ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XI (1967)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XI (1967)

W. WALISZEWSKI (Łódź)

On an axiomatic characterization of certain quasi-algebras with one partial operation

0. The purpose of this paper is to give an axiomatic system which characterizes the class of all quasi-algebras (see [1]) of the form $\langle C, \circ_C \rangle$, where C is the set of functions of non-empty domain and $\circ_C$ is the composition of functions of C. It is proved that the same system of axioms characterizes the class of all quasi-algebras of the form $\langle C, \circ_C' \rangle$, where C is a set of non-empty binary relations and $\circ_C'$ is the relative product of relations of C.

For a given function f, D_f denotes the domain of f. An inverse image of the set M given by f is denoted by $f^{-1}[M]$. If $M \subset D_f$, the image of M is denoted by $f[M]$. If the function $\circ$ is a binary partial operation, then the value $\circ(\langle x, y \rangle)$ is denoted by $x \circ y$.

1. Let us consider arbitrary quasi-algebras $\langle A, \circ \rangle$, where $\circ$ is a partial operation defined in the set A and satisfying the following system of axioms (see [2]):

$$\bigwedge_{x,y,z}\big(\langle x, y\rangle, \langle x \circ y, z\rangle, \langle y, z\rangle, \langle x, y \circ z\rangle \in D_\circ \Rightarrow (x \circ y) \circ z = x \circ (y \circ z)\big), \tag{1.0}$$

$$\bigwedge_{x,y,z}(\langle x, y\rangle, \langle x \circ y, z\rangle \in D_\circ \Rightarrow \langle y, z\rangle, \langle x, y \circ z\rangle \in D_\circ), \tag{1.1}$$

$$\bigwedge_{x,y,z}(\langle y, z\rangle, \langle x, y \circ z\rangle \in D_\circ \Rightarrow \langle x, y\rangle, \langle x \circ y, z\rangle \in D_\circ). \tag{1.2}$$

Any such quasi-algebra will be called a *generalized semi-group*. Evidently, axioms (1.0), (1.1) and (1.2) are independent.

Let C be an arbitrary set of functions with non-empty domains. For any pair $\langle g, f\rangle$ of functions of C such that

$$f^{-1}[D_g] \neq \emptyset \tag{1}$$

there exists exactly one function $g \circ_C f$ defined for $x \in f^{-1}[D_g]$ by the formula:

$$(g \circ_C f)(x) = g\big(f(x)\big). \tag{2}$$

The pair $\langle C, \circ_C\rangle$ is a quasi-algebra (see [1]) if and only if for every pair $\langle g, f\rangle$ of functions of C and satisfying condition (1), the function $g \circ_C f$ belongs to C. The domain of partial operation $\circ_C$ is defined as

$$D_{\circ C} = \{\langle g, f\rangle \colon\ g, f \in C \wedge f^{-1}[D_g] \neq \emptyset\}. \tag{3}$$

Any such quasi-algebra will be called a *generalized semi-group of functions.*

THEOREM 1. *A quasi-algebra* $\langle A, \circ\rangle$ *is a generalized semi-group if and only if there exists a generalized semi-group of functions isomorphic with* $\langle A, \circ\rangle$.

Proof. The sufficiency is obvious and we shall prove the necessity. Suppose that a quasi-algebra $\langle A, \circ\rangle$ satisfies the axioms (1.0), (1.1) and (1.2). Denote by B the set $A \cup \{e\}$, where e is an element which does not belong to the set A. Consider the partial operation $\odot$ defined as follows:

$$x \odot y = \begin{cases} x \circ y & \text{for} \quad \langle x, y\rangle \in D_\circ, \\ x & \text{for} \quad \langle x, y\rangle \in B \times \{e\}, \\ y & \text{for} \quad \langle x, y\rangle \in \{e\} \times A. \end{cases} \tag{4}$$

The set $D_\odot = D_\circ \cup B \times \{e\} \cup \{e\} \times A$ is the domain of the partial operation $\odot$. By an easy verification we conclude that the pair $\langle B, \odot\rangle$ is a quasi-algebra satisfying the axioms (1.0), (1.1) and (1.2).

Consider the function f defined by formula:

$$\big(f(u)\big)(x) = u \odot x \quad \text{for} \quad x \in D_\odot(u), \tag{5}$$

where

$$D_\odot(u) = \{x \colon\ \langle u, x\rangle \in D_\odot\} \quad \text{for} \quad u \in B. \tag{6}$$

First, we prove that the function f is a one-to-one mapping. Indeed, let $u, v \in B$ and $f(u) = f(v)$. It follows from (4) and (6) that $e \in D_\odot(u)$. Then by (5),

$$u = u \odot e = \big(f(u)\big)(e) = \big(f(v)\big)(e) = v \odot e = v.$$

Let $H = f[B]$. Then H is a set of functions and the functions of H map certain non-empty parts of the set B into the set B. We shall prove that

$$\bigwedge_{u,v}\Big(\langle u, v\rangle \in D_\odot \Leftrightarrow \big(u, v \in B \wedge \langle f(u), f(v)\rangle \in D_{\circ_H}\big)\Big) \tag{1.3}$$

and

$$\bigwedge_{u,v}\big(\langle u, v\rangle \in D_\odot \Rightarrow f(u \odot v) = f(u) \circ_H f(v)\big). \tag{1.4}$$

Indeed, suppose that $\langle u, v\rangle \in D_\odot$. Since $e \in D_\odot(v)$ we have $\big(f(v)\big)(e) = v \odot e = v \in D_\odot(u)$. Then

$$e \in \big(f(v)\big)^{-1}[D_\odot(u)] = \big(f(v)\big)^{-1}[D_{f(u)}].$$

Therefore $\big(f(v)\big)^{-1}[D_{f(u)}] \neq \emptyset$. In other words $\langle f(u), f(v)\rangle \in D_{\circ_H}$.

Let $x \in D_{f(u \odot v)}$. Then $x \in D_\odot(u \odot v)$. Consequently, $\langle u \odot v, x\rangle \in D_\odot$. Since $\langle B, \odot\rangle$ satisfies the axiom (1.1), we have $\langle v, x\rangle \in D_\odot$ and $\langle u, v \odot x\rangle \in D_\odot$. Therefore $\big(f(v)\big)(x) \in D_\odot(u)$ and

$$x \in \big(f(v)\big)^{-1}[D_{f(u)}] = D_{f(u) \circ_H f(v)}.$$

Moreover,

$$\big(f(u \odot v)\big)(x) = (u \odot v) \odot x = u \odot (v \odot x) = \big(f(u)\big)\big(f(v)\big)(x) = \big(f(u) \circ_H f(v)\big)(x).$$

Consider now an arbitrary $x \in D_{f(u) \circ_H f(v)}$. Then x belongs to $\big(f(v)\big)^{-1}[D_{f(u)}]$. Consequently, $x \in D_{f(v)}$ and $\big(f(v)\big)(x) \in D_{f(u)}$. In view of (6) and (5) we obtain $\langle v, x\rangle \in D_\odot$ and $\langle u, v \odot x\rangle \in D_\odot$. Since the quasi-algebra $\langle B, \odot\rangle$ satisfies the axiom (1.2), then $\langle u \odot v, x\rangle \in D_\odot$. In other words, $x \in D_\odot(u \odot v)$ and $f(u \odot v) = f(u) \circ_H f(v)$.

Now let $u, v \in B$ and $\langle f(u), f(v)\rangle \in D_{\circ_H}$. Therefore

$$\big(f(v)\big)^{-1}[D_{f(u)}] \neq \emptyset,$$

and there exists $x \in D_\odot(v)$ such that $v \odot x \in D_\odot(u)$. Consequently, $\langle u, v \odot x\rangle \in D_\odot$ and $\langle u, v\rangle \in D_\odot$. Propositions (1.3) and (1.4) have thus been proved. In other words, the quasi-algebras $\langle B, \odot\rangle$ and $\langle H, \circ_H\rangle$ are isomorphic. Let $C = f[A]$. We conclude from (1.3) and (1.4) that the pair $\langle C, \circ_C\rangle$ is a generalized semi-group of functions isomorphic with the quasi-algebra $\langle A, \circ\rangle$. This concludes the proof.

2. Let us now consider an arbitrary set C of non-empty binary relations. Any pair $\langle R, S\rangle$ of binary relations belonging to C and such that

$$\bigvee_{x,y,u}(\langle x, u\rangle \in R \wedge \langle u, y\rangle \in S) \tag{7}$$

determines the following binary relation:

$$R \circ'_C S = \{\langle x, y\rangle \colon \bigvee_u(\langle x, u\rangle \in R \wedge \langle u, y\rangle \in S)\}. \tag{8}$$

The ordered pair $\langle C, \circ'_C\rangle$ is a quasi-algebra if and only if for any ordered pair $\langle R, S\rangle$ of binary relations of set C satisfying (7) the binary relation $R \circ'_C S$ defined by (8) belongs to C. The domain of the partial operation $\circ'_C$ is the set

$$D_{\circ'_C} = \{\langle R, S\rangle \colon R, S \in C \wedge \bigvee_{x,y,u}(\langle x, u\rangle \in R \wedge \langle u, y\rangle \in S)\}.$$

Any such quasi-algebra will be called a *generalized semi-group of binary relations*.

THEOREM 2. *A quasi-algebra* $\langle A, \circ \rangle$ *is a generalized semi-group if and only if there exists a generalized semi-group of binary relations isomorphic with* $\langle A, \circ \rangle$.

Proof. Suppose that the quasi-algebra $\langle A, \circ \rangle$ satisfies the axioms (1.0), (1.1) and (1.2). We denote by $\circ'$ the partial operation defined by formula:

$$x \circ' y = y \circ x \quad \text{for} \quad \langle y, x \rangle \in D_{\circ}. \tag{9}$$

Since the quasi-algebra $\langle A, \circ \rangle$ satisfies the axioms (1.0), (1.1) and (1.2) these axioms are also satisfied by the quasi-algebra $\langle A, \circ' \rangle$. It follows from Theorem 1 that there exists a generalized semi-group $\langle C, \circ_C \rangle$ of functions isomorphic with $\langle A, \circ' \rangle$, and the formulas (2) and (7) yield

$$\bigwedge_{g,f}(\langle g, f \rangle \in D_{\circ_C} \Leftrightarrow \langle f, g \rangle \in D_{\circ C} \Rightarrow g \circ_C f = f \circ'_C g). \tag{2.1}$$

It follows from (2.1) that the quasi-algebra $\langle A, \circ \rangle$ is isomorphic with the quasi-algebra $\langle C, \circ'_C \rangle$. The sufficiency is obvious.

THEOREM 3. *If a quasi-algebra* $\langle A, \circ \rangle$ *is a generalized semi-group, then there exists an extension of* $\langle A, \circ \rangle$ *to a semi-group.*

Proof. Suppose that the quasi-algebra $\langle A, \circ \rangle$ satisfies the axioms (1.0), (1.1) and (1.2). It follows from Theorem 2 that there exists a generalized semi-group $\langle C, \circ'_C \rangle$ of binary relations isomorphic with the quasi-algebra $\langle A, \circ \rangle$. We denote by $\bar{C}$ the smallest set of binary relations containing the set C and closed with respect to the relative product of binary relations. The set $\bar{C}$ with the relative product is a semi-group. Thus, there exists an extension of the quasi-algebra $\langle A, \circ \rangle$ to a semi-group.

References

[1] J. Słomiński, *A theory of extension of quasi-algebras to algebras*, Rozprawy Matematyczne 40 (1964), pp. 1-64.

[2] W. Waliszewski, *Categories, groupoids, pseudogroups and analytical structures*, Rozprawy Matematyczne 45 (1965), pp. 1-40.

[3] — *On a generalization of the notion of a semi-group*, Coll. Math. 15 (1965), p. 191-194.