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Some properties of oscillatory solutions of certain differential equations of second order

We shall consider the second-order ordinary differential equation

$$[p(t)x']'+q(t,x) = 0, \tag{0.1}$$

where $x = x(t)$, p is a positive continuous function on the interval $\mathscr{I} = \{t\colon a \leqslant t < \infty\}$, and q is a real-valued continuous function on the strip $\mathscr{D} = \{(t,x)\colon t \in \mathscr{I}\}$.

A solution $x(t)$ of (0.1) is called *oscillatory* if for every t in $\mathscr{I}$ there are α, β such that $\beta > \alpha > t$, $x(\alpha) = 0$ and $x(\beta) \neq 0$. A solution $x(t)$ of (0.1) is called *S-oscillatory* if (i) it is oscillatory, (ii) the zeros of x have no accumulation point in $\mathscr{I}$, (iii) for every two points α, β such that $a \leqslant \alpha < \beta$ and $x'(\alpha) = x'(\beta) = 0$ there exists a γ such that $\alpha < \gamma < \beta$ and $x(\gamma) = 0$, i.e., the zeros of x and x' follow alternately.

Equation (0.1) is called *oscillatory* [*S-oscillatory*] if every non-zero solution of it is oscillatory [S-oscillatory, respectively]. It is well known that the equation

$$x''+q(t)x = 0, \tag{0.2}$$

where $q(t) \geqslant c > 0$, is S-oscillatory; on the other hand, the solution $x(t) = 3\sin t+\sin 3t$ of the equation

$$x''+\frac{15-18\sin^2 t}{1+2\cos^2 t}\,x = 0$$

is oscillatory and is not S-oscillatory.

We shall show (in Lemma 2) that under certain assumptions any oscillatory solution of (0.1) is S-oscillatory. Comparing (0.1) with another differential equations $[P(t)x']'+Q(t)x = 0$ we get certain conditions sufficient in order that (0.1) be S-oscillatory. The results obtained here overlap some results of R. V. Petropavlovskaia [13].

In the second part of the paper we obtain a condition sufficient in order that the series $\sum_{n=0}^{\infty}(t_{2n+2}-t_{2n})$ be convergent, where $t_0, t_2, \ldots$ denote

the successive zeros of an oscillatory solution $x(t)$. This condition is a generalization of one given by E. Gagliardo [10].

The last part of the paper concerns the values of an S-oscillatory solution $x(t)$ at the successive zeros of its derivative and the values of $x'(t)$ at the successive zeros of $x(t)$. Such investigation were initiated by G. Floquet [9] in 1883 who dealt with the solutions of (0.2) in the case where q is periodic. In 1919 W. F. Osgood [12] investigated this problem for nonperiodic q. Further papers have been published by M. Biernacki [1], Z. Butlewski [2], [4], [5], [6], C. Taam [15], and others (cf. [14] and [17]). The results obtained in § 3 are more general than those in [1], [2], [4], and are somewhat different from those in [6] and [15].

§ 1. Let $\mathscr{P} = \{p\colon p \in C(\mathscr{I}),\ p(t) > 0$ for t in $\mathscr{I}\}$ and let $\mathscr{Q}_1$ be the set of all functions q in $C(\mathscr{D})$ such that

$$|q(t, x_1) - q(t, x_2)| \leqslant m(t)\,|x_1 - x_2| \tag{1.1}$$

for some function m in $C(\mathscr{I})$ depending on q. Here $C(\mathscr{I})$ denotes the set of all real-valued continuous functions on $\mathscr{I}$.

We begin by noticing that if $p \in \mathscr{P}$ and $q \in \mathscr{Q}_1$, then for any initial conditions $x(a) = c_0$, $x'(a) = c_1$ there exists a unique solution of (0.1) satisfying these conditions and defined for all t in $\mathscr{I}$ (cf. L. Tonelli [16], R. Conti [7]).

LEMMA 1. *Suppose that $p \in \mathscr{P}$, $q \in \mathscr{Q}_1$, $P \in \mathscr{P}$, $Q \in C(\mathscr{I})$, the inequalities*

$$xq(t, x) \geqslant Q(t)x^2, \qquad p(t) \leqslant P(t), \tag{1.2}$$

$$[P(t) - p(t)]^2 + [xq(t, x) - Q(t)x^2]^2 \not\equiv 0 \tag{1.3}$$

are satisfied in $\mathscr{D}$ and $u(t)$ is a solution of the differential equation

$$[P(t)u']' + Q(t)u = 0 \tag{1.4}$$

such that $u(\alpha) = u(\beta) = 0$, where $a \leqslant \alpha < \beta$. Furthermore, suppose that $u(t) \neq 0$ for some t in (α, β). Then for every non-zero constant c the function $x(t) = cu(t)$ is not a solution of the differential equation (0.1).

Proof. Suppose the contrary. Then $[P(t)u'(t)]' = -Q(t)u(t)$ and $[p(t)u'(t)]' = -c^{-1}q\big(t, cu(t)\big)$, and the derivative of the function

$$V(t) = [P(t) - p(t)]\,u(t)\,u'(t)$$

would be nonnegative and not identically zero, because

$$V'(t) = [P(t) - p(t)][u'(t)]^2 + c^{-2}\{cu(t)q[t, cu(t)] - Q(t)[cu(t)]^2\}.$$

Consequently, $V(\beta) - V(\alpha)$ would be positive, contradicting $V(\alpha) = V(\beta) = 0$.

LEMMA 2. *Let $x(t)$ be an oscillatory solution of* (0.1) *and let $p \in \mathscr{P}$ and $q \in \mathscr{Q}_2$, where*

$$\mathscr{Q}_2 = \{q\colon\ q \in \mathscr{Q}_1,\ q(t,0) = 0,\ q(t,x) \neq 0 \text{ for } t \text{ in } \mathscr{I} \text{ and } x \neq 0\}.$$

Then $x(t)$ is S-oscillatory.

Proof. Since $q(t,0) = 0$ for t in $\mathscr{I}$, the function 0 is a solution of (0.1); consequently, in virtue of the uniqueness of the solution, $x(t)$ does not vanish identically on any interval. Similarly, since $q(t,x) \neq 0$ for $x \neq 0$, $x(t)$ does not vanish on any interval. An argument used by Z. Butlewski [3] shows that the zeros of $x(t)$ must not have any accumulation point in $\mathscr{I}$. Indeed, if there existed a monotone sequence τ_1, $\tau_2, \ldots$ of zeros of $x(t)$ convergent to some τ in $\mathscr{I}$, then $x(\tau)$ would be equal to 0,

$$x'(\tau) = \lim_{n\to\infty} \frac{x(\tau)-x(\tau_n)}{\tau-\tau_n}$$

would also be equal to zero, and $x(t)$ would be identically 0. Thus, the zeros of $x(t)$ can be arranged into an increasing sequence tending to ∞. Now, if α, β are two successive zeros of $x'(t)$, then

$$\int_\alpha^\beta q\big(t, x(t)\big)dt = p(\alpha)x'(\alpha) - p(\beta)x'(\beta) = 0$$

and $q\big(t, x(t)\big)$ must vanish at a point γ between α and β; since $q(\gamma, x) \neq 0$ if $x \neq 0$, $x(\gamma)$ must vanish as well. Thus $x(t)$ is S-oscillatory.

THEOREM 1. *Suppose that $p \in \mathscr{P}$, $P \in \mathscr{P}$, $q \in \mathscr{Q}_2$, $Q \in C(\mathscr{I})$, the equation* (1.4) *is oscillatory and* (1.2) *holds. Then the equation* (0.1) *is S-oscillatory.*

Proof. If (1.3) is not satisfied, then (0.1) is S-oscillatory by Lemma 2 Now, if (1.3) does not hold, we argue as follows. Suppose that α, β are two successive zeros of a non-zero solution $u(t)$ of the equation (1.4). We shall show that any solution $x(t)$ of (0.1) has at least one zero in $\langle\alpha, \beta\rangle$. Indeed, if $x(t) \neq 0$ for t in $\langle\alpha, \beta\rangle$, then the function

$$S(t) = u(t)\left[P(t)u'(t) - p(t)x'(t)\frac{u(t)}{x(t)}\right]$$

is differentiable on $\langle\alpha, \beta\rangle$. Taking into account (1.2) and (0.1), we infer that

$$S'(t) = p(t)\left[\frac{x'(t)u(t) - x(t)u'(t)}{x(t)}\right]^2 + [P(t)-p(t)][u'(t)]^2 +$$
$$+ \{x(t)q[t, x(t)] - Q(t)[x(t)]^2\}\left[\frac{u(t)}{x(t)}\right]^2.$$

Consequently, by (1.2) and (1.3), $S'(t)$ is nonnegative and is not identically 0 because, by Lemma 1, $x(t)$ is not identically equal to $cu(t)$ for any $c \neq 0$, and this contradicts $S(\alpha) = S(\beta) = 0$. We have thus shown that $x(t)$ vanishes somewhere in $\langle \alpha, \beta \rangle$; therefore $x(t)$ is oscillatory and hence S-oscillatory (by Lemma 2).

COROLLARY 1. *Suppose that* $p \in \mathscr{P}$, $q \in \mathscr{Q}_2$, $\lim\limits_{t\to\infty} p(t) < \infty$, *and there exists a positive function* φ *in* $C^{(2)}(\mathscr{I})$ *such that*

$$\int_a^\infty \left\{\varphi(t)\varphi''(t) + [\varphi(t)]^2 [p(t)]^{-1} \inf_{|x|<\infty} \left[\frac{q(t,x)}{x}\right]\right\} dt = \infty \quad \textit{and} \quad \int_a^\infty [\varphi(t)]^{-2} dt = \infty.$$

Then equation (0.1) *is S-oscillatory.*

Indeed, $p(t) \leqslant K$ for some constant K and we may substitute $P(t) = K$, $Q(t) = KQ_1(t)$, and apply the known criteria for the equation $x'' + Q_1(t)x = 0$ to be oscillatory (cf. [11] or [8]).

§ 2. Suppose now that $x(t)$ is an S-oscillatory solution of equation (0.1), $p \in \mathscr{P}$, and $q \in \mathscr{Q}_3$, where

$$\mathscr{Q}_3 = \{q\colon\ q \in \mathscr{Q}_2,\ xq(t,x) > 0 \text{ for } x \neq 0\}.$$

Furthermore let $t_0, t_2, t_4, \ldots$ denote the successive zeros of $x(t)$ and let $t_1, t_3, t_5, \ldots$ denote the successive zeros of $x'(t)$ ($t_{2n} < t_{2n+1} < t_{2n+2}$, for $n = 0, 1, 2, \ldots$). We shall use the following abbreviations:

$$\begin{aligned} y(t) &= p(t)x'(t), & Y_{2n} &= |y(t_{2n})|, \\ X_{2n+1} &= |x(t_{2n+1})|, & X_{2n} &= |x'(t_{2n})|. \end{aligned} \tag{2.1}$$

The following theorem is a generalization of a result of E. Gagliardo [10] who proved it for the linear equation $x'' + Q(t)x = 0$.

THEOREM 2. *Let* $0 < p_1 < p(t) \leqslant p_2 < \infty$, *where* p_1, p_2 *are constants and* $p \in \mathscr{P}$. *Let* $x(t)$ *be an S-oscillatory solution of* (0.1) *such that the integral*

$$\int_a^\infty m(t)\,dt \tag{2.2}$$

is convergent, where m *is any continuous function satisfying* (1.1). *Then the series*

$$\sum_{n=0}^\infty \frac{1}{t_{2n+2} - t_{2n}} \tag{2.3}$$

is convergent.

Proof. Multiplying either side of (0.1) by $y(t)$ and integrating from α to β we get

$$\tfrac{1}{2}[y(\beta)]^2-\tfrac{1}{2}[y(\alpha)]^2 = -\int_{\alpha}^{\beta} q[t, x(t)]p(t)x'(t)dt.$$

Let $\alpha = t_{2n}$ and $\beta = t_{2n+1}$. Then

$$Y_{2n}^2 = 2\int_{t_{2n}}^{t_{2n+1}} q[t, x(t)]p(t)x'(t)dt \quad (n=0,1,2,\ldots); \tag{2.4}$$

similarly, setting $\alpha = t_{2n+1}$, $\beta = t_{2n+2}$, we get

$$Y_{2n+2}^2 = -2\int_{t_{2n+1}}^{t_{2n+2}} q[t, x(t)]p(t)x'(t)dt \quad (n = 0,1,2,\ldots). \tag{2.5}$$

Since $x(t)$ is S-oscillatory, $x(t)x'(t) \geqslant 0$ for t in $\langle t_{2n}, t_{2n+1}\rangle$ and $x(t)x'(t) \leqslant 0$ for t in $\langle t_{2n+1}, t_{2n+2}\rangle$. Integrating either side of (0.1) over $\langle t_{2n}, t_{2n+2}\rangle$ we get

$$\int_{t_{2n}}^{t_{2n+2}} q[t, x(t)]dt = Y_{2n}+Y_{2n+2} > p_1(X_{2n}+X_{2n+2}) \quad \text{if} \quad x(t) \geqslant 0$$

and

$$\int_{t_{2n}}^{t_{2n+2}} q[t, x(t)]dt = -Y_{2n}-Y_{2n+2} < -p_1(X_{2n}+X_{2n+2}) \quad \text{if} \quad x(t) \leqslant 0.$$

Hence, in virtue of $X_{2n} = |x'(t_{2n})| = Y_{2n}/p(t_{2n})$,

$$\int_{t_{2n}}^{t_{2n+2}} |q[t, x(t)]|dt > p_1(X_{2n}+X_{2n+2}). \tag{2.6}$$

If $x(t) \geqslant 0$ for t in $\langle t_{2n}, t_{2n+2}\rangle$, then

$$[p(t)x'(t)]' = -q[t, x(t)] \leqslant 0,$$

and therefore $p(t)x'(t) \leqslant Y_{2n}$ for t in $\langle t_{2n}, t_{2n+1}\rangle$, $x'(t) \leqslant p_2X_{2n}/p_1$ for t in $\langle t_{2n}, t_{2n+1}\rangle$, and $-x'(t) \leqslant p_2X_{2n+2}/p_1$ for t in $\langle t_{2n+1}, t_{2n+2}\rangle$. Integrating the respective sides of these inequalities over $\langle t_{2n}, t\rangle$, where $t \leqslant t_{2n+1}$, we get

$$x(t) \leqslant x(t_{2n})+\frac{p_2}{p_1}X_{2n}(t_{2n+1}-t) \leqslant \frac{p_2}{p_1}X_{2n}(t_{2n+1}-t_{2n}).$$

Similarly integrating over $\langle t, t_{2n+2}\rangle$, where $t \geqslant t_{2n+1}$, we get

$$x(t) \leqslant x(t_{2n+2})+\frac{p_2}{p_1}X_{2n+2}(t_{2n+2}-t) \leqslant \frac{p_2}{p_1}X_{2n+2}(t_{2n+2}-t_{2n}).$$

Consequently,

$$x(t) \leqslant \frac{p_2}{p_1} \max(X_{2n}, X_{2n+2})(t_{2n+2}-t_{2n}) \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Analogously, if $x(t) \leqslant 0$ for t in $\langle t_{2n}, t_{2n+2}\rangle$, we get

$$x(t) \geqslant -\frac{p_2}{p_1} \max(X_{2n}, X_{2n+2})(t_{2n+2}-t_{2n}).$$

Thus, if $t_{2n} \leqslant t \leqslant t_{2n+2}$, then

$$|x(t)| \leqslant \frac{p_2}{p_1} \max(X_{2n}, X_{2n+2})(t_{2n+2}-t_{2n}). \tag{2.7}$$

Combining (2.6), (2.7) and $q \in \mathscr{Q}_3$, we get

$$\int_{t_{2n}}^{t_{2n+2}} m(t)\,dt \geqslant \int_{t_{2n}}^{t_{2n+2}} \left| \frac{q[t, x(t)]}{x(t)} \right| dt > \frac{p_1^2}{p_2} \frac{1}{t_{2n+2}-t_{2n}}$$

for $n = 0, 1, 2, \ldots$, and therefore if the integral (2.2) is convergent, so is the series (2.3).

§ 3. Let $\mathscr{Q}_4 = \{q\colon q \in \mathscr{Q}_3, q(t, -x) = -q(t, x) \text{ in } \mathscr{D}\}$, and let $\{t_n\}$, $\{X_n\}$, $\{Y_n\}$ be as above.

LEMMA 3. *Suppose that $x(t)$ is an S-oscillatory solution of* (0.1) *where $p \in \mathscr{P}$, $q \in \mathscr{Q}_4$, and that*

$$\text{if } a \leqslant t < s, \text{ then } [p(s)q(s, x) - p(t)q(t, x)]x \geqslant 0. \tag{3.1}$$

Then

$$p(t_{2n}) \int_0^{X_{2n+1}} q(t_{2n}, \xi)\,d\xi \leqslant \tfrac{1}{2} Y_{2n}^2 \leqslant p(t_{2n+1}) \int_0^{X_{2n+1}} q(t_{2n+1}, \xi)\,d\xi, \tag{3.2}$$

$$p(t_{2n+1}) \int_0^{X_{2n+1}} q(t_{2n+1}, \xi)\,d\xi \leqslant \tfrac{1}{2} Y_{2n+2}^2 \leqslant p(t_{2n+2}) \int_0^{X_{2n+1}} q(t_{2n+2}, \xi)\,d\xi. \tag{3.3}$$

Proof. We have shown that $x(t)x'(t) \geqslant 0$ for t in $\langle t_{2n}, t_{2n+1}\rangle$; hence, by (3.1),

$$p(t_{2n})q[t_{2n}, x(t)]x'(t) \leqslant p(t)q[t, x(t)]x'(t) \leqslant p(t_{2n+1})q[t_{2n+1}, x(t)]x'(t)$$

in $\langle t_{2n}, t_{2n+1}\rangle$. Substituting $x(t) = \xi$ ($x(t)$ being monotone in this interval), we get

$$p(t_{2n}) \int_0^{x(t_{2n+1})} q(t_{2n}, \xi)\,d\xi \leqslant \int_{t_{2n}}^{t_{2n+1}} p(t)q[t. x(t)]x'(t)\,dt \leqslant p(t_{2n+1}) \int_0^{x(t_{2n+1})} q(t_{2n+1}, \xi)\,d\xi.$$

Since

$$\int_0^{\gamma} q(t, x)\,dx = \int_0^{-\gamma} q(t, -x)\,dx,$$

we get (3.2). The proof of (3.3) is similar.

LEMMA 4. *Suppose that* $p \in \mathscr{P}$, $q \in \mathscr{Q}_4$, *that*

$$\text{(3.4)} \qquad \text{if } a \leqslant t < s, \text{ then } [p(s)q(s, x) - p(t)q(t, x)]x \leqslant 0,$$

and that $x(t)$ *is an S-oscillatory solution of* (0.1). *Then*

$$\text{(3.5)} \qquad p(t_{2n+1}) \int_0^{X_{2n+1}} q(t_{2n+1}, \xi)\,d\xi \leqslant \tfrac{1}{2} Y_{2n}^2 \leqslant p(t_{2n}) \int_0^{X_{2n+1}} q(t_{2n}, \xi)\,d\xi,$$

$$\text{(3.6)} \quad p(t_{2n+2}) \int_0^{X_{2n+1}} q(t_{2n+2}, \xi)\,d\xi \leqslant \tfrac{1}{2} Y_{2n+2}^2 \leqslant p(t_{2n+2}) \int_0^{X_{2n+1}} q(t_{2n+1}, \xi)\,d\xi.$$

The proof is similar to that of Lemma 3.

THEOREM 3. *Let* $p \in \mathscr{P}$, $q \in \mathscr{Q}_4$, *and let* $x(t)$ *be an S-oscillatory solution of* (0.1). *If* (3.1) *is satisfied, then*

$$X_{2n-1} \geqslant X_{2n+1} \quad \text{and} \quad Y_{2n} \leqslant Y_{2n+2};$$

if (3.4) *is satisfied, then*

$$X_{2n-1} \leqslant X_{2n+1} \quad \text{and} \quad Y_{2n} \geqslant Y_{2n+2}.$$

Proof. If $q \in \mathscr{Q}_4$ and (3.1) holds, then from (3.2) and (3.3) it follows that $Y_{2n} \leqslant Y_{2n+2}$ for $n = 0, 1, 2, \ldots$ On the other hand, from (3.5) and (3.6) it follows that if $q \in \mathscr{Q}_4$ and (3.4) holds, then $Y_{2n} \geqslant Y_{2n+2}$. Substituting $n-1$ instead of n in (3.3), we get

$$\tfrac{1}{2} Y_{2n}^2 \leqslant p(t_{2n}) \int_0^{X_{2n-1}} q(t_{2n}, \xi)\,d\xi \quad \text{for} \quad n = 1, 2, \ldots$$

Combining this inequality with (3.2) we infer that

$$p(t_{2n}) \int_0^{X_{2n+1}} q(t_{2n}, \xi)\,d\xi \leqslant p(t_{2n}) \int_0^{X_{2n-1}} q(t_{2n}, \xi)\,d\xi \quad \text{for} \quad n = 1, 2, \ldots$$

In virtue of the properties of the functions p and q we get $X_{2n-1} \geqslant X_{2n+1}$ if (3.1) is valid and, analogously, $X_{2n-1} \leqslant X_{2n+1}$ if (3.4) is valid, $n = 1, 2, \ldots$ This concludes the proof.

COROLLARY 2. *Assume successively that* $p \in \mathscr{P}$, p *is nondecreasing,* $q \in \mathscr{Q}_4$, (3.4) *is valid, and that* $q(t, x) \geqslant q_1 x \geqslant 0$ *for* $x \geqslant 0$ *and* t *in* $\mathscr{I}$, q_1 *being a positive constant. Then every S-oscillatory solution of* (0.1) *has the*

following properties:

$$X_{2n} \geqslant X_{2n+2} \quad \textit{and} \quad X_{2n+1} \leqslant X_{2n+3} \quad \textit{for } n = 0, 1, 2, \ldots$$

and

$$|x(t)| \leqslant X_0 \sqrt{p(t_0)/q_1} \quad \textit{for } t \textit{ in } \mathscr{I}.$$

Indeed, from Theorem 3 it follows that the sequence $\{X_{2n}\}$ is nonincreasing. From (3.5) it also follows that

$$\int_0^{X_{2n+1}} q(t_{2n+1}, \xi)\, d\xi \leqslant \tfrac{1}{2} p(t_0) X_0^2.$$

Hence $X_{2n+1}^2 \leqslant p(t_0) X_0^2/q_1$ and we get the conclusion. A similar argument may applied if the inequalities are reversed.

COROLLARY 3. *Suppose that $p \in \mathscr{P}$, p is nonincreasing, $q \in \mathscr{Q}_4$, and $q(t, x) \leqslant q_2 x < \infty$ for $x \geqslant 0$ and t in $\mathscr{I}$ (q_2 being a constant), and* (3.1) *holds. Then any S-oscillatory solution of* (0.1) *has the following properties:*

$$X_{2n} \leqslant X_{2n+2}, \quad X_{2n+1} \geqslant X_{2n+3}, \quad \textit{and} \quad X_{2n+1} \geqslant X_0 \sqrt{p(t_0)/q_2}$$

for $n = 0, 1, 2, \ldots$

THEOREM 4. *Let $p \in \mathscr{P}$, $q \in \mathscr{Q}_4$, let*

$$0 < p_1 \leqslant p(t) \leqslant p_2 = \lambda p_1, \tag{3.7}$$

where $1 \leqslant \lambda < \infty$ and p_1 and p_2 are constants, and $t \in \mathscr{I}$, let (3.1) *holds, and let*

$$0 \leqslant q_1 x \leqslant q(t, x) \leqslant q_2 x = \mu q_1 x, \tag{3.8}$$

where $1 \leqslant \mu < \infty$, q_1 and q_2 are constants, $q_1 > 0$, $t \in \mathscr{I}$ and $x \geqslant 0$. Then any solution $x(t)$ of (0.1) *is S-oscillatory, $\{X_{2n+1}\}$ is nonincreasing $\lim\limits_{n\to\infty} X_{2n+1} > 0$, and*

$$0 \leqslant \log \frac{X_{2n-1}}{X_{2n+3}} \leqslant \log \lambda\mu, \quad n = 1, 2, 3, \ldots \tag{3.9}$$

This yields an estimation of the logarithmic decrement of $\{X_{2n+1}\}$.

Proof. The function $\varphi(t) \equiv 1$ fulfils the hypotheses of Corollary 1; in particular,

$$\int_a^\infty [p(t)]^{-1} \inf_{|x|<\infty} \left[\frac{q(t, x)}{x}\right] dt = \infty.$$

Consequently, equation (0.1) is S-oscillatory. Moreover, by Theorem 3, $\{X_{2n+1}\}$ is nonincreasing and bounded away from zero. In turn, by (3.2) and (3.4) it follows that

$$p(t_{2n-1}) \int_0^{X_{2n-1}} q(t_{2n-1}, \xi)\, d\xi \leqslant p(t_{2n+1}) \int_0^{X_{2n+1}} q(t_{2n+1}, \xi)\, d\xi .$$

Combining this inequality with the hypotheses of the theorem we get $X_{2n-1} \leqslant \sqrt{\lambda\mu} \leqslant X_{2n+1}$ and hence (3.9). This concludes the proof.

THEOREM 5. *Assume that* $p \in \mathscr{P}$, $q \in \mathscr{Q}_4$, *and assume* (3.4), (3.7) *and* (3.8). *Then any solution* $x(t)$ *of* (0.1) *is* S-*oscillatory and bounded for* t *in* $\mathscr{I}$, $\{X_{2n-1}\}$ *is nonincreasing, and the logarithmic increment of* $\{X_{2n-1}\}$ *satisfies the inequalities*

$$0 \leqslant \log \frac{X_{2n+3}}{X_{2n-1}} \leqslant \log \lambda\mu \quad \textit{for} \quad n = 1, 2, \ldots$$

The proof is similar to that of Theorem 4.

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