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Theory of pursuit in gravitational and electromagnetic fields

0. Introduction. The notion of the pursuit game was originally defined by H. Steinhaus [6] in 1925. Further contributions to this theory are due to J. Mycielski [1], [2], C. Ryll-Nardzewski [5] and A. Zięba [1], [7]-[9].

In the present paper we give the extension of ideas of A. Zięba to the case of relativistic physics. The basic result of this paper can be formulated mathematically as follows: Let R be a Riemannian space-time with the gravitational field $g^{\mu\nu}$ and electromagnetic field A_ν.

We consider the classes U and V of admissible strategies; they consist of all integrable functions of two variables (positions in space-time of escaping and pursuing players) x^μ, $y^\mu \in R$ with values in R satisfying the metric conditions:

$$(1)\qquad \begin{aligned} g^{\mu\nu}(u_\mu - aA_\mu)(u_\nu - aA_\nu) &= -1, \qquad u_\mu \in U, \\ g^{\mu\nu}(v_\mu - bA_\mu)(v_\nu - bA_\nu) &= -1, \qquad v_\mu \in V, \end{aligned}$$

where a and b denote respectively the ratios of the charge and the mass for the escaping and pursuing players.

The equilibrium conditions of the conflicting situation between two players in R are given by the theorem: If $\sigma(x^\mu, y^\mu)$ is a nonnegative and differentiable function for x^μ, $y^\mu \in R$ and the functions $u_\mu^* \in U$ and $v_\mu^* \in V$ are such that for every $u_\mu \in U$, x^μ, $y^\mu \in R$ we have

$$(2)\qquad ug^{\mu\nu}p_\mu u_\nu + vg^{\mu\nu}q_\mu v_\nu^* \geqslant -1$$

and for every $v_\mu \in V$, x^μ, $y^\mu \in R$ we have

$$(3)\qquad ug^{\mu\nu}p_\mu u_\nu^* + vg^{\mu\nu}q_\mu v_\nu \leqslant -1$$

and if for every $x^\mu \in R$ we have

$$\sigma(x^\mu, x^\mu) = 0,$$

then u_μ^* and v_μ^* are the optimal strategies, and $\sigma(x^\mu, y^\mu)$ is the value of the game in the initial position x^μ, y^μ.

The symbols p_μ and q_μ appearing in the above formulas are defined as $p_\mu = \partial\sigma/\partial x^\mu$ and $q_\mu = \partial\sigma/\partial y^\mu$ and the scalar functions u and v with values in R are given numbers which bound the strategies in classes U and V.

If the equation

$$ug^{\mu\nu}p_\mu u_\nu^* + vg^{\mu\nu}q_\mu v_\nu^* = -1$$

is satisfied, we say that the pursuit forms a uniformly closed game.

From the conditions (1), (2) and (3) we can uniquely determine the Hamiltonian. For the system of two objects pursuing each other in fields A_ν and $g^{\mu\nu}$ this Hamiltonian takes the form

$$\begin{aligned} H &= H(p_\nu, q_\nu;\ x^\nu, y^\nu) \\ &= u(\sqrt{-g^{\mu\nu}p_\mu p_\nu} - ag^{\mu\nu}p_\mu A_\nu) - v(\sqrt{-g^{\mu\nu}q_\mu q_\nu} + bg^{\mu\nu}q_\mu A_\nu). \end{aligned}$$

Next, it is shown that the optimal pursuit trajectories satisfy the canonical equations of Hamilton:

$$\begin{aligned} \frac{dx^\mu}{ds} &= \frac{\partial H}{\partial p_\mu}, & \frac{dp_\mu}{ds} &= -\frac{\partial H}{\partial x^\mu}, \\ \frac{dy^\mu}{ds} &= \frac{\partial H}{\partial q_\mu}, & \frac{dq_\mu}{ds} &= -\frac{\partial H}{\partial y^\mu}, \end{aligned} \tag{4}$$

and that the value $\sigma(x^\mu, y^\mu)$ of the game satisfies the equation of Hamilton-Jacobi:

$$H(x^\mu, y^\mu;\ \partial\sigma/\partial x^\mu, \partial\sigma/\partial y^\mu) = 1.$$

The pursuit equations (4) were solved by the method of canonical transformations in the case of the homogeneous Maxwellian electromagnetic field.

The canonical formalism, built for the relativistic theory of pursuit, is consistent with Einstein's principle of special and general theory of relativity.

1. Notions and principles of a two-person pursuit game. If the pursuing and escaping elements carry electrical charges in a given electromagnetic field with potential A_μ, we face the following question: How should one define the basic pursuit equation in order to make it possible to study the pursuit in the presence of electromagnetic and gravitational fields? We shall attempt to answer this question in the present paper. Using the pursuit equation derived here, we shall discuss the pursuit game in a homogeneous electrical field.

Before we start the derivation of the fundamental pursuit equation in the electromagnetic and gravitational field, we remind the basic notions of the relativistic theory of pursuit games [3].

Let us denote by X the player who directs the pursuit, and by Y the player who directs the evasion. We assume that the pursuit and evasion take place in the Minkowski space-time (in general, in Riemann space-time).

Let the element of the game for player X be the moving point with coordinates $x^\mu(s_1)$, and for player Y — the moving point with coordinates $y^\mu(s_2)$. The values of the proper time for players X and Y are denoted respectively by s_1 and s_2.

One of the most important notions of the theory of pursuit games is the notion of four-velocities, as functions of proper time s_i $(i = 1, 2)$ and instantenuous positions of players, i.e. x^μ and y^μ. Thus, for the pursuing and escaping player we have

$$u^\mu = u^\mu(x^\nu, y^\nu, s_1), \qquad v^\mu = v^\mu(x^\nu, y^\nu, s_2), \qquad \mu = 1, 2, 3, 4. \tag{5}$$

The choice of the directions of the four-velocities u^μ and v^μ depends upon the decisions of players X and Y. We shall call them respectively the *methods of pursuit* and *evasion* or shortly — the *strategies of the game.*

The absolute values of these strategies (the four-velocities defined with respect to proper time) are bounded by side conditions

$$u_\mu u^\mu = -1, \qquad v_\mu v^\mu = -1, \tag{6}$$

imposed by the metric in the Minkowski space-time (or generally, the Riemann space-time). In formulas (6) and in the sequel we choose units in such a way that the velocity of light in vacuum $c = 1$.

Thus, the conditions (6) guarantee that the game will be in the interior of the light cone.

If the strategies u^μ and v^μ are given by players X and Y for the pursuing and escaping element, then the motion of the game is described by the following system of ordinary differential equations

$$\begin{aligned} \frac{dx^\mu}{ds_1} &= u^\mu(x^\nu, y^\nu, s_1), \\ \frac{dy^\mu}{ds_2} &= v^\mu(x^\nu, y^\nu, s_2), \end{aligned} \qquad (\mu = 1, 2, 3, 4) \tag{7}$$

with arbitrary initial conditions s_i^0 $(i = 1, 2)$, x_0^ν, y_0^ν, which we shall also include into the rules of the game.

The system (7) is simply the parametric form of differential equations for lines of the vector field (5). These lines, or the integrals of equations (7) give the trajectories of pursuit and evasion.

The second important notion in the relativistic theory of pursuit is the notion of pursuit time. The definition of this notion in the classical case presents no difficulty, since the rate of all clocks in all inertial systems is the same. In the relativistic case, however, different clocks, connected with different systems have different rates. For that reason we introduce the notion of the proper time of the given clock.

We see, therefore that if we define the pursuit time for the proper time of different players, the minimax principle will not be satisfied. In fact, let $\sigma_1 = s_1^1 - s_1^0$ denote the proper time of the pursuit, and let $\sigma_2 = s_2^1 - s_2^0$ denote the proper time of the evasion. We have no right to claim that the minimal pursuit time is equal to the maximal evasion time (since in this case the rates of the clocks of both players would have to be identical, which contradicts the principles of the theory of relativity). We remind that s_i^0 and s_i^1 $(i = 1, 2)$ denote respectively the initial and terminal proper time for each player.

We think that it will be most reasonable if we define the pursuit time with respect to a standard clock, which we shall call the referee's clock. This clock can move in the given coordinate system, but it will be most convenient to study the pursuit in such a coordinate system, for which the referee's clock rests. We shall denote the proper time of this clock by s.

If we denote the initial proper time (of the standard clock) by s^0, and if we treat the terminal proper time of this clock as a functional depending upon the strategies and initial conditions, i.e.

$$s^1 = s^1[u^\mu(x_0^\nu, y_0^\nu, s_1^0),\ v^\mu(x_0^\nu, y_0^\nu, s_2^0);\ x_0^\nu, y_0^\nu, s_i^0],$$

then the difference $\sigma = s^1 - s^0$ will be called the *pursuit time* in the referee's clock.

In theory of games the functional $\sigma = \sigma[u^\mu, v^\mu, x^\mu, y^\mu, s_i]$ is called the *payoff function* corresponding to the given strategies u^μ and v^μ and rules of the game x^μ, y^μ, s_i. We dropped the index zero, since the initial conditions are chosen arbitrarily in the Minkowski (or, generally, Riemann) space-time, satisfying the metric conditions

$$(8) \qquad g_{\mu\nu} dx^\mu dx^\nu = -(ds_1)^2, \qquad g_{\mu\nu} dy^\mu dy^\nu = -(ds_2)^2,$$

where $g_{\mu\nu}$ is the fundamental metric tensor.

If we express the proper time s_i $(i = 1, 2)$ of player X and Y by the referee's proper time s, i.e. if we give the functional relation

$$(9) \qquad s_i = s_i(s) \qquad (i = 1, 2),$$

then the system of differential equations (7) for the pursuit and evasion trajectories will take the form

$$(10) \qquad \begin{aligned} \frac{dx^\mu}{ds} &= uu^\mu\{x^\nu[s_1(s)], y^\nu[s_2(s)], s_1(s)\}, \\ \frac{dy^\mu}{ds} &= vv^\mu\{x^\nu[s_1(s)], y^\nu[s_2(s)], s_2(s)\}, \end{aligned}$$

where

$$\frac{ds_1}{ds} = u(s), \qquad \frac{ds_2}{ds} = v(s). \tag{11}$$

Solving the system (10) with given functions (9) or (11) and initial conditions s^0, x_0^ν, y_0^ν we obtain the pursuit trajectories (in the parametric form)

$$x^\mu = x^\mu(s), \qquad y^\mu = y^\mu(s),$$

expressed in the referee's proper time.

We see that the left-hand sides of equations (10) give the strategies of pursuit and evasion in the proper time of the standard clock. The absolute values of these strategies are equal to given functions $u(s)$ and $v(s)$. In fact, if we square equations (10) and use conditions (6) we get immediately

$$\dot{x}_\mu \dot{x}^\mu = -u^2, \qquad \dot{y}_\mu \dot{y}^\mu = -v^2, \tag{12}$$

where the dot stands for the derivative with respect to s.

We may now interpret the physical sense of functions u and v: they give the values of four-velocities $\dot{x}^\mu$ and $\dot{y}^\mu$ in terms of the referee's clock. We may add that relation (12) can be easily obtained from metric conditions (8) and notations (11).

The basic problem of the theory of pursuit games is to find the best methods of pursuit and evasion, or the optimal strategies for each of the players. Such an optimal strategy for each of the players is the minimax strategy, based on the minimax criterion which can be found in paper [7].

The best (minimax) strategies of pursuit and evasion will be denoted by:

u_μ^*, v_μ^* for the optimal covariant strategies,

u_*^μ, v_*^μ for the optimal countercovariant strategies.

The payoff function σ, corresponding to strategies (u_*^μ, v_*^μ) or strategies (u_μ^*, v_μ^*) common for all initial conditions, will be called the *optimal pursuit time*. It follows from this definition that $\sigma = \sigma(x^\mu, y^\mu, s_i)$. We should remember here that the values of the parameters (x^μ, y^μ, s_i) determine the arbitrary initial conditions for the system (7). In view of the relation $x^\mu(s_1)$ and $y^\mu(s_2)$ the optimal pursuit time is a function of s_1 and s_2.

If we now express the proper time for each player by the referee's proper time, we get from (9)

$$\sigma(s) = \sigma\{s_i(s), x^\mu[s_1(s)], y^\mu[s_2(s)]\}. \tag{13}$$

We shall give now a generalization of the fundamental equation of the theory of pursuit game.

We shall start from the equation of Zięba ([9])

$$\frac{d\tau}{dt} = -1, \tag{14}$$

which gives a necessary and sufficient condition for the considered class of pursuit games in the three-dimensional space to be uniformly closed, with the optimal methods of pursuit and evasion.

The optimal pursuit time $\tau(t) = \tau[t, x_i(t), y_i(t)]$ $(i = 1, 2, 3)$ and the usual time t is understood here in the Newton sense. These times, that is τ and t are, naturally, the invariants of Galilean transformation, well known in the classical mechanics.

We generalize the equation (14) for the relativistic case in a natural way:

$$\frac{d\sigma}{ds} = -1, \tag{15}$$

where σ denotes the optimal pursuit time, measured in the referee's clock, whose proper time we denoted by s.

In view of (13) the last equation may be written in explicit form

$$\frac{\partial\sigma}{\partial s_1}\cdot\frac{ds_1}{ds} + \frac{\partial\sigma}{\partial s_2}\cdot\frac{ds_2}{ds} + \frac{\partial\sigma}{\partial x^\mu}\cdot\frac{dx^\mu}{ds_1}\cdot\frac{ds_1}{ds} + \frac{\partial\sigma}{\partial y^\mu}\cdot\frac{dy^\mu}{ds_2}\cdot\frac{ds_2}{ds} = -1$$

or

$$u(p_0+p_\mu u_*^\mu)+v(q_0+q_\mu v_*^\mu) = -1, \tag{16}$$

where we used formulas (7) and (11) and introduced the Monge notations:

$$p_0 = \frac{\partial\sigma}{\partial s_1}, \quad q_0 = \frac{\partial\sigma}{\partial s_2}, \quad p_\mu = \frac{\partial\sigma}{\partial x^\mu}, \quad q_\mu = \frac{\partial\sigma}{\partial y^\mu} \quad (\mu = 1, 2, 3, 4). \tag{17}$$

Equation (15) plays the role of a necessary condition (in the Riemann space-time) for the relativistic class of pursuit games to be uniformly closed with optimal strategies of pursuit and evasion, denoted with stars in equation (16).

In the sequel we shall study only such pursuit games for which σ does not depend upon s_1 and s_2 in the explicit way. In this case we have $p_0 = q_0 = 0$ and equation (16) takes the form

$$up_\mu u_*^\mu + vq_\mu v_*^\mu = -1.$$

If the player X chooses the optimal method of pursuit u_*^μ and the player Y chooses an arbitrary method of evasion v^μ (or vice versa, if X chooses u^μ and Y chooses the optimal method of evasion v_*^μ), then the pursuit time σ for these strategies statisfies the conditions

$$up_\mu u_*^\mu + vq_\mu v^\mu \geqslant -1, \quad up_\mu u^\mu + vq_\mu v_*^\mu \leqslant -1, \tag{18}$$

which we shall call the equilibrium conditions for the conflicting situation. These conditions express the relativistic generalization of the corresponding conditions of Zięba in the classical theory of pursuit games ([9]).

The relations (18) state that the linear form of u^μ and v^μ

$$up_\mu u^\mu + vq_\mu v^\mu \tag{19}$$

tor $u^\mu = u_*^\mu$ assumes its maximum for $v^\mu = v_*^\mu$, and for $v^\mu = v_*^\mu$ it assumes fhe minimum for $u^\mu = u_*^\mu$. It should be emphasized here that the extremal properties of the form (19) are attained only under the side conditions (6). Without these conditions the above form has no extremum.

Thus, if we want to get the optimal (extremal) values for the considered strategies with absolute values bounded by (6) we have to apply the method of Lagrange multipliers to the form (19). The strategies u_*^μ and v_*^μ determined in this manner allow us in turn to find the trajectories of optimal pursuit and evasion $x_*^\mu(s)$ and $y_*^\mu(s)$ from the system (7) or (10) of ordinary differential equations.

Thus, the pursuit problem is basically reduced to the problem of finding the optimal methods of the game u_*^μ and v_*^μ and the payoff function σ, or the optimal pursuit time.

All our formulas (except for the classical equation of Zięba (14)) hold in the special and general theory of relativity, hence they are consistent with the Einstein's principle of general covariance.

2. The basic equation for the pursuit in the electromagnetic field. Suppose that in the Minkowski space-time we are given the electromagnetic field with the given potential A_μ. This field, for the given density j_ν of electric current satisfies, naturally, the d'Alembert equation

$$\partial_\mu \partial_\mu A_\nu = -4\pi j_\nu,$$

and Lorentz's gauge conditions

$$\partial_\nu A_\nu = 0,$$

where $\partial_\mu \equiv \partial/\partial x_\mu$.

The Lagrangean of the system of two material points with charges e_1 and e_2 and rest masses m_1 and m_2 in presence of the outside electromagnetic field A_ν has the form

$$L = \tfrac{1}{2} m_1 u_\nu u_\nu + e_1 A_\nu u_\nu + \tfrac{1}{2} m_2 v_\nu v_\nu + e_2 A_\nu v_\nu,$$

where the four-velocities u_ν and v_ν are restricted by metric conditions

$$u_\nu u_\nu = -1, \quad v_\nu v_\nu = -1. \tag{20}$$

From the definition of generalized momenta we get

$$P_\mu \equiv \frac{\partial L}{\partial u_\mu} = m_1 u_\mu + e_1 A_\mu, \quad Q_\mu \equiv \frac{\partial L}{\partial v_\mu} = m_2 v_\mu + e_2 A_\mu. \tag{21}$$

It follows that the generalized four-momentum of each point charge is the sum of mechanical and field momentum. Therefore the four-vectors P_μ and Q_μ determine the directions of motion of mass points in the given electromagnetic field A_μ.

Dividing the expressions (21) by m_1 and m_2 we get, respectively,

$$U_\mu = u_\mu + \alpha A_\mu, \quad V_\mu = v_\mu + \beta A_\mu,$$

where we denoted

$$U_\mu = \frac{1}{m_1} P_\mu, \quad V_\mu = \frac{1}{m_2} Q_\mu, \quad \alpha = \frac{e_1}{m_1}, \quad \beta = \frac{e_2}{m_2}.$$

The four-vectors U_μ and V_μ which determine the directions of motion of two mass points with proper charges α and β in the field A_μ, are restricted, in view of (20), by the conditions

$$(U_\nu - \alpha A_\nu)(U_\nu - \alpha A_\nu) = -1, \quad (V_\nu - \beta A_\nu)(V_\nu - \beta A_\nu) = -1. \tag{22}$$

Now, in order to change the physical interpretation into the game theoretical interpretation it suffices to identify the mass points with the elements of the game, and the four-vectors U_μ and V_μ with the generalized strategies.

The pursuit trajectories in the electromagnetic field can be obtained from the system of ordinary differential equations

$$\begin{aligned} \frac{dx_1}{U_1} &= \frac{dx_2}{U_2} = \frac{dx_3}{U_3} = \frac{dx_4}{U_4}, \\ \frac{dy_1}{V_1} &= \frac{dy_2}{V_2} = \frac{dy_3}{V_3} = \frac{dy_4}{V_4}, \end{aligned} \tag{23}$$

which determine the lines of vector fields $U_\mu(x_\nu, y_\nu)$ and $V_\mu(x_\nu, y_\nu)$. The system (23) can also be written in parametric form

$$\begin{aligned} \frac{dx_\mu}{ds_1} &= U_\mu[x_\nu(s_1), y_\nu(s_2)], \\ \frac{dy_\mu}{ds_2} &= V_\mu[x_\nu(s_1), y_\nu(s_2)], \end{aligned} \tag{24}$$

or, if we express it in the referee's proper time, we get

$$\text{(25)}\qquad \begin{aligned} \frac{dx_\mu}{ds} &= u(s)\,U_\mu\{x_\nu[s_1(s)],\, y_\nu[s_2(s)]\},\\ \frac{dy_\mu}{ds} &= v(s)\,V_\mu\{x_\nu[s_1(s)],\, y_\nu[s_2(s)]\}. \end{aligned}$$

If we define the pursuit time in electromagnetic field on the set of initial conditions of system (24) or system (25), then the equilibrium conditions for the conflicting situation, for the optimal pursuit time σ and optimal generalized strategies U_μ^* and V_μ^* will take the form

$$\text{(26)}\qquad \frac{d\sigma}{ds} = \begin{cases} up_\nu U_\nu^* + vq_\nu V_\nu \geqslant -1,\\ up_\nu U_\nu^* + vq_\nu V_\nu^* = -1,\\ up_\nu U_\nu + vq_\nu V_\nu^* \leqslant -1, \end{cases}$$

where U_ν and V_ν are arbitrary strategies.

The symbols p_ν and q_ν denote the Monge notations for the payoff function $\sigma = \sigma(x_\mu, y_\mu)$.

The relations (26) hold under the assumption that the function σ does not depend explicitely on s, s_1 and s_2, i.e.

$$\frac{\partial\sigma}{\partial s} = \frac{\partial\sigma}{\partial s_1} = \frac{\partial\sigma}{\partial s_2} = 0.$$

The condition $\dfrac{\partial\sigma}{\partial s} = 0$ can hold only if $u(s) = \text{const}$, $v(s) = \text{const}$, or, in view of (11), it suffices to assume that $s_1(s) = us$, $s_2(s) = vs$.

It follows from the inequality (26) that the linear form

$$\text{(27)}\qquad up_\nu U_\nu + vq_\nu V_\nu$$

treated as a form with respect to U_ν and V_ν assumes, under the side conditions (22), the extremum for the optimal strategies U_ν^* and V_ν^*.

Applying the method of indefinite Lagrange multipliers, we obtain

$$\text{(28)}\qquad \begin{cases} F(U_\nu, V_\nu; \lambda_1, \lambda_2) = up_\nu U_\nu + vq_\nu V_\nu + \lambda_1[(U_\nu - aA_\nu)(U_\nu - aA_\nu) + 1] +\\ \qquad\qquad + \lambda_2[(V_\nu - \beta A_\nu)(V_\nu - \beta A_\nu) + 1],\\ \dfrac{\partial F}{\partial U_\mu} = up_\mu + 2\lambda_1(U_\mu - aA_\mu) = 0,\\ \dfrac{\partial F}{\partial V_\mu} = vq_\mu + 2\lambda_2(V_\mu - \beta A_\mu) = 0,\\ \dfrac{\partial F}{\partial \lambda_1} = (U_\mu - aA_\mu)(U_\mu - aA_\mu) + 1 = 0,\\ \dfrac{\partial F}{\partial \lambda_2} = (V_\mu - \beta A_\mu)(V_\mu - \beta A_\mu) + 1 = 0. \end{cases}$$

From the above system of ten algebraic equations with ten unknowns $(U_\mu^*, V_\mu^*, \lambda_1, \lambda_2)$ one can eliminate the Lagrange multipliers λ_1 and λ_2 and obtain the coordinates of the four-vectors U_μ^* and V_μ^*, which determine the directions of the optimal pursuit in the electromagnetic field A_μ. The two first vector equations of the system (28) yield, after simple transformations

$$U_\mu^* - aA_\mu = -\frac{up_\mu}{2\lambda_1}, \quad V_\mu^* - \beta A_\mu = -\frac{vq_\mu}{2\lambda_2}. \tag{29}$$

From (29) and from the two last scalar equations of the system (28) we get

$$u^2 p_\nu p_\nu + 4\lambda_1^2 = 0, \quad v^2 q_\nu q_\nu + 4\lambda_2^2 = 0,$$

or

$$2\lambda_1 = \pm u\sqrt{-p_\nu p_\nu}, \quad 2\lambda_2 = \pm v\sqrt{-q_\nu q_\nu}. \tag{30}$$

In this way, using (29) and (30) we can express the optimal strategies of the game by the partial derivatives of the payoff function

$$U_\mu^* = \mp \frac{p_\mu}{\sqrt{-p_\nu p_\nu}} + aA_\mu, \quad V_\mu^* = \mp \frac{q_\mu}{\sqrt{-q_\nu q_\nu}} + \beta A_\mu. \tag{31}$$

Because of the signs in (31) we can obtain from (26) four different equations, which give the minimum of the form (27) for the best method of evasion under the optimal pursuit, and best method of pursuit under the optimal evasion. We restrict ourselves to presenting only one of them, namely the equation

$$u(\sqrt{-p_\nu p_\nu} - ap_\nu A_\nu) - v(\sqrt{-q_\nu q_\nu} + \beta q_\nu A_\nu) = 1, \tag{32}$$

where u and v are given numbers, and the required payoff function $\sigma = \sigma(x_\mu, y_\mu)$ is related with p_μ and q_μ by Monge notations (17).

We see that the problem of the two-person pursuit game in the electromagnetic field A_μ can be reduced to solving the first order partial differential equation (32) for the optimal pursuit time σ. If we know the integral surface of this equation we can, using (17) and (31), determine the optimal directions of pursuit and evasion. This in turn, in view of the system (25) of ordinary differential equations, allows us to determine the trajectories of pursuit and evasion, under the given initial conditions.

The optimal trajectories of pursuit and evasion are essentially contained in the characteristics of the equation (32), which we shall call the basic equation of pursuit in the electromagnetic field, or the "Hamilton-Jacobi" equation.

This equation is equivalent to the canonical system of Hamilton differential equations, which can most easily be solved by the method of canonical transformations. The optimal pursuit time σ plays the role of generating function for these transformations; here the Monge notations p_μ and q_μ have to be identified with the "generalized momenta" canonically conjugated with the positions x_μ and y_μ. In other words, canonical variables p_μ, q_μ, x_μ and y_μ should be treated as independent variables in the phase space.

3. Canonical form of pursuit equations. Let us return to the discussion of the Hamilton-Jacobi equation (32), i.e. the equation

(33)
$$H(p_\mu, q_\mu; x_\mu, y_\mu) \equiv u(\alpha p_\nu A_\nu - \sqrt{-p_\nu p_\nu}) + v(\beta q_\nu A_\nu + \sqrt{-q_\nu q_\nu}) + 1 = 0.$$

It is a first order partial differential equation for the unknown function $\sigma = \sigma(x_\mu, y_\mu)$ which appears in Monge notations.

According to the theory of Cauchy, integration of equation (33) in the given field $A_\mu(x_\nu)$ can be reduced to the problem of integration of an auxiliary system of ordinary differential equations, which in this case coincides with the canonical system of Hamilton:

$$\frac{dx_\mu}{ds} = \frac{\partial H}{\partial p_\mu}, \qquad \frac{dp_\mu}{ds} = -\frac{\partial H}{\partial x_\mu},$$
$$\frac{dy_\mu}{ds} = \frac{\partial H}{\partial q_\mu}, \qquad \frac{dq_\mu}{ds} = -\frac{\partial H}{\partial y_\mu}. \tag{34}$$

The Hamilton function H is given in the phase space $(p_\mu, q_\mu, x_\mu, y_\mu)$ by the left-hand side of equation (33).

It is easy to verify that the Hamilton equations are simply the canonical form of Euler-Lagrange equation for the functional

$$J[p_\mu, q_\mu; x_\mu, y_\mu] = \int (p_\nu \dot{x}_\nu + q_\nu \dot{y}_\nu - H)\, ds.$$

In fact, if we define the Lagrange function L as

$$L = p_\nu \dot{x}_\nu + q_\nu \dot{y}_\nu - H,$$

then the Euler-Lagrange equations

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{p}_\mu} - \frac{\partial L}{\partial p_\mu} = 0, \qquad \frac{d}{ds}\frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial L}{\partial x_\mu} = 0,$$
$$\frac{d}{ds}\frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} = 0, \qquad \frac{d}{ds}\frac{\partial L}{\partial \dot{y}_\mu} - \frac{\partial L}{\partial y_\mu} = 0,$$

for the variational principle

$$\delta \int L\, ds = 0, \tag{35}$$

will automatically reduce to the Hamilton equation (34).

It can be seen from (33), (34) and (35) that the basic pursuit equation can appear in three equivalent forms:

1) the form of the Hamilton-Jacobi equation,

2) the form of Hamilton equations,

3) the form of variational principle of Hamilton.

In the present paper we attempted to stress as much as possible the similarity between the foundations of the theory of pursuit game and the corresponding principles of motion in theoretical mechanics. The analogies existing between the equations of pursuit and the equations of motion allow us in many cases to apply the well-known (from the analytic mechanics) and well-developed methods of integrations.

The most commonly used method for integrating the system of equations (34) is the so-called method of Hamilton-Jacobi, based on the canonical transformations. These transformations give the exact relation between the integral surface of the equation (33) and the first integrals of the canonical system (34). The essence of this method is contained in the following theorem of Jacobi:

If $\sigma(x_\mu, y_\mu; a_\mu, c_\mu)$ is any complete integral of the Hamilton-Jacobi equation (33), *then the first integrals of the Hamilton system of equations* (34) *can be written in the form*

$$\frac{\partial\sigma}{\partial a_\mu} = b_\mu, \qquad \frac{\partial\sigma}{\partial c_\mu} = d_\mu, \tag{36}$$

$$\frac{\partial\sigma}{\partial x_\mu} = p_\mu, \qquad \frac{\partial\sigma}{\partial y_\mu} = q_\mu, \tag{37}$$

where a_μ and c_μ as well as b_μ and d_μ are arbitrary constants.

The first integrals of motion (36) completely determine the optimal trajectories of pursuit in the configurational space (x_μ, y_μ), and, together with formulas (37), determine the trajectories in the phase space $(x_\mu, y_\mu; p_\mu, q_\mu)$.

Now we shall give the solution of Hamilton equations in the homogeneous electrical field. Natuially, we shall obtain this solution with the use of the method of Hamilton-Jacobi.

4. Pursuit game in the homogeneous electric field. Suppose we are given the electric field with potential φ or strength $\vec{E}$ in the Minkowski space-time. We assume that we have two elements of the game moving in this space and carrying electric charges. The information functions will be the following

$$A_1 = A_2 = A_3 = 0, \quad A_4 = i\varphi, \quad u = \text{const}, \quad v = \text{const}, \quad u \neq v. \tag{34}$$

In this section we shall study the pursuit game in the homogeneous electrostatic field, i.e. such electric field, whose strength vector $\vec{E}$ is constant:

$$\vec{E} = \text{const.} \tag{39}$$

This strength can be expressed by the scalar electric potential φ according to the following relation, known from electrostatics

$$\vec{E} = -\nabla\varphi. \tag{40}$$

If we choose the coordinate system in such a way that one of the axes will coincide with the direction of the field $\vec{E}$, then we obtain from (39) and (40):

$$\varphi(x_1) = -Ex_1, \quad \varphi(y_1) = -Ey_1. \tag{41}$$

If we now separate the variables in the basic pursuit equation (32):

$$\sigma(x_\mu, y_\mu) = \sigma_1(x_\mu)+\sigma_2(y_\mu),$$

we get two independent equations with partial derivatives

$$\begin{aligned} u\left(\sqrt{-\frac{\partial\sigma_1}{\partial x_\mu}\cdot\frac{\partial\sigma_1}{\partial x_\mu}} -\alpha A_\mu\frac{\partial\sigma_1}{\partial x_\mu}\right) &= A, \\ v\left(\sqrt{-\frac{\partial\sigma_2}{\partial y_\mu}\cdot\frac{\partial\sigma_2}{\partial y_\mu}} +\beta A_\mu\frac{\partial\sigma_2}{\partial y_\mu}\right) &= B, \end{aligned} \tag{42}$$

where the separation constants satisfy the condition $A-B = 1$.

Let us introduce the auxiliary notations

$$m = \frac{A}{u}, \quad n = \frac{B}{v}, \quad mu-nv = 1.$$

The equations (42), after introducing (38) and (41) take the form

$$\begin{aligned} \sqrt{-\frac{\partial\sigma_1}{\partial x_\nu}\cdot\frac{\partial\sigma_1}{\partial x_\nu}} +i\alpha Ex_1\frac{\partial\sigma_1}{\partial x_4} &= m, \\ \sqrt{-\frac{\partial\sigma_2}{\partial y_\nu}\cdot\frac{\partial\sigma_2}{\partial y_\nu}} -i\beta Ey_1\frac{\partial\sigma_2}{\partial y_4} &= n. \end{aligned} \tag{43}$$

Since $x_2, x_3, x_4; y_2, y_3, y_4$ are cyclic variables, we shall try to find the integral of the equations (43) in the form

$$\begin{aligned} \sigma_1(x_\mu; a_\mu) &= a_1+\sigma_{x_1}(x_1; a_2, a_3, a_4)+a_2x_2+a_3x_3+a_4x_4, \\ \sigma_2(y_\mu; c_\mu) &= c_1+\sigma_{y_1}(y_1; c_2, c_3, c_4)+c_2y_2+c_3y_3+c_4y_4. \end{aligned}$$

The functions σ_{x_1} and σ_{y_1} satisfy, in view of (43), the equations

$$(44)\qquad \begin{aligned} &\sqrt{-\left(\frac{\partial\sigma_{x_1}}{\partial x_1}\right)^2-(a_2^2+a_3^2+a_4^2)+i\alpha Ea_4x_1} = m,\\ &\sqrt{-\left(\frac{\partial\sigma_{y_1}}{\partial y_1}\right)^2-(c_2^2+c_3^2+c_4^2)-i\beta Ec_4y_1} = n. \end{aligned}$$

If we now introduce the notations

$$(45)\qquad \begin{aligned} a_4 &= ia_0, & a_2^2+a_3^2+a_4^2 &= -a^2, & m-i\alpha Ea_4x_1 &= x,\\ c_4 &= ic_0, & c_2^2+c_3^2+c_4^2 &= -c^2, & n+i\beta Ec_4y_1 &= y, \end{aligned}$$

and integrate the equations (44), we get

$$(46)\qquad \sigma_x = \pm\frac{1}{\alpha Ea_0}\int\sqrt{a^2-x^2}\,dx,\qquad \sigma_y = \pm\frac{1}{\beta Ec_0}\int\sqrt{c^2-y^2}\,dy.$$

Therefore,

$$(47)\qquad \begin{aligned} \sigma_x &= \pm\frac{x\sqrt{a^2-x^2}+a^2\arcsin(x/a)}{2\alpha Ea_0} \qquad (a>0),\\ \sigma_y &= \pm\frac{y\sqrt{c^2-y^2}+c^2\arcsin(y/c)}{2\beta Ec_0} \qquad (c>0). \end{aligned}$$

The pursuit trajectories can be determined by the Jacobi theorem from the equations:

$$(48)\qquad \begin{aligned} \frac{\partial\sigma}{\partial a_2} &\equiv \frac{\partial\sigma_x}{\partial a_2}+x_2 = b_2, & \frac{\partial\sigma}{\partial c_2} &\equiv \frac{\partial\sigma_y}{\partial c_2}+y_2 = d_2,\\ \frac{\partial\sigma}{\partial a_3} &\equiv \frac{\partial\sigma_x}{\partial a_3}+x_3 = b_3, & \frac{\partial\sigma}{\partial c_3} &\equiv \frac{\partial\sigma_y}{\partial c_3}+y_3 = d_3,\\ \frac{\partial\sigma}{\partial a_4} &\equiv \frac{\partial\sigma_x}{\partial a_4}+x_4 = b_4, & \frac{\partial\sigma}{\partial c_4} &\equiv \frac{\partial\sigma_y}{\partial c_4}+y_4 = d_4. \end{aligned}$$

When differentiating the functions σ_x and σ_y with respect to a_i and c_i $(i = 2, 3, 4)$ one should use formulas (45) in (47). We shall proceed in a different way, namely, instead of determining the derivatives directly from (47), we determine them indirectly from (46). To do this we first differentiate the expressions (46) with respect to parameters a_i and c_i $(i = 2, 3, 4)$, and then integrate them with respect to x and y

getting

$$
(49)\qquad
\begin{aligned}
\frac{\partial\sigma_x}{\partial a_2} &= -\frac{ia_2}{\alpha E a_4}\arcsin\frac{x}{a},\\
\frac{\partial\sigma_x}{\partial a_3} &= -\frac{ia_3}{\alpha E a_4}\arcsin\frac{x}{a},\\
\frac{\partial\sigma_x}{\partial a_4} &= -\frac{i}{\alpha E a_4^2}\left[(a^2+a_4^2)\arcsin\frac{x}{a}+m\sqrt{a^2-x^2}\right],\\
\frac{\partial\sigma_y}{\partial c_2} &= -\frac{ic_2}{\beta E c_4}\arcsin\frac{y}{c},\\
\frac{\partial\sigma_y}{\partial c_3} &= -\frac{ic_3}{\beta E c_4}\arcsin\frac{y}{c},\\
\frac{\partial\sigma_y}{\partial c_4} &= -\frac{i}{\beta E c_4^2}\left[(c^2+c_4^2)\arcsin\frac{y}{c}+n\sqrt{c^2-y^2}\right].
\end{aligned}
$$

Substituting now expressions (49) into (48), we get the final form of the trajectories of the optimal pursuit

$$
\begin{aligned}
x_2 &= b_2+\frac{ia_2}{\alpha E a_4}\arcsin\frac{x}{a},\\
x_3 &= b_3+\frac{ia_3}{\alpha E a_4}\arcsin\frac{x}{a},\\
x_4 &= b_4+\frac{i}{\alpha E a_4^2}\left[(a^2+a_4^2)\arcsin\frac{x}{a}+m\sqrt{a^2-x^2}\right],\\
y_2 &= d_2+\frac{ic_2}{\beta E c_4}\arcsin\frac{y}{c},\\
y_3 &= d_3+\frac{ic_3}{\beta E c_4}\arcsin\frac{y}{c},\\
y_4 &= d_4+\frac{i}{\beta E c_4^2}\left[(c^2+c_4^2)\arcsin\frac{y}{c}+n\sqrt{c^2-y^2}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
a &= i\sqrt{a_2^2+a_3^2+a_4^2}, &\quad x &= m-i\alpha E a_4 x_1,\\
c &= i\sqrt{c_2^2+c_3^2+c_4^2}, &\quad y &= n+i\beta E c_4 y_1.
\end{aligned}
$$

The arbitrary constants a_i, b_i, c_i, d_i $(i = 2, 3, 4)$ and m, n $(mu-nv = 1)$ should be determined from initial conditions.

5. Basic pursuit equation in the electromagnetic and gravitational fields. In the third section we formulated three equivalent forms of pursuit equations in the given electromagnetic field A_μ. These equations are, naturally, invariant under the Lorentz transformation, hence they satisfy the special relativity principle of Einstein.

If we want our formalism of the theory of pursuit game to satisfy the principle of general covariance, it is sufficient simply to change the Minkowski space-time into the Riemann space-time with general gravitational field with potential $g_{\mu\nu}$.

The basic pursuit equation in the gravitational field can be obtained in a very simple manner, namely it suffices to write the equation (32) in the form

$$u(\sqrt{-p_\nu p^\nu}-\alpha p_\nu A^\nu)-v(\sqrt{-q_\nu q^\nu}+\beta q_\nu A^\nu) = 1, \tag{50}$$

true in the arbitrary coordinate system.

From the Einstein equivalence principle it follows that every equation in a non inertial system has the same form as in the corresponding gravitational field $g_{\mu\nu}$. If we use the formulas from tensor analysis

$$p^\mu = g^{\mu\nu}p_\nu, \qquad q^\mu = g^{\mu\nu}q_\nu, \qquad A^\mu = g^{\mu\nu}A_\nu,$$

we obtain from (50) the "Hamilton-Jacobi" equation for the optimal pursuit time (payoff function) in the Riemann space-time

$$u(\sqrt{-g^{\mu\nu}p_\mu p_\nu}-\alpha g^{\mu\nu}p_\mu A_\nu)-v(\sqrt{-g^{\mu\nu}q_\mu q_\nu}+\beta g^{\mu\nu}q_\mu A_\nu) = 1. \tag{51}$$

This is the basic pursuit equation in the given electromagnetic field A_μ and gravitational field $g^{\mu\nu}$.

The metric tensor $g^{\mu\nu}$ satisfies the gravitational equation of Einstein

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R = -8\pi T^{\mu\nu}, \tag{52}$$

where $R^{\mu\nu}$ is the Riemann tensor, R denotes its invariant and $T^{\mu\nu}$ is the energy-momentum tensor. We expressed the equations (52) in the natural measurement units, where the velocity of light in vacuum c and the Newton gravitational constant k are equal one.

The pursuit equation in the presence of only gravitational field simplifies to

$$u\sqrt{-g^{\mu\nu}p_\mu p_\nu}-v\sqrt{-g^{\mu\nu}q_\mu q_\nu} = 1.$$

The solution of this equation for the case of the Schwarzschild gravitational field can be found in paper [4].

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