ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE X (1967)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE

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Construction of a fundamental solution for certain elliptic differential equations of order 4

1. In this paper we deal with the construction of fundamental solutions for the equation

$$L^2u(X)+ku(X) = 0 \tag{1.1}$$

where $X = (x_1, \ldots, x_n)$, k is a constant, and operator L, defined by formula

$$Lu(X) = \sum_{i,k}^{n} a_{ik}u_{ik}, \qquad u_{ik} = \frac{\partial^2 u}{\partial x_i \partial x_k},$$

with $a_{ik} = a_{ki}$ independent of X, is elliptic. We consider first the cases $n = 2$, $n = 3$, and then the general case. For the sake of convenience, we write $k = C^4$ or $k = -C^4$, C being positive.

In the paper [4] a construction of a fundamental solution for the equation $\Delta^2u(X)+ku(X) = 0$ was given. Applying these results, we shall find the fundamental solutions for equation (1.1).

2. In the sequel, $\sum a_{ij}\xi_i\xi_j$ will stand for a positive-definite quadratic form $(i, j = 1, \ldots, n)$, with constant coefficients, a^{ij} will denote the elements of the matrix inverse to (a_{ij}). Then the form $\sum a^{ij}\xi_i\xi_j$ is also positive definite. E_R will denote the ellipsoid $\sum a^{ij}(x_i-x_i^0)(x_j-x_j^0) = R^2$, and r_{XY} will denote the "elliptic distance" of the points $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$, namely

$$r_{XY} = \Big[\sum_{i,j}^{n} a^{ij}(x_i-y_i)(x_j-y_j)\Big]^{1/2}.$$

Obviously, $r_{XY} > 0$ if $X \neq Y$.

We now find a solution of the equation (1.1), of the form $u(X) = U(r_{XX_0})$, $X_0 = (x_1^0, \ldots, x_n^0)$ being a fixed point.

LEMMA 1 (see [1]). *Let the function* $U(r)$ *be of class* C^2 *in* $\langle 0, \infty)$, *and let* $u(X) = U(r_{XX_0})$, *then*

$$L(u) = \sum_{i,k}^{n} a_{ik}u''_{ik} = U''(r)+(n-1)r^{-1}U'(r), \qquad r>0.$$

We omit the simple proof.

LEMMA 2 (see [4]). *Each solution* $u(X)$ *of class* $C^{(4)}$ *of any of the equations* $Lu(X)-C^2u(X) = 0$ *or* $Lu(X)+C^2u(X) = 0$, *satisfies the equation*

$$L^2u(X)-C^4u(X) = 0.$$

Let v be a function of class C^1 in a region D, let S be a surface in D with continuous normal (in the sequel such surfaces will be briefly called surfaces), then the transversal derivative (with respect to S) of v is defined as ([2], p. 140)

$$\frac{\partial v}{\partial \nu} = \sum_{i=1}^{n} v'_i \sum_{j=1}^{n} a_{ij}\cos(n, x_j).$$

Of course, it is defined only for points on the surface S. In every case we shall deal with the transversal derivative under the sign of the surface integral (over a surface of type of the sphere), n will denote the normal line to the surface, directed towards the interior of the surface, at a corresponding point of the surface.

In Lemmas 3, 4, 6, $\dfrac{\partial}{\partial \nu}$ will denote the transversal derivative with respect to an arbitrary surface lying in the region in which the involved functions are defined.

LEMMA 3. *Let the quadratic form* $\sum\limits_{i,k=1}^{n} a_{ik}y_iy_k$ *be positive-definite; then there exists a positive constant* k *such that for each surface* $S \subset D$ *we have* $\left|\dfrac{\partial r}{\partial \nu}\right| < k$.

Proof. It is enough to prove that the functions $\partial r/\partial x_i$ are bounded, since

$$\frac{\partial r}{\partial \nu} = \sum \frac{\partial r}{\partial x_i} \sum a^{ij}\cos(n, x_j).$$

This follows from the estimation

$$\left|\frac{\partial r}{\partial x_i}\right| = \left|\sum_{j=1}^{n} r^{-1}a^{ij}(x_j-x_j^0)\right| \leqslant \sum_{j=1}^{n} |a^{ij}|r^{-1}|x_j-x_j^0|$$

$$\leqslant M\sum_{j=1}^{n} |x_j-x_j^0|A^{-1}\Big[\sum_{p=1}^{n}(x_p-x_p^0)^2\Big]^{-1/2} \leqslant nA^{-1}M = k_1,$$

where $k_1 > 0$, $M = \max|a^{ij}|$, and A denotes the minimum of the quadratic form $\sum a^{ij}u_i u_j$ on the unit sphere.

LEMMA 4. *If $w(r)$ is a differentiable function, then*

$$\frac{\partial w(r)}{\partial \nu} = w'(r)\frac{\partial r}{\partial \nu}.$$

We omit the proof.

From Lemma 4 it follows

LEMMA 5. *If $w(r) = O(r^a)$, then $\dfrac{\partial w(r)}{\partial \nu} = O(r^a)$; if $w'(r) = o(r^a)$, then $\dfrac{\partial w(r)}{\partial \nu} = o(r^a)$. The symbols $O(r^a)$ and $o(r^a)$ relate to the case $r \to 0^+$.*

LEMMA 6. *If*

$$G(X) = \Big(\sum_{j=1}^{n}\Big(\sum_{k=1}^{n} a^{ik}(x_k - x_k^0)\Big)^2\Big)^{1/2},$$

then

$$\frac{\partial \ln r}{\partial \nu} = \frac{1}{G(X)} = O(r^{-1}).$$

The proof is analogous to that of Lemma 1.

LEMMA 7. *The surface integral*

$$R^{2-n}\iint\limits_{\partial E_R}\frac{dS}{G(X)}$$

is independent of R.

Proof. Let

$$\text{(T)} \qquad x_i - x_i^0 = \sum_{j=1}^{n} A_{ij}(\bar{x}_j - x_j^0), \qquad i = 1, \ldots, n$$

be an orthogonal transformation converting the quadratic form $\sum\limits_{i,j=1}^{n} a^{ij}(x_i - x_i^0)(x_j - x_j^0)$ into the canonical form $\sum\limits_{j=1}^{n} B_j(\bar{x}_j - x_j^0)^2$. Then (T) transforms the ellipsoid

$$\text{(2.1)} \qquad \sum_{i,j=1}^{n} a^{ij}(x_i - x_i^0)(x_j - x_j^0) = R^2$$

into the ellipsoid

$$\text{(2.2)} \qquad \sum_{j=1}^{n} B_j(\bar{x}_j - x_j^0)^2 = R^2.$$

Let the equations

$$\begin{aligned} \bar{x}_1-x_1^0 &= C_1R\cos\varphi_1, \\ \bar{x}_2-x_2^0 &= C_2R\sin\varphi_1\cos\varphi_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{x}_n-x_n^0 &= C_nR\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\sin\varphi_{n-1}, \end{aligned}$$

where

$$0 \leqslant \varphi_1 \leqslant \pi, \quad \ldots, \quad 0 \leqslant \varphi_{n-2} \leqslant \pi, \quad 0 \leqslant \varphi_{n-1} \leqslant 2\pi,$$

stand for the parametric description of the ellipsoid (2.2). Let us write the parametric equations of the ellipsoid (2.1) in the form

$$x_i-x_i^0 = RH_i(\varphi_1, \ldots, \varphi_{n-1}), \quad i = 1, \ldots, n$$

and let

$$G(X) = Rg(\varphi_1, \ldots, \varphi_{n-1}) > 0, \quad dS = R^{n-1}J(\varphi_1, \ldots, \varphi_{n-1}).$$

Then

$$\begin{aligned} R^{2-n}\iint\limits_{\partial E_R}\frac{dS}{G(X)} &= R^{2-n}\int\limits_0^{\pi}\ldots\int\limits_0^{2\pi}\frac{R^{n-1}J(\varphi_1, \ldots, \varphi_{n-1})}{Rg(\varphi_1, \ldots, \varphi_{n-1})}\,d\varphi_1\ldots d\varphi_{n-1} \\ &= \int\limits_0^{\pi}\ldots\int\limits_0^{2\pi}\frac{J(\varphi_1, \ldots, \varphi_{n-1})}{g(\varphi_1, \ldots, \varphi_{n-1})}\,d\varphi_1\ldots d\varphi_{n-1}. \end{aligned}$$

LEMMA 8 (see [4]). *Let* $u(X) = U(r)$, $n = 3$; *then*

$$L^2u(X) = U^{(4)}(r)+4r^{-1}U^{(3)}(r).$$

LEMMA 9 (see [4]). *Let the function* $U(r)$ *satisfy the differential equation*

$$U^{(4)}(r)+4r^{-1}U^{(3)}(r)+kU(r) = 0, \quad r > 0;$$

then the function $v(r) = rU(r)$ *satisfies the equation*

$$v^{(4)}(r)+kv(r) = 0.$$

Lemma 8 and Lemma 9 follow from Lemma 1.

DEFINITION 1. A function $U(r)$ is called a *fundamental solution* of equation (1.1) in a bounded region D if 1° the boundary ∂D of the region D is of class C_σ^1 (this means that ∂D is the union of a finite number of surfaces of class C^1), 2° the function $U(r_{XY})$ is of class $C^{(4)}$ with respect to $X \in \bar{D}$ and $Y \in \bar{D}$ except the set where $X = Y$, 3° $U(r_{XY})$ satisfies in D equation (1.1) in each variable separately, 4° for each function $v(X)$ of class $C^{(3)}$ in $\bar{D}$ and $C^{(4)}$ in D satisfying equation (1.1) in D, we have

$$v(X) = \gamma_n\iint\limits_{\partial D}\left(U\frac{\partial Lv}{\partial\nu}-\frac{\partial U}{\partial\nu}Lv+LU\frac{\partial v}{\partial\nu}-\frac{\partial LU}{\partial\nu}v\right)dS_Y \tag{2.3}$$

for each $X \in D$, γ_n being a constant depending on the dimension of D.

3. We proceed to the construction of fundamental solutions of (1.1) in the case $n = 2$ and $k = C^4$ and $k = -C^{-4}$.

(a) The case $k = -C^4$. Equation (1.1) takes on the form

$$L^2u(x, y) - C^4u(x, y) = 0 \tag{3.1}$$

and can be written in the operator form $(L-C^2)(L+C^2)\,u = 0$.

We shall construct the fundamental solution with aid of the integrals of the equations

$$Lu - C^2u = 0, \tag{3.2}$$

$$Lu + C^2u = 0. \tag{3.3}$$

In virtue of Lemmas 1 and 2, the integrals $U(r)$ of (3.2) and (3.3) satisfy the respective ordinary equations

$$U''(r) + r^{-1}U'(r) + C^2U(r) = 0, \tag{3.4}$$

$$U''(r) + r^{-1}U'(r) - C^2U(r) = 0. \tag{3.5}$$

Equation (3.5) may be obtained from (3.4) by substituting Ci instead of C. Let us introduce a new variable by

$$R = Cr; \tag{3.6}$$

then we obtain the equation

$$R^2U''(R) + RU'(R) + R^2U(R) = 0. \tag{3.7}$$

Analogously, from (3.5) we obtain

$$R^2U''(R) + RU'(R) - R^2U(R) = 0. \tag{3.8}$$

These are the Bessel equations with the value 0 of the parameter s (see [6]). A particular solution of (3.7) is the Bessel function of the second kind and of order zero:

$$U_1(R) = Y_0(R) = \frac{2}{\pi}\ln\frac{R}{2} + O(1). \tag{3.9}$$

By (3.5), (3.6), (3.7), (3.8) and Lemma 2, the functions $Y_0(CR)$ and $Y_0(CiR)$ are solutions of (3.1).

THEOREM 1 (see [4]). *The fundamental solution of equation* (3.1) *is of the form*

$$U_1(r) - U_2(r) = U(r), \tag{3.10}$$

where

$$U_1(r) = Y_0(Cr) = L_1(r) + \frac{2}{\pi}\ln\left(\frac{Cr}{2}\right) = \frac{2}{\pi}\ln\left(\frac{Cr}{2}\right) + O(1), \tag{3.11}$$

$$U_2(r) = \operatorname{Re} Y_0(CiR) = \frac{2}{\pi}\ln\frac{Cr}{2} + L_2(r) = \frac{2}{\pi}\ln\left(\frac{Cr}{2}\right) + O(1). \tag{3.12}$$

Proof. By (3.10), (3.11) and (3.12)

$$U(r) = L_2(r) - L_1(r) = O(1). \tag{3.13}$$

It follows from (3.13) and Lemmas 3, 4, 5, that

$$\frac{\partial U(r)}{\partial\nu} = \frac{\partial L_2(r)}{\partial\nu} - \frac{\partial L_1(r)}{\partial\nu} = o(r). \tag{3.14}$$

By (3.11), (3.12), and by Lemma 2,

$$LU(r) = -C^2\big(Y_0(CR)+\mathrm{Re}\,Y_0(CiR)\big) = \frac{-4C^2}{\pi}\ln\frac{Cr}{2}+O(1). \tag{3.15}$$

From Lemmas 4 and 5 and from (3.15) we get

$$\frac{\partial LU(r)}{\partial\nu} = \big(LU(r)\big)'\frac{\partial r}{\partial\nu} = \left(\frac{-4C^2}{\pi r}+o(r)\right)\frac{r}{G(X)}. \tag{3.16}$$

Let $v(x, y)$ be of class $C^{(4)}$ in $\bar{D}$ and satisfy equation (3.1), and let E_R^* denote the interior of the ellipse

$$\sum_{i,j=1}^{n} a^{ij}(x_i-x_i^0)(x_j-x_j^0) = R^2 \tag{3.17}$$

contained in D. Applying the fundamental formula

$$\iint_D (UL^2v - vL^2U)\,dx\,dy + \int_{\partial D}\left[U\frac{\partial Lv(Y)}{\partial\nu}+LU(r)\frac{\partial v(Y)}{\partial\nu}-v(Y)\frac{\partial LU(r)}{\partial\nu}-Lv\frac{\partial U}{\partial\nu}\right]dS_Y$$

(see [5]) to the functions $U(r)$ and $v(x, y)$ and to the set $D-E_R^*$ we obtain

$$\begin{aligned}\iint_{D-E_R^*}\big(U(r)L^2v(Y)-v(Y)L^2U(r)\big)d\xi\,d\eta &= -\int_Z U(r)\frac{\partial Lv(Y)}{\partial\nu}dS_Y - \\ &-\int_Z v(Y)\frac{\partial LU(r)}{\partial\nu}dS_Y - \int_Z\left[LU(r)\frac{\partial v(Y)}{\partial\nu}-Lv(Y)\frac{\partial U(r)}{\partial\nu}\right]dS_Y,\end{aligned} \tag{3.18}$$

where $Z = \partial(D-E_R^*)$. The left-hand side in this formula is equal to zero, since $U(r)$ and $v(x, y)$ satisfy equation (1.1). Thus, we obtain

$$\begin{aligned}\int_{\partial D}\left(U\frac{\partial Lv}{\partial\nu}-v\frac{\partial LU}{\partial\nu}+LU\frac{\partial v}{\partial\nu}-Lv\frac{\partial U}{\partial\nu}\right)dS_Y+\int_{\partial E_R}U\frac{\partial Lv}{\partial\nu}dS_Y - \\ -\int_{\partial E_R}\frac{\partial U}{\partial\nu}Lv\,dS_Y+\int_{\partial E_R}LU\frac{\partial v}{\partial\nu}dS_Y-\int_{\partial E_R}v\frac{\partial LU}{\partial\nu}dS_Y = 0.\end{aligned} \tag{3.19}$$

Applying the mean-value theorem to the curvilinear integrals in (3.19) and applying (3.13), (3.14), (3.15), (3.16) and Lemmas 6 and 7, we get

$$I_1 = \int\limits_{\partial E_R} U(r)\frac{\partial Lv}{\partial \nu}dS_Y = a_1 R U(r)\frac{\partial Lv(Q)}{\partial \nu} = a_1 R(L_2-L_1)\frac{\partial Lv(Q)}{\partial \nu}$$

$$= a_1 R O(1)\frac{\partial Lv(Q)}{\partial \nu} \to 0, \quad R\to 0,$$

where $Q\epsilon\partial E_R$, and a_1 denotes the perimetre of the unit ellipse E_1;

$$I_2 = \int\limits_{\partial E_R} Lv(Y)\frac{\partial U(r)}{\partial \nu}ds_Y = a_1 R Lv(Q_1)\frac{\partial U(R)}{\partial \nu} = a_1 R Lv(Q_1)\left(\frac{\partial L_2}{\partial \nu}-\frac{\partial L_1}{\partial \nu}\right)$$

$$= a_1 R Lv(Q_1) O(r) \to 0, \quad Q_1\epsilon\partial E_R;$$

$$I_3 = \int\limits_{\partial E_R} LU(r)\frac{\partial v(Y)}{\partial \nu}ds_Y = a_1 R\left(\frac{-4C^2}{\pi}\ln\frac{CR}{2}+O(1)\right)\frac{\partial v(Q_2)}{\partial \nu}\to 0, \quad Q_2\epsilon\partial E_R;$$

$$I_4 = \int\limits_{\partial E_R} v(Y)\frac{\partial LU(r)}{\partial \nu}ds_Y = \int\limits_{\partial E_R} v(Y)\left(\frac{-4C^2}{\pi r}+O(r)\right)\frac{r}{G(X)}ds_Y$$

$$= \int\limits_{\partial E_R} v(Y)\left(\frac{-4C^2}{\pi r}\right)\frac{r}{G(X)}ds_Y + \int\limits_{\partial E_R} v(Y)O(r)\frac{r}{G(X)}ds_Y$$

$$= \frac{-4C^2}{\pi}v(Q_3)\int\limits_{\partial E_R}\frac{ds_Y}{G(X)} + \int\limits_{\partial E_R} v(Q_3)O(r)\frac{r}{G(X)}ds_Y \longrightarrow$$

$$\frac{-4C^2}{\pi}v(x,y)\int\limits_{\partial E_R}\frac{ds_Y}{G(X)}, \quad Q_3\epsilon\partial E_R.$$

These formulas and (3.19) give formula (2.3), where

$$\gamma_2 = -\pi\left(4C^2\int\limits_{\partial E_R}\frac{ds_Y}{G(X)}\right)^{-1}. \tag{3.20}$$

By Definition 1 and by the regularity of the function $U(r)$, the function $U(r)$ defined by (3.10) is a fundamental solution of (3.1).

(b) The case $k = C^4$. Equation (1.1) takes on the form

$$L^2u(x,y)+C^4u(x,y) = 0. \tag{3.21}$$

THEOREM 2 (see [4]). *The fundamental solution of* (3.21) *is the function*

$$U(r) = \operatorname{Im} Y_0(Cr\sqrt{i}) = \frac{2}{\pi}\left(\frac{\pi}{4}-\frac{C^2r^2}{2}\ln\frac{Cr}{2}+k\left(\frac{Cr}{2}\right)^2\right)+O(r^{4-\varepsilon}), \tag{3.22}$$

where ε *is an arbitrary positive number.*

Proof. By Lemmas 1, 3, 4, 5, and by (3.22),

$$\frac{\partial U}{\partial \nu} = U'(r)\frac{\partial r}{\partial \nu} = \left(B_1 \ln\frac{Cr}{2} + O(r)\right)\frac{r}{G(X)},$$

$$U''(r) = \frac{2}{\pi}\left(-C^2\ln\frac{Cr}{2} + \frac{C^2 r}{2} - \frac{3}{2}C^2\right) + O(r),$$

$$U'''(r) = \frac{-2C^2}{\pi r} + O(r),$$

$$LU(r) = U''(r) + r^{-1}U'(r) = B_2\ln\frac{Cr}{2} + O(r),$$

$$\frac{dLU(r)}{dr} = U'''(r) + r^{-1}U''(r) - r^{-2}U'(r) = B_2 r^{-1} + O(r),$$

$$\frac{\partial LU(r)}{\partial \nu} = \frac{dLU(r)}{dr}\cdot\frac{\partial r}{\partial \nu} = \left(B_3 r^{-1} + O(r)\right)\frac{r}{G(X)} = B_3\frac{1}{G(X)} + O(r),$$

where B_1, B_2, B_3 are positive constants.

Starting with these formulas we may argue similarly as in the proof of Theorem 1; in this case

$$\gamma_2 = -\pi\left(4C^2\int\frac{ds_Y}{G(X)}\right)^{-1}.$$

4. In this section we deal with the case $n = 3$. In virtue of Lemmas 8 and 9, equation (1.1) takes on the form

$$V^{(4)}(r) + kV(r) = 0. \tag{4.1}$$

(a) The case $k = -C^4$. Equation (4.1) is then of the form

$$V^{(4)}(r) - C^4 V(r) = 0. \tag{4.2}$$

By Lemma 9, the functions $U_1(r) = r^{-1}e^{Cr}$ and $U_2(r) = r^{-1}\cos Cr$ are solutions of (1.1).

THEOREM 3 (see [4]). *The function*

$$U(r) = r^{-1}(e^{Cr} - \cos Cr) \tag{4.3}$$

is a fundamental solution of (1.1).

Proof. It is enough to check that the function $U(r)$ satisfies formula (2.3). Let a function $v(Y)$ of class $C^{(4)}$ in $\overline{D}$ satisfy equation (1.1) and let the ellipsoid E_R with the centre X be defined by equation (2.1). Applying the fundamental formula (see [5])

$$\iiint\limits_{D-E_R}(UL^2v - vL^2U)\,dx\,dy\,dz + \iint\limits_{\partial(D-E_R)}\left[U(r)\frac{\partial}{\partial\nu}Lv(Y) - Lv(Y)\frac{\partial}{\partial\nu}U(r) + \right.$$
$$\left. + LU(r)\frac{\partial}{\partial\nu}v(Y) - v(Y)\frac{\partial}{\partial\nu}LU(r)\right]dS_Y = 0$$

to the functions $U(r)$ and $v(Y)$ in the set $D-E_R$ we obtain

$$(4.4)\quad \iiint\limits_{D-E_R} \big(U(r)L^2v(Y)-v(Y)L^2U(r)\big)d\xi\, d\eta\, d\zeta$$

$$= -\iint\limits_{\partial(D-E_R)} \left(U(r)\frac{\partial}{\partial\nu}Lv(Y)-Lv(Y)\frac{\partial}{\partial\nu}U(r)\right)dS_Y-$$

$$-\iint\limits_{\partial(D-E_R)} \left(LU(r)\frac{\partial}{\partial\nu}v(Y)-v(Y)\frac{\partial}{\partial\nu}LU(r)\right)dS_Y.$$

Arguing similarly as in the proof of Theorem 1 and using the formulas (4.3), (3.2), (3.3) and the Lemmas 3, 4, 5 we obtain

$$I_1 = \iint\limits_{\partial E_R} U(r)\frac{\partial}{\partial\nu}Lv(Y)dS_Y = a_2R^2R^{-1}(e^{CR}-\cos CR)\frac{\partial}{\partial\nu}\big(Lv(Q)\big) \to 0,$$

where $Q\in\partial E_R$ and a_2 denotes the area of the ellipsoid E_1;

$$I_3 = \iint\limits_{\partial E_R} LU(r)\frac{\partial v(Y)}{\partial\nu}dS_Y$$

$$= a_2R^2R^{-1}C^2(e^{CR}+\cos CR)\frac{\partial v(Q_2)}{\partial\nu} \to 0, \qquad Q_2\in\partial E_R,$$

$$I_2 = \iint\limits_{\partial E_R} Lv(Y)\frac{\partial U(r)}{\partial\nu}dS_Y$$

$$= a_2R^2Lv(Q_1)(-R^{-2}e^{CR}+R^{-1}Ce^{CR}+R^{-2}\cos CR+CR^{-1}\sin CR)\frac{\partial r}{\partial\nu} \to 0,$$

$$Q_1\in\partial E_R,$$

$$I_4 = \iint\limits_{\partial E_R} v(Y)\frac{\partial}{\partial\nu}LU'(r)dS_Y$$

$$= v(Q_3)(C^2R^{-2}e^{CR}-C^3R^{-1}e^{CR}+C^2R^{-2}\cos CR+C^3R^{-1}\sin CR)\iint\limits_{\partial E_R}\frac{\partial r}{\partial\nu}dS_Y,$$

$$Q_3\in\partial E_R.$$

By Lemma 7,

$$\iint\limits_{\partial E_R}\frac{\partial r}{\partial\nu}dS_Y = \iint\limits_{\partial E_R}\frac{RdS_Y}{G(X)} = \int\limits_0^{2\pi}\int\limits_0^{\pi}\frac{R^3J(\theta,\varphi)}{Rg(\theta,\varphi)}d\theta\, d\varphi$$

$$= R^2\int\limits_0^{2\pi}\int\limits_0^{\pi}\frac{J(\theta,\varphi)}{g(\theta,\varphi)}d\theta\, d\varphi,$$

whence

$$I_4 \to 2C^2 y(x, y, z) \int_0^{2\pi} \int_0^{\pi} \frac{J(\theta, \varphi)}{g(\theta, \varphi)}\, d\theta\, d\varphi .$$

The above formulas and (4.4) give (2.3) with

$$\gamma_3 = \left(2C^2 \int_0^{2\pi} \int_0^{\pi} \frac{J(\theta, \varphi)}{g(\theta, \varphi)}\, d\theta\, d\varphi\right)^{-1}.$$

(b) The case $k = C^4$. The equation (4.1) is then of the form

$$V^{(4)}(r) + C^4 V(r) = 0 .$$

Particular integrals of this equation are the functions

$$U_1(r) = r^{-1} e^{Ar} \sin Ar, \qquad U_2(r) = r^{-1} e^{-Ar} \sin Ar,$$

where $A = (C\sqrt{2})/2$. By Lemma 9, they are also integrals of (1.1).

THEOREM 4 (see [4]). *The function*

$$U(r) = r^{-1} \sin Ar (e^{Ar} - e^{-Ar})$$

is a fundamental solution of equation (1.1), *where* $A = (C\sqrt{2})/2$.

Proof. An argument similar to this in the proof of Theorems 1 and 2 shows that the surface integrals I_1, I_2, I_3 tend to zero, and

$$I_4 = a_2 R^2 R^{-1} 2A^3 v(Q_3)\Big((e^{AR} + e^{-AR})(-R^{-1} \cos AR - A \sin Ar) +$$

$$+ A(e^{AR} - e^{-AR}) \cos AR\Big) \iint_{\partial E_R} \frac{\partial r}{\partial v}\, dS_Y \to -2C^2 v(x, y, z) \int_0^{2\pi} \int_0^{\pi} \frac{J(\theta, \varphi)}{g(\theta, \varphi)}\, d\theta\, d\varphi .$$

We thus obtain formula (2.3) with

$$\gamma_3 = -\left(2C^2 \int_0^{2\pi} \int_0^{\pi} \frac{J(\theta, \varphi)}{g(\theta, \varphi)}\, d\theta\, d\varphi\right)^{-1}.$$

5. We shall now deal with the case $n > 3$, $k = -C^4$. Equation (1.1) takes on the form

$$L^2 u(X) - C^4 u(X) = 0, \tag{5.1}$$

where $X = (x_1, \ldots, x_n)$. By Lemmas 1 and 2, the integrals $U(r)$ of equation (5.1) satisfy the differential equation

$$U''(r) + r^{-1}(n-1)\, U'(r) + C^2 U(r) = 0 \tag{5.2}$$

or

$$U''(r)+(n-1)r^{-1}U'(r)-C^2U(r) = 0. \tag{5.3}$$

The functions $u(X) = U(r_{XX_0})$ are solutions of (5.1). The substitution $R = Cr$ converts equation (5.2) into

$$U''(R)+\big(1-2(2-n)/2\big)R^{-1}U'(R)+U(R) = 0, \tag{5.2a}$$

which is a special case of the equation (see [3])

$$U''(z)+(1-2\alpha)z^{-1}U'(z)+\big((\beta\gamma z^{\gamma-1})^2+(\alpha^2-s^2\gamma^2)z^{-2}\big)U(z) = 0, \tag{5.4}$$

where

$$\alpha = (2-n)/2, \qquad z = R, \qquad \gamma = \beta = 1, \qquad s = -\gamma = (n-2)/2.$$

The integrals of (5.4) are the functions (see [3])

$$U(z) = z^{\alpha}Z_s(\beta z^{\gamma}), \tag{5.5}$$

where $s > 0$ and

$$Z_s(z) = Y_s(z) = \frac{2}{\pi}I_s(z)\ln\frac{z}{2} - \tag{5.6}$$

$$-\frac{1}{\pi}\sum_{k=0}^{s-1}\frac{(s-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-s}-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{s-2k}}{k!(s+k)!}\big(f(k+1)+f(k+s+1)\big),$$

where

$$I_s(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{s+2k}}{k!(s+k)!},$$

and, in the case of non-integral s,

$$Z_s(z) = I_{-s}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k-s}}{\Gamma(k+1)\Gamma(-s+k+1)}. \tag{5.7}$$

Consequently, the function

$$U(r) = (Cr)^{-(n-2)/2}Z_s(Cr)$$

is a solution of equation (5.2).

We shall single out three cases:

$$\left.\begin{array}{l} 1^\circ\ s = (n-2)/2 = 2p+1 \\ 2^\circ\ s = (n-2)/2 = 2p \end{array}\right\}\ p \text{ being a positive integer,}$$

3° $s = (n-2)/2$ n is odd.

Ad 1°. In this case by (5.6)

$$(5.8)\qquad Y_s(Cr) = \frac{-(2p)!}{\pi}\left(\frac{Cr}{2}\right)^{-2p-1} + O(r^{-2p-1})$$

(cf. [4], p. 222). By (5.5), (5.6), (5.8) and by Lemma 2 the function

$$(5.9)\qquad U_1(r) = (Cr)^{-2p-1}\left(\frac{\big(-(2p)!\big)}{\pi}\left(\frac{Cr}{2}\right)^{-2p-1} + O(r^{-2p-1})\right)$$

is a solution of equation (5.2).

Since the substitution of Ci instead of C converts equation (5.2) into (5.3), the function $\operatorname{Im}(Cir)^{-2p-1}Y_s(Cir)$, and hence the function $U_2(r) = (Cr)^{-2p-1}\operatorname{Im} Y_s(Cir)$, is a solution of (5.3). By (5.8),

$$Y_s(Cir) = -\frac{1}{\pi}(2p)!\left(\frac{Cir}{2}\right)^{-2p-1} + O(r^{1-2p}),$$

and

$$(5.10)\qquad U_2(r) = (Cr)^{-2p-1}\left((-1)^p\left(\frac{Cr}{2}\right)^{-2p-1}\frac{(2p)!}{\pi} + O(r^{1-2p}\right).$$

THEOREM 5 (see [4]). *If* $n > 3$, $s = (n-2)/2 = 2p+1$, *and* p *is integer, then the fundamental solution is represented by*

$$U(r) = C_1 U_1(r) + C_2 U_2(r),$$

where $C_1 = 1$, $C_2 = (-1)^p$ *and the functions* $U_1(r)$ *and* $U_2(r)$ *are defined by* (5.9) *and* (5.10), *i.e.*,

$$(5.11)\qquad U(r) = (Cr)^{-2p-1}[C_1 O(r^{-2p-1}) + C_2 O(r^{1-2p})].$$

Proof. By (5.11) and Lemmas 3, 4 and 5,

$$(5.12)\qquad U(r) = O(r^{4-n}),$$

and

$$(5.13)\qquad \frac{\partial U(r)}{\partial \nu} = U'(r)\frac{\partial r}{\partial \nu} = O(r^{3-n})\frac{r}{G(X)} = O(r^{3-n})$$

for $\dfrac{\partial r}{\partial \nu} = \dfrac{r}{G(X)} = O(1)$. By Lemma 2,

$$\begin{aligned}(5.14)\qquad LU(r) &= -C^2\big(C_1 U_1(r) - C_2 U_2(r)\big)\\ &= 2C^2(Cr)^{-4p-2}\frac{(2p)!}{\pi}\left(\frac{1}{2}\right)^{-2p-1} - C^2(Cr)^{-2p-1}\big(C_1 O(r^{-2p-1}) - C_2 O(r^{1-2p})\big)\\ &= O(r^{2-n}).\end{aligned}$$

By (5.14) and by Lemmas 3 and 4

$$\frac{\partial LU(r)}{\partial \nu} = \big(LU(r)\big)' \frac{\partial r}{\partial \nu} = \big(\beta_n r^{1-n} + O(r^{3-n})\big)\frac{\partial r}{\partial \nu}, \tag{5.15}$$

where

$$\beta_n = -\frac{1}{\pi} C^{4-n} 2^{(n+2)/2} \left(\frac{n-2}{2}\right)!. \tag{5.16}$$

Since the function $U(r)$ defined by (5.11) satisfies conditions 2° and 3° of Definition 1, it remains to check that it also satisfies condition 4°, i.e., that formula (2.3) is satisfied with $\gamma_n = a_n \beta_n$, a_n denoting the area of the surface of the n-dimensional unit ellipsoid E_1. An argument similar to that applied in the proof of Theorem 1 leads to formula (3.19), the involved surface integrals being $(n-1)$-fold. Applying the mean-value theorem to these surface integrals we obtain in virtue of (5.15), (5.12), (5.13) and (5.14)

$$I_1 = \iint\limits_{\partial E_R} U(r) \frac{\partial Lv(Y)}{\partial \nu} dS_Y = a_n R^{n-1} O(R^{4-n}) = a_n O(R^3) \to 0,$$

$$I_2 = \iint\limits_{\partial E_R} Lv(Y) \frac{\partial U(r)}{\partial \nu} dS_Y = a_n R^{n-1} O(R^{3-n}) = a_n O(R^2) \to 0,$$

$$I_3 = \iint\limits_{\partial E_R} LU(r) \frac{\partial v(Y)}{\partial \nu} dS_Y = a_n R^{n-1} O(R^{2-n}) = a_n O(R) \to 0,$$

$$\begin{aligned} I_4 &= \iint\limits_{\partial E_R} v(Y) \frac{\partial LU(r)}{\partial \nu} dS_Y = \big(\beta_n R^{1-n} + O(R^{3-n})\big) v(Q) \iint\limits_{\partial E_R} \frac{\partial r}{\partial \nu} dS_Y \\ &= \beta_n \iint\limits_{\partial E_R} \frac{R}{G(X)} dS_Y R^{1-n} v(Q) + O(R^{3-n}) \\ &= \beta_n R^{n-1} \int_0^{2\pi} \ldots \int_0^{\pi} \frac{J(\varphi_1, \ldots, \varphi_n)}{g(\varphi_1, \ldots, \varphi_n)} d\varphi_1 \ldots d\varphi_n R^{1-n} v(Q) + O(R^{3-n}) \\ &= \beta_n a_n v(Q) + O(R^{3-n}) \to a_n \beta_n v(X), \qquad \theta \in \partial E_R, \end{aligned}$$

where

$$a_n = \int_0^{2\pi} \ldots \int_0^{\pi} \frac{J(\varphi_1, \ldots, \varphi_n)}{g(\varphi_1, \ldots, \varphi_n)} d\varphi_1 \ldots d\varphi_n. \tag{5.17}$$

The formulas we just obtained and formula (3.19) give formula (2.3) with $\gamma_n = (a_n \beta_n)^{-1}$, where β_n is defined by (5.16) and a_n is determined by (5.17), and hence our assertion follows.

In case 2° we deduce from formulas (5.5), (5.6) and from Lemma 2 that the functions

$$U_1(r) = (Cr)^{-2p}\left(\frac{-(2p-1)!}{\pi}\left(\frac{Cr}{2}\right)^{-2p}+O(r^{2-2p})\right) \tag{5.18}$$

and

$$U_2(r) = (Cr)^{-2p}\left(-(-1)^p\frac{(2p-1)!}{\pi}\left(\frac{Cr}{2}\right)^{-2p}+O(r^{1-2p})\right) \tag{5.19}$$

are integrals of (5.1). The fundamental solution may now be obtained from the functions $U_1(r)$ and $U_2(r)$ analogously to the foregoing cases. This leads to

THEOREM 6 (see [4]). *In the case where* $s = 2p$ *and* p *is integer, the function*

$$U(r) = (-1)\,U_1(r)+(-1)^p U_2(r)$$

is a fundamental solution of (5.1) *the functions* $U_1(r)$ *and* $U_2(r)$ *being defined by* (5.18) *and* (5.19).

The proof is similar to that of Theorems 1 and 5; the constant γ_n is equal to $\dfrac{1}{a_n\beta_n}$, where $\beta_n = \dfrac{(n-2)(n-4)}{\pi}\,C^{4-n}2^{(n-2)/2}$ and a_n is determined by (5.17).

In the case 3° we single out two sub-cases

3°A) $s = (n-2)/2 = (2q+1)/2 = 2p+\frac{1}{2}$ for $q = 2p$,

3°B) $s = (n-2)/2 = (2q+1)/2 = 2p+\frac{3}{2}$ for $q = 2p+1$.

By Lemma 2 and by formulas (5.5) and (5.7), the functions

$$U_1(r) = u_1(Cr) \tag{5.20}$$

$$= \frac{1}{\Gamma(1)\Gamma(1-s)}\,2^{(n-2)/2}(Cr)^{2-n}-\frac{1}{\Gamma(2)\Gamma(2-s)}\,2^{(n-6)/2}(Cr)^{4-n}+O(r^{6-n}),$$

$$u_1(Cir) = \frac{2^{(n-2)/2}(Cr)^{2-n}i^{2-n}}{\Gamma(1)\Gamma(1-s)}+\frac{2^{(n-6)/2}(Cr)^{2-n}i^{4-n}}{\Gamma(2)\Gamma(2-s)}+O(r^{6-n}),$$

Re $u_1(Cir)$, and $\operatorname{Im} u_1(Cir)$ are integrals of (5.2).

Ad 3°A). Let $U_2(r) = \operatorname{Im} u_1(Cir)$ and let

$$U(r) = -\big(U_1(r)+U_2(r)\big)-O(r^{6-n}). \tag{5.21}$$

Ad 3°B). The solutions of (5.2) are functions defined by (5.21) and the function

$$U_3(r) = \operatorname{Im} u_3(Cir) = \frac{2^{(n-2)/2}(Cr)^{2-n}}{\Gamma(1)\Gamma(1-s)}+\frac{2^{(n-6)/2}(Cr)^{4-n}}{\Gamma(2)\Gamma(2-s)}+O(r^{6-n}).$$

The function

$$(5.22)\qquad U(r) = U_3(r) - U_1(r) = \frac{2^{(n-4)/2}(Cr)^{4-n}}{\Gamma(2)\Gamma(2-s)} + O(r^{6-n})$$

is a solution of (5.2).

THEOREM 7 (see [4]). *Let* $s = 2p+\frac{1}{2}$ *or* $s = 2p+\frac{3}{2}$, *p being a positive integer, the functions $U(r)$ defined by* (5.21) *and* (5.22) *are fundamental solutions of* (5.1), *respectively.*

The proof is analogous to that of Theorem 5. In both cases $\gamma_n = (\alpha_n\beta_n)^{-1}$, where $\beta_n = 2^{(n-4)/2}C^{4-n}/\Gamma(2)\Gamma(2-s)$ and α_n is determined by (5.17).

6. The case $n > 3$, $k = C^4$. The equation (1.1) takes on the form

$$(6.1)\qquad L^2u(X) + C^4u(X) = 0, \qquad X(x_1, \ldots, x_n),$$

it may be obtained from (5.1) by substitution of $C\sqrt{i}$ instead of C.

We shall single out the following cases

1°A) $s = 2p+1 = 8q+1$ for $p = 4q$,
1°B) $s = 2p+1 = 8q+3$ for $p = 4q+1$,
1°C) $s = 2p+1 = 8q+5$ for $p = 4q+2$,
1°D) $s = 2p+1 = 8q+7$ for $p = 4q+3$,
2°A) $s = 2p = 4q$ for $p = 2q$,
2°B) $s = 2p = 4q+2$ for $p = 2q+1$,
3°A) $s = q+\frac{1}{2} = 2p+\frac{1}{2}$ for $q = 2p$,
3°B) $s = q+\frac{1}{2} = 2p+\frac{3}{2}$ for $q = 2p+1$.

Ad 1°A). By (5.6)

$$(6.2)\qquad Y_s(Cr\sqrt{i}) = -\frac{1}{\pi}(8q)!\left(\frac{Cr\sqrt{i}}{2}\right)^{-8q-1} - \frac{1}{\pi}(8q-1)!\left(\frac{Cr\sqrt{i}}{2}\right)^{-8q} -$$

$$-\frac{1}{\pi}\sum_{k=2}^{8q}\frac{(8q-k)!}{k!}\left(\frac{Cr\sqrt{i}}{2}\right)^{2k-8q-1} + \frac{2}{\pi}I_s(Cr\sqrt{i})\ln\frac{Cr\sqrt{i}}{2} -$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(Cr\sqrt{i}/2)^{s+2k}}{k!(s+k)!}\big(f(k+1)+f(k+s+1)\big).$$

In virtue of (5.5) and (6.2) the function

$$F_1(r) = (Cr\sqrt{i})^{-8q-1}Y_s(Cr\sqrt{i})$$

$$= -\frac{1}{\pi}(8q)!\,2^{8q+1}(Cr\sqrt{i})^{-16q-2} - \frac{1}{\pi}(8q-1)!\,2^{8q-1}(Cr)^{-16q} + O(r^{6-n})$$

and the function

$$(6.3)\qquad U_1(r) = -\mathrm{Re}\,F_1(r) = \frac{1}{\pi}(8q-1)!\,2^{8q-1}(Cr)^{-16q}+O(r^{6-n})$$

are solutions of (6.1).

Ad 1°B). The function

$$F_2 = (Cr\sqrt{i})^{-8q-3}Y_s(Cr\sqrt{i})$$

$$= -\frac{1}{\pi}(8_q+2)!\,2^{8q+3}(Cr)^{-16q-6}i+\frac{1}{\pi}(8q+1)!\,2^{8q+1}(Cr)^{16q-4}+O(r^{6-n})$$

and the function

$$(6.4)\qquad U_2(r) = \mathrm{Re}\,F_2(r) = \frac{1}{\pi}(8q+1)!\,2^{8q+1}(Cr)^{-16q-4}+O(r^{6-n})$$

are solutions of (6.1).

Ad 1°C). The function

$$F_3(r) = (Cr\sqrt{i})^{-8q-5}Y_s(Cr\sqrt{i})$$

$$= \frac{1}{\pi}(8q+4)!\,2^{8q+5}(Cr)^{-16q-10}i-\frac{1}{\pi}(8q+3)!\,2^{8q+3}(Cr)^{-16q-8}+O(r^{6-n})$$

and the function

$$(6.5)\qquad U_3(r) = -\mathrm{Re}\,F_3(r) = \frac{1}{\pi}(8q+3)!\,2^{8q+3}(Cr)^{-16q-8}+O(r^{6-n})$$

are solutions of (6.1).

Ad 1°D). The function

$$F_4(r) = (Cr\sqrt{i})^{-8q-7}Y_s(Cr\sqrt{i})$$

$$= -\frac{1}{\pi}(8q+6)!\,2^{8q+7}(Cr)^{-16q-14}i+\frac{1}{\pi}(8q+5)!\,2^{8q+5}(Cr)^{-16q-12}+O(r^{6-n})$$

and the function

$$(6.6)\qquad U_4(r) = \mathrm{Re}\,F_4(r) = \frac{1}{\pi}(8q+5)!\,2^{8q+5}(Cr)^{-16q-12}+O(r^{6-n})$$

are solutions of (6.1).

THEOREM 8 (see [4]). *Let* $n > 3$. *If* $s = (n-2)/2 = 8q+1$, $8q+3$, $8q+5$, $8q+7$, *then the functions* (6.3), (6.4), (6.5) *and* (6.6) *are fundamental solutions of* (6.1), *respectively.*

Proof. We shall only consider the case $s = 8q+1$; the remaining ones can be proved similarly. By (6.3)

$$U_1(r) = C_1 r^{4-n} + O(r^{6-n}),$$

$$\frac{\partial U_1(r)}{\partial \nu} = U'(r)\frac{\partial r}{\partial \nu} = O(r^{5-n})\frac{\partial r}{\partial \nu} + C_2 r^{3-n}\frac{\partial r}{\partial \nu}$$

$$= O(r^{5-n})\frac{r}{G(X)} + C_2 r^{3-n}\frac{r}{G(X)} = O(r^{6-n}) + C_2\frac{r^{4-n}}{G(X)},$$

$$LU_1(r) = C_3 r^{2-n} + O(r^{4-n}).$$

By Lemma 4

$$\frac{\partial LU(r)}{\partial \nu} = \big(LU(r)\big)'\frac{\partial r}{\partial \nu} = \big(C_4 r^{1-n} + O(r^{3-n})\big)O(1)$$

where C_1, C_2, C_3 are constants and $C_4 = 2^{(n+2)/2}C^{4-n}\big((n-2)/2\big)!$ and $\gamma_n = (C_4 a_n)^{-1}$ and a_n is determined by (5.17).

The proof runs down like that of Theorem 5.

The next theorem will deal with the case 2°A) and 2°B).

If $s = 2p = 4q$, then the function

$$F_5(r) = (Cr\sqrt{i})^{-4q} Y_s(Cr\sqrt{i})$$

$$= -\frac{1}{\pi}(4q-1)!\,2^{4q}(Cr)^{-8q} - \frac{1}{\pi}(4q-2)!\,2^{2-8q}i + O(r^{6-n}),$$

and the function

$$U_5(r) = -\operatorname{Im} F_5(r) = \frac{1}{\pi}(4q-2)!\,2^{4q-2}(Cr)^{2-8q} + O(r^{6-n}) \tag{6.7}$$

are solutions of (6.1).

If $s = 2p = 4q+2$ the function

$$F_6(r) = (Cr\sqrt{i})^{-4q-2} Y_s(Cr\sqrt{i})$$

$$= \frac{1}{\pi}(4q+1)!\,2^{4q+2}(Cr)^{-8q-4} + \frac{1}{\pi}(4q)!\,2^{4q}(Cr)^{-8q-2}i + O(r^{6-n})$$

and the function

$$U_6(r) = \operatorname{Im} F_6(r) = \frac{1}{\pi}(4q)!\,2^{4q}(Cr)^{-8q-2} + O(r^{6-n}) \tag{6.8}$$

are solutions of (6.1).

THEOREM 9 (see [4]). *Let $n > 3$. If $s = (n-2)/2 = 4q$ or $s = (n-2)/2 = 4q+2$, then the functions* (6.7) *and* (6.8) *are respectively the fundamental solutions of* (6.1).

Proof. We shall only consider the case 2°A), the second one being analogous. By (6.7)

$$U_5(r) = D_1 r^{4-n} + O(r^{6-n}),$$

$$\frac{\partial U_5(r)}{\partial \nu} = U_5'(r)\frac{\partial r}{\partial \nu} = \big(D_2 r^{3-n} + O(r^{5-n})\big)\frac{r}{G(X)} = D_2\frac{r^{4-n}}{G(X)} + O(r^{6-n}),$$

$$LU_5(r) = D_3 r^{2-n} + O(r^{4-n}).$$

By Lemma 4

$$\frac{\partial LU_5(r)}{\partial \nu} = \big(LU(r)\big)'\frac{\partial r}{\partial \nu} = \big(D_4 r^{1-n} + O(r^{3-n})\big)O(1)$$

where D_1, D_2, D_3 are constants and

$$D_4 = \frac{1}{\pi}\big((n-6)/2\big)!\,2^{(n-4)/2}C^{4-n}(4-n)(2-n), \qquad \gamma_n = (D_4 a_n)^{-1},$$

and a_n is denoted by (5.17).

The proof is analogous to that of Theorem 5.

Let now $s = q+\frac{1}{2} = 2p+\frac{1}{2}$. The function

$$z^{-s}Y_s(z) = \frac{z^{-2s}2^s}{\Gamma(1)\Gamma(1-s)} + \frac{z^{-2s+2}2^{s-2}}{\Gamma(2)\Gamma(2-s)} + z^{-s}\sum_{k=2}^{\infty}\frac{(-1)^k z^{2k-s}}{\Gamma(k+1)\Gamma(k-s+1)}$$

is a solution of (5.4), whence the function

$$F_6(r) = (Cr\sqrt{i})^{-s}Y_s(Cr\sqrt{i}) = \frac{2^s(Cr\sqrt{i})^{-2q-1}}{\Gamma(1)\Gamma(1-s)} - \frac{2^{s-2}(Cr\sqrt{i})^{1-2q}}{\Gamma(2)\Gamma(2-s)} + O(r^{6-n})$$

and the function

$$U_6(r) = (-1)^p\operatorname{Re}F_6(r) = -\frac{2^{s-1/2}(Cr)^{-4p-1}}{\Gamma(1)\Gamma(1-s)} + \frac{2^{s-5/2}(Cr)^{1-4p}}{\Gamma(2)\Gamma(2-s)} + O(r^{6-4})$$

satisfy (6.1).

Also the function

$$\begin{aligned} F_7(r) &= (Cri^{3/2})^{-s}Y_s(Cri^{3/2}) \\ &= \frac{2^s(Cr)^{-4p-1}}{\Gamma(1)\Gamma(1-s)}\,i^{-(3/2)(4p+1)} + \frac{2^{s-2}(Cr)^{1-4p}}{\Gamma(2)\Gamma(2-s)}\,i^{-(1-4p)(3/2)} + O(r^{6-n}) \end{aligned}$$

and the function

$$U_7(r) = (-1)^p\operatorname{Im}F_7(r) = \frac{2^{s-1/2}(Cr)^{-4p-1}}{\Gamma(1)\Gamma(1-s)} + \frac{2^{s-5/2}(Cr)^{1-4p}}{\Gamma(2)\Gamma(2-s)} + O(r^{6-n})$$

are solutions of (6.1).

If follows that the function

$$U(r) = U_6(r)+U_7(r) = \frac{2^{s-3/2}(Cr)^{1-4p}}{\Gamma(2)\Gamma(2-s)}+O(r^{6-n}) \tag{6.9}$$

is a solution of (6.1).

Let $s = q+\frac{1}{2} = 2p+\frac{3}{2}$, the function

$$F_8(r) = (Cr\sqrt{i})^{-s}Y_s(Cr\sqrt{i})$$

$$= \frac{2^s(Cr)^{-4p-1}(\sqrt{i})^{-4p-1}}{\Gamma(1)\Gamma(1-s)} - \frac{2^{s-2}(Cr)^{1-4p}(\sqrt{i})^{1-4p}}{\Gamma(2)\Gamma(2-s)} +O(r^{6-n})$$

and the function

$$U_8(r) = (-s)^p\mathrm{Re}F_8(r) = \frac{2^{s-1/2}(Cr)^{-4p-3}}{\Gamma(1)\Gamma(1-s)} + \frac{2^{s-5/2}(Cr)^{-4p-1}}{\Gamma(2)\Gamma(2-s)}+O(r^{6-n})$$

are solutions of (6.1).

The function

$$F_9(r) = (Cri^{3/2})^{-s}Y_s(Cri^{3/2})$$

$$= \frac{2^s(Cr)^{-4p-3}i^{(-4p-3)(3/2)}}{\Gamma(1)\Gamma(1-s)} - \frac{2^{s-2}(Cr)^{-4p-1}i^{(-4p-1)(3/2)}}{\Gamma(2)\Gamma(2-s)} +O(r^{6-n})$$

and the function

$$U_9(r) = (-1)^p\mathrm{Re}F_9(r) = \frac{2^{s-1/2}(Cr)^{-4p-3}}{\Gamma(1)\Gamma(1-s)} - \frac{2^{s-5/2}(Cr)^{-4p-1}}{\Gamma(2)\Gamma(2-s)} +O(r^{6-n})$$

are also solutions of (6.1).

Hence the function

$$U(r) = U_8(r)-U_9(r) = \frac{2^{s-3/2}(Cr)^{-4p-1}}{\Gamma(2)\Gamma(2-s)} +O(r^{6-n}) \tag{6.10}$$

is a solution of (6.1).

Now,

$$U(r) = E_1r^{4-n}+O(r^{6-n}),$$

$$\frac{\partial U}{\partial \nu} = U'(r)\frac{\partial r}{\partial \nu} = \big(E_2r^{3-n}+O(r^{5-n})\big)\frac{r}{G(X)} = E_2\frac{r^{4-n}}{G(X)}+O(r^{6-n}),$$

$$LU(r) = E_3r^{2-n}+O(r^{4-n}).$$

By Lemma 4,

$$\frac{\partial LU(r)}{\partial \nu} = \big(LU(r)\big)'\frac{\partial r}{\partial \nu} = \big(E_4r^{1-n}+O(r^{3-n})\big)O(1),$$

where E_1, E_2, E_3 are constants and $E_4 = \frac{2^{n/2}C^{2-n}(2-n)}{\Gamma(1)\Gamma\big((4-n)/2\big)}$, and $\gamma_n = (E_4 a_n)^{-1}$, and a_n is determined by (5.17).

By the already familiar procedure we may prove the following

THEOREM 10 (see [4]). *Let* $n > 3$. *If* $s = (n-2)/2 = 2p+1$ *or* $s = (n-2)/2 = 2p+\frac{3}{2}$ *then the functions* (6.9) *and* (6.10) *are respectively the fundamental solutions of* (6.1).

7. In this section we shall apply the fundamental solutions $U(r)$ of (1.1) to the boundary problems of Lauricelli [5] and of Riquier [5] for equation (1.1).

Let $H(X, Y)$ be a function of X, Y defined in $\bar{D}\times\bar{D}$ and satisfying the conditions

1. $H(X, Y)\in C^{(4)}$ for $(X, Y)\in D\times D$,
2. $H(X, Y)\in C^{(3)}$ for $(X, Y)\in \bar{D}\times\bar{D}$,
3. $H(X, Y) = U(r)$ for $Y\in\partial D$,
4. $\frac{dH}{d\nu} = \frac{dU}{d\nu}$ for $Y\in\partial D$,
5. as a function of Y, $H(X, Y)$ satisfies equation (1.1).

DEFINITION 2. The function

$$G(X, Y) = U(r) - H(X, Y) \tag{7.1}$$

is called the *Green function of type* (L) for equation (1.1) in the (bounded) domain D.

We shall now give a formula solving the Lauricelli problem for equation (1.1). Taking in the fundamental formula

$$\int\int\int_D (uL^2v - vL^2u)\,dx_1\ldots dx_n + \int\int_{\partial D}\left(Lu\frac{dv}{d\nu} - v\frac{dLu}{d\nu} + u\frac{dLv}{d\nu} - Lv\frac{du}{d\nu}\right)dS_Y = 0 \tag{7.2}$$

a solution of (1.1) of class $C^{(4)}$ in D as $u(Y)$, and as $v(Y)$ the function $\gamma_n H(X, Y)$ we obtain from (7.2) and by the identity $uL^2H - HL^2u = 0$

$$0 = \gamma_n \int\int_{\partial D}\left(Lu\frac{dH}{d\nu} - H\frac{dLU}{d\nu} + U\frac{dLu}{d\nu} - Lu\frac{dU}{d\nu}\right)dS_Y. \tag{7.3}$$

Upon setting $v(Y)$ as $u(Y)$ into (2.3) we get

$$u(X) = \gamma_n \int\int_{\partial D}\left(LU\frac{du}{d\nu} - u\frac{dLU}{d\nu} + U\frac{dLu}{d\nu} - Lu\frac{dU}{d\nu}\right)dS_Y. \tag{7.4}$$

If we add formulas (7.3) and (7.4), then

$$u(X) = \gamma_n \iint_{\partial D} \Big[(LU - LH)\frac{du}{d\nu} + \Big(\frac{dLH}{d\nu} - \frac{dLU}{d\nu}\Big)u + \\ + \Big(\frac{dH}{d\nu} - \frac{dU}{d\nu}\Big)Lu + (U-H)\frac{dLu}{d\nu}\Big]\, dS_Y .$$

By conditions 3 and 4 and by Definition 2 we obtain

$$u(X) = \gamma_n \iint_{\partial D} \Big(LG\frac{du}{d\nu} - \frac{dLG}{d\nu}u\Big)\, dS_Y . \tag{7.5}$$

THEOREM 11. *Suppose that there exist Green function of type* (L). *Let the functions* $f(Y)$ *and* $h(Y)$ *be continuous on* ∂D, *the boundary of* D, *then the functions*

$$u(X) = \gamma_n \iint_{\partial D} \Big[h(Y) LG(X, Y) - f(Y)\frac{dLG(X, Y)}{d\nu}\Big]\, dS_Y$$

solves the boundary problem of Lauricelli:

$$u(X) = f(Y), \quad \frac{du(Y)}{d\nu} = h(Y) \quad \textit{for} \quad Y \in \partial D.$$

The Green function for the Riquier problem (R) is defined as follows. Let $H_1(X, Y)$ be a function of X, Y defined in $D \times D$ and satisfying the following conditions:

6. $H_1(X, Y) \in C^{(4)}$ for $(X, Y) \in D \times D$,
7. $H_1(X, Y) \in C^{(3)}$ for $(X, Y) \in \bar{D} \times \bar{D}$,
8. $LH_1 = LU(r)$ for $Y \in \partial D$,
9. $\dfrac{dLH_1}{d\nu} = \dfrac{dLU}{d\nu}$ for $Y \in \partial D$,
10. as a function of Y, $H_1(X, Y)$ satisfies equation (1.1).

DEFINITION 3. The function $G_1(X, Y) = U(r) - H_1(X, Y)$ is called the *Green function of type* (R) for equation (1.1) in the (bounded) domain D.

Analogously as above,

$$u(X) = -\gamma_n \iint_{\partial D} \Big[G_1(X, Y)\frac{dLu}{d\nu} - \frac{dG_1(X, Y)}{d\nu} Lu\Big]\, dS_Y ,$$

which implies the following

THEOREM 12. *Suppose that there exists Green function of type* (R). *If* $f_1(Y)$ *and* $h_1(Y)$ *are continuous function on* ∂D *then the formula*

$$u(X) = \gamma_n \iint_{\partial D} \left[h_1(Y) \frac{dG_1(X, Y)}{d\nu} - \frac{dLG_1(X, Y)}{d\nu} f_1(Y) \right] dS_Y$$

solves the equation (1.1) *with the Riquier boundary values*

$$u(Y) = f_1(Y), \quad Lu(Y) = h_1(Y) \quad \text{for} \quad Y \in \partial D.$$

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