ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XI (1967)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XI (1967)

J. MUSIAŁEK (Kraków)

On some differential inequalities of elliptic type

In this paper we deal with the partial differential inequalities

(I) $$L(u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p(r)\frac{\partial u}{\partial x_i}\right) \geqslant f(u),$$

(II) $$\sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} + p(X)u \geqslant f(u)g(X)$$

where $r = (x_1^2 + \ldots + x_n^2)^{1/2}$, $X = (x_1, \ldots, x_n)$, and in the second inequality the involved second-order differential operator is elliptic with constant coefficients. We give sufficient conditions for local solvability of these inequalities. The particular case where $L = \Delta$, the Laplacian, was considered by Osserman [3] who gave also necessary conditions for solvability of the inequality.

We first deal with inequality (I).

1. As an auxiliary tool we shall consider the differential equation

(III) $$\left(z^{n-1}p(z)\varphi'(z)\right)' = z^{n-1}f\left(\varphi(z)\right), \qquad n \geqslant 2$$

with the following assumptions:

(A) 1° $f(t)$ is a positive continuous non-decreasing function defined on the real axis, 2° $p(z)$ is defined for $z \geqslant 0$, $p(z) \in C^1$, $\inf_{z\geqslant 0} p(z) > 0$.

We are interested in the solutions φ of this equation satisfying the initial conditions

(1.1) $$\lim_{z\to 0+}\varphi(z) = a, \quad \lim_{z\to 0+}\varphi'(z) = 0,$$

and defined in an interval $(0, R)$. Integrating both sides of equation (III) we obtain

$$\int_0^s \left(t^{n-1}p(t)\varphi'(t)\right)' dt = \int_0^s t^{n-1}f\left(\varphi(t)\right) dt$$

whence

$$\int_0^x \varphi'(s)\,ds = \int_0^x \frac{ds}{s^{n-1}p(s)} \int_0^s t^{n-1} f(\varphi(t))\,dt$$

and

$$\varphi(x) = a + \int_a^x \frac{ds}{s^{n-1}p(s)} \int_0^s t^{n-1} f(\varphi(t))\,dt. \tag{1.2}$$

This equation is obviously equivalent to equation (III) with the imposed initial conditions.

THEOREM 1. *Equation* (III) *has a solution satisfying the conditions* (1.1) *in a certain interval* $(0, R)$.

Proof. We shall apply the fixed-point theorem of Schauder [4]. Let $C[0, b]$ denote the Banach space of continuous functions $\psi = \psi(x)$ defined on the interval $[0, \dot{R}]$ with the usual norm $\|\psi\| = \max\limits_{0\leqslant x\leqslant b} |\psi(x)|$.

Let us write further

$$\|\psi\|_1 = \|\psi\| + \sup_{0\leqslant x<y\leqslant b} \left|\frac{\psi(y)-\psi(x)}{y-x}\right| + \sup_{0\leqslant x<y\leqslant b} \left|\frac{\psi(y)-\psi(x)}{y(y-x)}\right|$$

and let $M(R, b)$ denote the set of these functions $\psi \in C[0, b]$ for which $\|\psi\|_1 \leqslant M$.

The set $M(R, b)$ is obviously closed and convex. The functions of this set satisfy the Lipschitz condition with the constant M, and $\varphi'_+(0) = 0$ for $\varphi \in M(R, b)$. Thus $M(R, b)$ is a compact subset of $C[0, b]$, by the theorem of Arzelá.

Now, let us consider the integral operator

$$\varphi \to K\varphi = a + \int_0^x \frac{ds}{s^{n-1}p(s)} \int_0^s t^{n-1} f(\varphi(t))\,dt.$$

Let $\boldsymbol{a}$ denote the function equal to a in $[0, b]$, let $L(\delta) = \max\limits_{|x|\leqslant\delta} |f(x)|$, $\mu = \inf\limits_{x\geqslant 0} p(x)$, and let φ be in the ball $B = \{\varphi\colon \|\varphi - \boldsymbol{a}\| \leqslant 1\}$. Then $\|\varphi\| \leqslant \|\boldsymbol{a}\|+1$ and

$$|(K\varphi)(x) - a\,| \leqslant \int_0^x \frac{1}{s^{n-1}p(s)} \cdot \frac{s^n}{n} K(\|\varphi\|)\,ds \leqslant L(\|\boldsymbol{a}\|+1)\frac{b^2}{2n\mu}.$$

Thus for $0 < b \leqslant \big(2n\mu/L(\|\boldsymbol{a}\|+1)\big)^{1/2} = R$ we have $|(K\varphi)(x) - a| \leqslant 1$, so we see that the operator K maps the ball B into itself. Let us write for brevity $\chi(x) = (K\varphi)(x)$, then for $0 \leqslant x < y \leqslant b$, $\psi \in B$

$$\left|\frac{\chi(y)-\chi(x)}{y-x}\right| = \left|\frac{1}{y-x}\int\limits_x^y \frac{ds}{s^{n-1}p(s)}\int\limits_0^s t^{n-1}f(\varphi(t))\,dt\right|$$

$$= \left|\frac{1}{s_0^{n-1}p(s_0)}\int\limits_0^s t^{n-1}f(\varphi(t))\,dt\right| \leqslant \frac{s_0}{np(s_0)}\,L(\|\boldsymbol{a}\|+1) \leqslant \frac{b}{n\mu}\,L(\|\boldsymbol{a}\|+1)$$

where $x < s_0 < y$, and similarly

$$\left|\frac{\chi(y)-\chi(x)}{y(y-x)}\right| \leqslant \frac{s_0}{y}\cdot\frac{L(\|\boldsymbol{a}\|+1)}{np(s_0)} \leqslant \frac{L(\|\boldsymbol{a}\|+1)}{np(s_0)}.$$

Thus $\|K\psi\| \leqslant 1+\|\boldsymbol{a}\|+(1+b)\dfrac{L(\|\boldsymbol{a}\|+1)}{n\mu}$, i.e. $\psi\in R(M, b)$ for $\psi\in B$. So we have proved that K maps the ball B into a compact subset of B. By the fixed-point theorem of Schauder [4], there exists a $\psi\in B$ such that $K\varphi = \varphi$, i.e., φ satisfies the integral equation (1.2).

LEMMA 1. *Let the assumptions* (A) *be satisfied, let φ satisfy equation* (III) *in* $(0, R)$ *with the condition* (1.1) *and let* $\lim\limits_{z\to R}\varphi(z) = \infty$. *Let* $u(X) = u(x_1, \ldots, x_n)$ *be a function continuous in the ball* $x_1^2+\ldots+x_n^2 \leqslant R^2$, *of class* C^2 *and satisfying the equation*

$$L(u) = \sum_{i=1}^{n}\big(p(r)u_i'\big)_i' = f(u)$$

in the interior of the ball. Then $u(X)$ satisfies the inequality $u(X) \leqslant \varphi(r)$ in the ball.

Proof. We shall prove that the function $v(X) = u(X)-\varphi(r)$ is non-positive for $0 \leqslant r < R$. Indeed, suppose that $v(X_0) > 0$ for a certain X_0 in the ball $B_R = \{X\colon x_1^2+\ldots+x_n^2 < R^2\}$. Since $\lim\limits_{r\to R}\varphi(r) = +\infty$, the function v assumes a positive maximum at least at one point of the set B_R, say X_1. Therefore $u(X) > \varphi(r)$ in a certain neighbourhood Q of X_1, which implies $f(u) > f(\varphi) > 0$. Consequently $L(u) = f(u) \geqslant f(\varphi) = L(\varphi)$ and $L(v) = L(u-\varphi) = L(u)-L(\varphi) \geqslant f(\varphi)-f(\varphi) = 0$ in Q; in other words,

$$\sum_{i=1}^{n}\big(p(r)v_i'\big)_i'+0\cdot v \geqslant 0 \quad \text{in } Q.$$

By a theorem of Hopf ([2], p. 159), we have $v(X) \equiv 0$ in Q, which is impossible since $X_1\in Q$.

LEMMA 2. *Let the assumptions* (A) *be satisfied, and let $f'(t)$ be continuous and non-negative. Then inequality* (I) *has solutions on the whole hyperplane $u = 0$ if and only if there is a constant a such that there exists a solution of equation* (III) *on the half-line $z > 0$ satisfying the conditions* (1.1).

Proof. If φ satisfies the conditions of Lemma 2, then the function $u(X) = \varphi(r)$ satisfies equation (III), and hence inequality (I).

To prove the necessity we shall prove that if there does not exist a solution of (III) on the whole half-line $z > 0$ satisfying the conditions (1.1) then there does not exist a solution of inequality (I) on the hyperplane $u = 0$. Indeed, let $[0, R)$ be a maximal interval of existence of a solution of (III) satisfying conditions (1.1) with a certain a. Then, by assumption (A), 1°, the function $z^{n-1}p(z)\varphi'(z)$ has a positive derivative for $z > 0$, since $\varphi'_+(0) = 0$, this function is positive non-decreasing, whence $\varphi(z) \to \infty$ as $z \to R^-$. This enables us to apply Lemma 1. Let $u(X)$ be a fixed solution of the inequality (I) and $\varphi_a(r)$ a one-parameter family $(-\infty < a < \infty)$ of solutions of equation (III) satisfying conditions (1.1). By Lemma 1, each solution of inequality (I) satisfies the inequality $u(X) \leqslant \varphi(r)$ for $0 \leqslant r < R$, in particular $u(0) \leqslant \varphi(0) = a$. Since we may choose an arbitrary real value for a, $u(0) = -\infty$ contradicts the regularity of $u(X)$.

LEMMA 3. *Let $f(u) \in C^1$, $f(u) > 0$, $f'(u) \geqslant 0$ for $-\infty < u < \infty$, let $p(r)$ be continuous and satisfy $0 < m \leqslant p(r) \leqslant M < \infty$ for $0 \leqslant r < R$, let a solution $\varphi(x)$ of* (III) *exist in $[0, R)$ with $x = R$ as a singular point. Then*

$$\int_0^\infty \Big(\int_0^v f(u)\,du\Big)^{1/2} dv < \infty.$$

Proof. The function $\varphi(x)$ is increasing since all functions in right-hand side of the equation (1.2) are positive: thus $\varphi(z) \to \infty$ as $z \to R^-$. From

$$\big(p(x)\varphi'(x)\big)' + (n-1)n^{-1}p(x)\varphi'(x) = f\big(\varphi(x)\big)$$

and from the assumptions relative to the functions $p(x)$, $f(u)$ and $\varphi'(x)$ we infer that $(p\varphi')' < f(\varphi)$ whence $p\varphi'(p\varphi')' < p\varphi' f(\varphi) < Mf(\varphi)\varphi'$ and

$$\frac{1}{2}\cdot\frac{d}{dx}\big(p(x)\varphi'(x)\big)^2 \leqslant Mf(\varphi)\frac{d\varphi(x)}{dx}. \tag{1.3}$$

We shall distinguish two cases:

1. $\varphi(0) \geqslant 0$. Integrating both sides of (1.3) we obtain

$$\big(m\varphi'(x)\big)^2 \leqslant 2M\int_a^{\varphi(x)} f(u)\,du \leqslant 2M\int_0^{\varphi(x)} f(u)\,du,$$

$$\frac{m}{\sqrt{2M}}\Big(\int_0^{\varphi(x)} f(u)\,du\Big)^{-1/2}\varphi'(x)\,dx < 1.$$

Integrating this inequality once more and taking into account that $\varphi(x) \to \infty$ as $x \to R^-$, we obtain

$$\frac{m}{\sqrt{2M}} \int_0^\infty \left(\int_0^v f(u)\,du \right)^{-1/2} dv < R. \tag{1.4}$$

2. $\varphi(0) < 0$. Arguing as above we are lead to the inequality

$$\begin{aligned}(m\varphi'(x))^2 &\leqslant 2M \int_a^{\varphi(x)} f(u)\,du = 2M \int_a^0 f(u)\,du + 2M \int_0^{\varphi(x)} f(u)\,du \\ &\leqslant 2M \int_0^{\varphi(x)} f(u)\,du + 2M \int_0^{\varphi(x)} f(u)\,du = 4M \int_0^{\varphi(x)} f(u)\,du\end{aligned}$$

valid for x sufficiently close to R, which leads again to the inequality (1.4).

LEMMA 4. *If* $f(u) \in C^1$, $f(u) > 0$, $f'(u) \geqslant 0$, $\int_0^\infty \left(\int_0^v f(u)\,du\right)^{-1/2} dv = \infty$, $0 < m = \inf_{0 \leqslant x < \infty} p(x) \leqslant \sup_{0 \leqslant x < \infty} p(x) = M < \infty$, $p(x) \in C^1$, *then the integral equation* (1.2) *admits a solution on the whole half-line* $[0, \infty)$.

Proof. Let $N > 0$ and let

$$f_N(u) = \begin{cases} f(u) & \text{for} \quad u \in [0, N), \\ f(N) + (u-N)^2 & \text{for} \quad u > N. \end{cases}$$

Then $f_N(u)$ tends to $f(u)$ as $N \to \infty$ uniformly on each compact set. Let $\varphi_N(x)$ denote the solution of the equation (1.2) with f replaced with $f_N(u)$. By Theorem 1 and Lemma 3, there exists a solution $\varphi_N(x)$ in the interval $[0, r_N]$, where

$$r_N = \frac{m_0}{\sqrt{2M_0}} \int_0^\infty \left[\int_0^v f_N(u)\,du \right]^{-1/2} dv < \infty,$$

$\varphi_N(x) = \varphi(x)$ in $\left(0, f(\varphi(N))\right)$, and $\varphi_N(x) \to \varphi(x)$ as $N \to \infty$. By our assumptions $r_N \to \infty$ as $N \to \infty$, whence $\varphi_N(x) \to \varphi(x)$ in the whole interval $[0, \infty)$.

THEOREM 2. *Let the function* $f(x)$ *be continuous and positive on the straight line, let* $f'(x) \geqslant 0$ *exist for* $x \geqslant t_0 > 0$, *let* $p(x) \in C^1$, $p'(x) \geqslant 0$ *for* $x \geqslant 0$, *let* $m_0 \leqslant p(r) < M_0$, *and let*

$$0 < I = \int_0^\infty \left(\int_0^t f(s)\,ds \right)^{-1/2} dt < \infty.$$

Then there does not exist a function $u(X) \in C^2$ *on the whole hyperplane* $u = 0$ *and a ball* S *such that* $L(u) > 0$ *on the hyperplane* $u = 0$, $L(u) \geqslant f(u)$ *for* $X \notin S$.

Proof. Let us suppose that such a function $u(X)$ exists. From this assumption we shall deduce that $I = \infty$. By the same theorem of Hopf, the function $u(X)$ attains the maximum on the boundary of ∂S of S. Let μ denote the minimum of u on ∂S. Let $g(t)$ be an auxilary function of class C^1, defined on the real line and satisfying the conditions (a) $g'(t) \geqslant 0$ for all t, (b) $g(t) \leqslant m$ for $t < t_1$, (c) $g(t) \leqslant f(t)$ for all t, (d) $g(t) \geqslant f(t)-1$ to $t \geqslant t_2 > t_1$, (e) $g(t) > 0$ for all t. Then $L(u) \geqslant g(u)$ for all u.

Replacing the function $f(t)$ by $g(t)$ we obtain from Lemmas 2 and 3

$$\int_0^\infty \Big(\int_0^t g(s)\,ds\Big)^{-1/2} dt = \infty$$

which implies that $I = \infty$. Indeed, by (d), $f(s) \leqslant g(s)+1 < 2g(s)$ for $s \geqslant t_3 > t_2$, whence

$$\int_{t_3}^t f(s)\,ds \leqslant 2\int_0^t g(s)\,ds, \qquad t > t_3$$

which gives

$$\Big(2\int_0^t g(s)\,ds\Big)^{-1/2} \leqslant \Big(\int_{t_3}^t f(s)\,ds\Big)^{-1/2} = \Big(\int_0^t f(s)\,ds - \int_0^{t_3} f(s)\,ds\Big)^{-1/2}$$
$$\leqslant \Big(\int_0^t f(s)\,ds - \tfrac{1}{2}\int_0^t f(s)\,ds\Big)^{-1/2} = \Big(\tfrac{1}{2}\int_0^t f(s)\,ds\Big)^{-1/2}$$

for all sufficiently large t. Hence

$$\infty = \int_{t_3}^\infty \Big(2\int_0^t g(s)\,ds\Big)^{-1/2} dt \leqslant \int_{t_3}^\infty \Big(\tfrac{1}{2}\int_0^t f(s)\,ds\Big)^{-1/2} dt$$

and $I = \infty$.

2. Now, we pass to inequality (II). Let $u(X)$ be a solution of (II) where $p(X)$ is continuous and $f(u)$ and $g(X)$ are continuous and let a_i be the solution of the system of linear equations

$$2\sum_{k=1}^n a_{ik}a_k + b_i = 0 \qquad (i = 1, \ldots, n) \tag{2.1}$$

and let us set $v(X) = u(X)\exp\big(-\sum_{i=1}^n a_i x_i\big)$.

LEMMA 5. *The function $v(X)$ satisfies the inequality*

$$\sum_{i,j=1}^n a_{ij} v''_{ij}(X) \geqslant C(X)v(X) + A(X) \tag{2.2}$$

where

$$C(X) = \sum_{j,k=1}^{n} a_{jk} a_j a_k - p(X),$$

$$A(X) = f\Big(v \exp\Big(\sum_{i=1}^{n} a_i x_i\Big)\Big) g(X) \exp\Big(-\sum_{i=1}^{n} a_i x_i\Big),$$

and vice versa: if the function $v(X)$ *satisfies inequality* (2.2) *then the function*

$$u(X) = v(X)\exp\Big(\sum a_i x_i\Big)$$

satisfies inequality (II).

Proof. Substituting the first and second derivatives of $u(X)$ to (II) we obtain

$$\exp\Big(\sum_{i=1}^{n} a_i x_i\Big)\Big(\sum_{j,k=1}^{n} a_{jk}(v''_{jk} + v'_j a_k + v'_k a_j + v a_k a_j) + \sum_{j=1}^{n} b_j(v'_j + v a_j)\Big) \geqslant g(X) f\Big(v \exp\Big(\sum_{i=1}^{n} a_i x_i\Big)\Big).$$

Dividing by $\exp\big(\sum a_i x_i\big)$ we get

$$\sum_{j,k=1}^{n} a_{jk} v''_{jk} + 2\sum_{j,k=1}^{n} a_{jk} v'_j a_k + \sum_{j=1}^{n} b_j v'_j + \\ + v\Big(\sum_{j,k=1}^{n} a_{jk} a_k a_j + \sum_{j=1}^{n} b_j a_j + p(X)\Big) \geqslant A(X),$$

$$\sum_{j,k=1}^{n} a_{jk} v''_{jk} + \sum_{j=1}^{n}\Big(2\sum_{k=1}^{n} a_{kj} a_k + b_j\Big) v'_j + \\ + v\Big(\sum_{j,k=1}^{n} a_{jk} a_k a_j + \sum_{j=1}^{n} b_j a_j + p(X)\Big) \geqslant A(X).$$

By (2.1),

$$\sum_{j,k=1}^{n} a_{jk} v''_{jk} + v\sum_{j=1}^{n} a_j\Big(2\sum_{k=1}^{n} a_{jk} a_k + b_j - \sum_{k=1}^{n} a_{jk} a_k + p(X)\Big) \geqslant A(X).$$

By (2.1) we obtain inequality (2.2). The proof of the second part of Lemma 5 follows in the same way.

Let the function $F(v)$ satisfy (in the whole space) the inequality

$$F(v) \geqslant C(X) v + A(X). \tag{2.3}$$

If the function v satisfies the equations

$$\sum_{i,j=1}^{n} a_{ij} v''_{ij} = F(v) \tag{2.4}$$

then it also satisfies (2.3) whence the function $u(X)$ satisfies (II). Therefore in order to prove existence of solutions of (II) it is enough to construct a solution of (2.4).

We shall need the following lemma whose proof may be found in [1].

LEMMA 6. *Let a^{ik} denote the elements of the matrix inverse to (a_{ik}), let $r = \left(\sum_{i,k=1}^{n} a^{ik} x_i x_k\right)^{1/2}$. If $v(X) = V(r)$ is a solution of the equation* (2.4) *then the function $V(r)$ satisfies the ordinary equation*

$$V''(r)+(n-1)r^{-1}V'(r) = C(X)V(r)+A(X).$$

In virtue of Lemmas 5 and 6, inequality (II) reduces to the inequality

$$r^{1-n}\left(r^{n-1}V'(r)\right)' \geqslant F(v) \geqslant C(X)V(r)+A(X).$$

Setting $V(r) = \varphi(r)$ we get

$$r^{1-n}\left(r^{n-1}\varphi'(r)\right)' \geqslant F\left(\varphi(r)\right).$$

THEOREM 3. *Let $F(v) \geqslant C(X)v+A(X)$. If the function $F(v)$ is continuous and positive and if $\int_0^\infty \left(\int_0^v F(s)\,ds\right)^{1/2} dv = \infty$, then there exists the solution of the inequality* (II).

Proof. By Theorem 2, equation (2.4) has a solution $V(r)$ depending on r. Then the function $V(r)$ satisfies (2.2) and the function $u(X) = V(r)\exp\left(\sum a_i x_i\right)$ satisfies (II).

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