists a positive function \( \varphi \) in \( C^3(\mathcal{I}) \) such that

\[
\int_a^\infty \left\{ \varphi(t)\varphi''(t) + [\varphi(t)]^2 [p(t)]^{-1} \inf_{|x|<\infty} \frac{q(t,x)}{x} \right\} dt = \infty \quad \text{and} \quad \int_a^\infty [\varphi(t)]^{-2} dt = \infty.
\]

Then equation \((0.1)\) is \( S \)-oscillatory.

Indeed, \( p(t) \leq K \) for some constant \( K \) and we may substitute \( P(t) = K, Q(t) = KQ_1(t) \), and apply the known criteria for the equation \( x'' + Q_1(t)x = 0 \) to be oscillatory (cf. [11] or [8]).

§ 2. Suppose now that \( x(t) \) is an \( S \)-oscillatory solution of equation \((0.1)\), \( p \in \mathcal{P} \), and \( q \in \mathcal{L}_2 \), where

\[
\mathcal{L}_2 = \{ q : q \in \mathcal{L}_2, xq(t, x) > 0 \ \text{for} \ x \neq 0 \}.
\]

Furthermore let \( t_0, t_2, t_4, \ldots \) denote the successive zeros of \( x(t) \) and let \( t_1, t_3, t_5, \ldots \) denote the successive zeros of \( x'(t) \) (\( t_{2n} < t_{2n+1} < t_{2n+2} \), for \( n = 0, 1, 2, \ldots \)). We shall use the following abbreviations:

\[
y(t) = p(t)x'(t), \quad X_{2n} = |y(t_{2n})|;
\]

\[
X_{2n+1} = |x(t_{2n+1})|, \quad X_{2n} = |x'(t_{2n})|.
\]

The following theorem is a generalization of a result of E. Gagliardo [10] who proved it for the linear equation \( x'' + Q(t)x = 0 \).

**Theorem 2.** Let \( 0 < p_1 < p(t) \leq p_2 < \infty \), where \( p_1, p_2 \) are constants and \( p \in \mathcal{P} \). Let \( x(t) \) be an \( S \)-oscillatory solution of \((0.1)\) such that the integral

\[
\int_a^\infty m(t) dt
\]

is convergent, where \( m \) is any continuous function satisfying \((1.1)\). Then the series

\[
\sum_{n=0}^\infty \frac{1}{t_{2n+2} - t_{2n}}
\]

is convergent.
Property of oscillatory solutions

Proof. Multiplying either side of (0.1) by \( y(t) \) and integrating from \( a \) to \( \beta \) we get

\[
\frac{1}{2} [y(\beta)]^2 - \frac{1}{2} [y(a)]^2 = - \int_a^\beta q[t, x(t)] p(t) x'(t) \, dt.
\]

Let \( a = t_{2n} \) and \( \beta = t_{2n+1} \). Then

\[
Y_{2n}^2 = 2 \int_{t_{2n}}^{t_{2n+1}} q[t, x(t)] p(t) x'(t) \, dt \quad (n = 0, 1, 2, \ldots);
\]

similarly, setting \( a = t_{2n+1} \), \( \beta = t_{2n+2} \), we get

\[
Y_{2n+2}^2 = -2 \int_{t_{2n+1}}^{t_{2n+2}} q[t, x(t)] p(t) x'(t) \, dt \quad (n = 0, 1, 2, \ldots).
\]

Since \( x(t) \) is \( S \)-oscillatory, \( x(t) x'(t) \geq 0 \) for \( t \) in \( (t_{2n}, t_{2n+1}) \) and \( x(t) x'(t) \leq 0 \) for \( t \) in \( (t_{2n+1}, t_{2n+2}) \). Integrating either side of (0.1) over \( (t_{2n}, t_{2n+1}) \) we get

\[
\int_{t_{2n}}^{t_{2n+1}} q[t, x(t)] \, dt = Y_{2n} + Y_{2n+2} > p_1 (X_{2n} + X_{2n+2}) \quad \text{if} \quad x(t) \geq 0
\]

and

\[
\int_{t_{2n}}^{t_{2n+1}} q[t, x(t)] \, dt = -Y_{2n} - Y_{2n+2} < -p_1 (X_{2n} + X_{2n+2}) \quad \text{if} \quad x(t) \leq 0.
\]

Hence, in virtue of \( X_{2n} = \left| x'(t_{2n}) \right| = Y_{2n}/p(t_{2n}) \),

\[
\int_{t_{2n}}^{t_{2n+2}} |q[t, x(t)]| \, dt > p_1 (X_{2n} + X_{2n+2}).
\]

If \( x(t) \geq 0 \) for \( t \) in \( (t_{2n}, t_{2n+1}) \), then

\[
[p(t)x'(t)]' = -q[t, x(t)] \leq 0,
\]

and therefore \( p(t)x'(t) \leq Y_{2n} \) for \( t \) in \( (t_{2n}, t_{2n+1}) \), \( x'(t) \leq p_2 X_{2n}/p_1 \) for \( t \) in \( (t_{2n}, t_{2n+1}) \), and \( -x'(t) \leq p_2 X_{2n+2}/p_1 \) for \( t \) in \( (t_{2n+1}, t_{2n+2}) \). Integrating the respective sides of these inequalities over \( (t_{2n}, t) \), where \( t \leq t_{2n+1} \), we get

\[
x(t) \leq x(t_{2n}) + \frac{p_2}{p_1} X_{2n} (t_{2n+1} - t) \leq \frac{p_2}{p_1} X_{2n} (t_{2n+1} - t_{2n}).
\]

Similarly integrating over \( (t, t_{2n+2}) \), where \( t \geq t_{2n+1} \), we get

\[
x(t) \leq x(t_{2n+2}) + \frac{p_2}{p_1} X_{2n+2} (t_{2n+2} - t) \leq \frac{p_2}{p_1} X_{2n+2} (t_{2n+2} - t_{2n}).
\]
Consequently,

\[ x(t) \leq \frac{P_2}{P_1} \max(X_{2n}, X_{2n+2})(t_{2n+2} - t_{2n}) \text{ for } n = 0, 1, 2, \ldots \]

Analogously, if \( x(t) \leq 0 \) for \( t \) in \( < t_{2n}, t_{2n+2} > \), we get

\[ x(t) \geq -\frac{P_2}{P_1} \max(X_{2n}, X_{2n+2})(t_{2n+2} - t_{2n}). \]

Thus, if \( t_{2n} \leq t \leq t_{2n+2} \), then

\[ |x(t)| \leq \frac{P_2}{P_1} \max(X_{2n}, X_{2n+2})(t_{2n+2} - t_{2n}). \]

Combining (2.6), (2.7) and \( q \in Q_3 \), we get

\[ \int_{t_{2n}}^{t_{2n+2}} m(t) dt \geq \int_{t_{2n}}^{t_{2n+2}} \left| \frac{q[t, x(t)]}{x(t)} \right| dt > \frac{P_1}{P_2} \frac{1}{t_{2n+2} - t_{2n}} \]

for \( n = 0, 1, 2, \ldots \), and therefore if the integral (2.2) is convergent, so is the series (2.3).

§ 3. Let \( Q_4 = \{q: q \in Q_3, q(t, -x) = -q(t, x) \text{ in } D\} \), and let \( \{t_n\}, \{X_n\}, \{Y_n\} \) be as above.

**Lemma 3.** Suppose that \( x(t) \) is an \( S \)-oscillatory solution of (0.1) where \( p \in P, q \in Q_4 \), and that

\[ (3.1) \text{ if } a \leq t < s, \text{ then } [p(s)q(s, x) - p(t)q(t, x)]x \geq 0. \]

Then

\[ (3.2) \quad p(t_{2n}) \int_0^{X_{2n+1}} q(t_{2n}, \xi) d\xi \leq \frac{1}{2} Y_{2n}^2 \leq p(t_{2n+1}) \int_0^{X_{2n+1}} q(t_{2n+1}, \xi) d\xi, \]

\[ (3.3) \quad p(t_{2n+1}) \int_0^{X_{2n+1}} q(t_{2n+1}, \xi) d\xi \leq \frac{1}{2} Y_{2n+2}^2 \leq p(t_{2n+2}) \int_0^{X_{2n+1}} q(t_{2n+2}, \xi) d\xi. \]

**Proof.** We have shown that \( x(t)x'(t) \geq 0 \) for \( t \) in \( < t_{2n}, t_{2n+1} > \); hence, by (3.1),

\[ p(t_{2n})q[t_{2n}, x(t)]x'(t) \leq p(t_{2n+1})q[t_{2n+1}, x(t)]x'(t) \]

in \( < t_{2n}, t_{2n+1} > \). Substituting \( x(t) = \xi (x(t) \text{ being monotone in this interval}) \), we get

\[ p(t_{2n}) \int_0^{X_{2n+1}} q(t_{2n}, \xi) d\xi \leq \int_{t_{2n}}^{t_{2n+1}} p(t)q[t, x(t)]x'(t) dt \leq p(t_{2n+1}) \int_0^{X_{2n+1}} q(t_{2n+1}, \xi) d\xi. \]
Since
\[ \int_0^\gamma q(t, x) \, dx = \int_0^{-\gamma} q(t, -x) \, dx, \]
we get (3.2). The proof of (3.3) is similar.

**Lemma 4.** Suppose that \( p \in \mathcal{P}, \, q \in \mathcal{L}_4 \), that

\[ (3.4) \quad \text{if } a \leq t < s, \text{ then } [p(s)q(s, x) - p(t)q(t, x)]x \leq 0, \]
and that \( x(t) \) is an \( S \)-oscillatory solution of (0.1). Then

\[ (3.5) \quad p(t_{2n+1}) \int_0^{x_{2n+1}} q(t_{2n+1}, \xi) \, d\xi \leq \frac{1}{2} Y_{2n}^2 \leq p(t_{2n}) \int_0^{x_{2n+1}} q(t_{2n}, \xi) \, d\xi, \]

\[ (3.6) \quad p(t_{2n+2}) \int_0^{x_{2n+1}} q(t_{2n+2}, \xi) \, d\xi \leq \frac{1}{2} Y_{2n+2}^2 \leq p(t_{2n+2}) \int_0^{x_{2n+1}} q(t_{2n+2}, \xi) \, d\xi. \]

The proof is similar to that of Lemma 3.

**Theorem 3.** Let \( p \in \mathcal{P}, \, q \in \mathcal{L}_4 \), and let \( x(t) \) be an \( S \)-oscillatory solution of (0.1). If (3.1) is satisfied, then

\[ X_{2n-1} \geq X_{2n+1} \quad \text{and} \quad Y_{2n} \leq Y_{2n+2}; \]

if (3.4) is satisfied, then

\[ X_{2n-1} \leq X_{2n+1} \quad \text{and} \quad Y_{2n} \geq Y_{2n+2}. \]

**Proof.** If \( q \in \mathcal{L}_4 \) and (3.1) holds, then from (3.2) and (3.3) it follows that \( Y_{2n} \leq Y_{2n+2} \) for \( n = 0, 1, 2, \ldots \). On the other hand, from (3.5) and (3.6) it follows that if \( q \in \mathcal{L}_4 \) and (3.4) holds, then \( Y_{2n} \geq Y_{2n+2} \). Substituting \( n-1 \) instead of \( n \) in (3.3), we get

\[ \frac{1}{2} Y_{2n}^2 \leq p(t_{2n}) \int_0^{x_{2n-1}} q(t_{2n}, \xi) \, d\xi \quad \text{for} \quad n = 1, 2, \ldots \]

Combining this inequality with (3.2) we infer that

\[ p(t_{2n}) \int_0^{x_{2n-1}} q(t_{2n}, \xi) \, d\xi \leq p(t_{2n}) \int_0^{x_{2n-1}} q(t_{2n}, \xi) \, d\xi \quad \text{for} \quad n = 1, 2, \ldots \]

In virtue of the properties of the functions \( p \) and \( q \) we get \( X_{2n-1} \geq X_{2n+1} \)
if (3.1) is valid and, analogously, \( X_{2n-1} \leq X_{2n+1} \) if (3.4) is valid, \( n = 1, 2, \ldots \). This concludes the proof.

**Corollary 2.** Assume successively that \( p \in \mathcal{P}, \, p \) is nondecreasing, \( q \in \mathcal{L}_4 \), (3.4) is valid, and that \( q(t, x) \geq q_1 x \geq 0 \) for \( x \geq 0 \) and \( t \) in \( \mathcal{I}, \) \( q_1 \) being a positive constant. Then every \( S \)-oscillatory solution of (0.1) has the
following properties:

\[ X_{2n} \geq X_{2n+2} \quad \text{and} \quad X_{2n+1} \leq X_{2n+3} \quad \text{for} \quad n = 0, 1, 2, \ldots \]

and

\[ |x(t)| \leq X_0 \sqrt{p(t_0)/q_1} \quad \text{for} \quad t \in \mathcal{I}. \]

Indeed, from Theorem 3 it follows that the sequence \( \{X_{2n}\} \) is non-increasing. From (3.5) it also follows that

\[ \int_0^{X_{2n+1}} q(t_{2n+1}, \xi) d\xi \leq \frac{1}{2} p(t_0) X_0^2. \]

Hence \( X_{2n+1}^2 \leq p(t_0) X_0^2/q_1 \) and we get the conclusion. A similar argument may apply if the inequalities are reversed.

**Corollary 3.** Suppose that \( p \in \mathcal{P}, \ p \) is nonincreasing, \( q \in \mathcal{Q}, \) and \( q(t, x) \leq q_2 x < \infty \) for \( x \geq 0 \) and \( t \in \mathcal{I} \) (\( q_2 \) being a constant), and (3.1) holds. Then any \( S \)-oscillatory solution of (0.1) has the following properties:

\[ X_{2n} \leq X_{2n+2}, \quad X_{2n+1} \geq X_{2n+3}, \quad \text{and} \quad X_{2n+1} \geq X_0 \sqrt{p(t_0)/q_2} \]

for \( n = 0, 1, 2, \ldots \)

**Theorem 4.** Let \( p \in \mathcal{P}, \ q \in \mathcal{Q}, \) let

\[ (3.7) \quad 0 < p_1 \leq p(t) \leq p_2 = \lambda p_1, \]

where \( 1 \leq \lambda < \infty \) and \( p_1 \) and \( p_2 \) are constants, and \( t \in \mathcal{I}, \) let (3.1) holds, and let

\[ (3.8) \quad 0 \leq q_1 x \leq q(t, x) \leq q_2 x = \mu q_1 x, \]

where \( 1 \leq \mu < \infty, \ q_1 \) and \( q_2 \) are constants, \( q_1 > 0, \ t \in \mathcal{I} \) and \( x \geq 0. \) Then any solution \( x(t) \) of (0.1) is \( S \)-oscillatory, \( \{X_{2n+1}\} \) is nonincreasing

\[ \lim_{n \to \infty} X_{2n+1} > 0, \]

and

\[ (3.9) \quad 0 \leq \log \frac{X_{2n+1}}{X_{2n+3}} \leq \log \lambda \mu, \quad n = 1, 2, 3, \ldots \]

This yields an estimation of the logarithmic decrement of \( \{X_{2n+1}\}. \)

**Proof.** The function \( \varphi(t) = 1 \) fulfils the hypotheses of Corollary 1; in particular,

\[ \int_0^\infty [p(t)]^{-1} \inf_{|x| < \infty} \left[ \frac{q(t, x)}{x} \right] dt = \infty. \]
Consequently, equation (0.1) is S-oscillatory. Moreover, by Theorem 3, \( \{X_{2n+1}\} \) is nonincreasing and bounded away from zero. In turn, by (3.2) and (3.4) it follows that
\[
\frac{X_{2n-1}}{X_{2n+1}} < \frac{p(t_{2n-1})}{p(t_{2n+1})} \int_0^\xi q(t_{2n-1}, \xi) d\xi \leq p(t_{2n+1}) \int_0^\xi q(t_{2n+1}, \xi) d\xi.
\]
Combining this inequality with the hypotheses of the theorem we get
\[
X_{2n-1} \leq V\sqrt{\lambda} \leq X_{2n+1}
\]
and hence (3.9). This concludes the proof.

**Theorem 5.** Assume that \( p \in \mathcal{P} \), \( q \in \mathcal{C}_4 \), and assume (3.4), (3.7) and (3.8). Then any solution \( x(t) \) of (0.1) is S-oscillatory and bounded for \( t \in \mathcal{J} \), \( \{X_{2n-1}\} \) is nonincreasing, and the logarithmic increment of \( \{X_{2n-1}\} \) satisfies the inequalities
\[
0 \leq \log \frac{X_{2n+3}}{X_{2n-1}} \leq \log V\sqrt{\lambda} \quad \text{for} \quad n = 1, 2, \ldots
\]

The proof is similar to that of Theorem 4.

**References**


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