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On Borel sets and B-measurable functions in metric spaces

In this note we give simple proofs of the theorems proved by D. Montgomery in [5]. In these proofs the paracompactness of metric spaces and the existence of a σ-locally finite base are used, instead of the operation $\mathscr{M}$ (see the proofs in [5], [2] or [3]). Such a proof of the theorem asserting that a set which is locally of an additive class a or of a multiplicative class $a > 0$ is itself of the same class (see Corollary 3 above) was given in [4] and [6], where this theorem was obtained from some general theorems on local properties.

By a space we always mean a metrizable space, ϱ is a metric for the considered space. The definitions of all the topological notions used in this note can be found in [1] or [3].

THEOREM 1. *The union of a locally finite family of sets of an (additive or multiplicative) class a is the set of the same class.*

Proof. We proceed by the induction with respect to a. The theorem is obvious for the additive class 0. We shall show that the validity of the theorem for an additive class a implies its validity for the multiplicative class a.

Let $\{A_s\}_{s\in S}$ be a locally finite family of sets of the multiplicative class a. Each point x of the considered space X has a neighborhood $U(x)$ which meets only a finite number of A_s. By paracompactness of metric spaces (see [1], p. 160), the covering $\{U(x)\}_{x\in X}$ has an open locally finite refinement $\{V_t\}_{t\in T}$. For every $t\in T$ there exists a finite set $S_t \subset S$ such that

$$V_t \cap \bigcup_{s\notin S_t} A_s = 0.$$

Hence the set

$$V_t \setminus \bigcup_{s\in S} A_s = \big(V_t \setminus \bigcup_{s\in S_t} A_s\big) \cap \big(V_t \setminus \bigcup_{s\notin S_t} A_s\big) = V_t \setminus \bigcup_{s\in S_t} A_s$$

is of the additive class a. By the hypothesis, the set

$$\bigcup_{t\in T} \big(V_t \setminus \bigcup_{s\in S} A_s\big) = X \setminus \bigcup_{s\in S} A_s$$

is of the same class. It follows that the set $\bigcup_{s\in S} A_s$ is of the multiplicative class a.

To complete the proof it is enough to show that the theorem holds for a locally finite family $\{A_s\}_{s\in S}$ of set of an additive class $a>0$, under the assumption of its validity for all the classes $<a$. Let $A_s=\bigcup_{n=1}^{\infty} A_{s,n}$, where $A_{s,n}$ is of class $<a$, and let $\gamma_1,\gamma_2,\ldots,$ be a sequence of ordinal numbers $<a$ such that for every $\gamma<a$ we have $\gamma\leqslant\gamma_m$ for some integer m (if $a=a_0+1$ one can take $\gamma_1=\gamma_2=\ldots=a_0$). For every pair (m,n) of integers let

$$S_{m,n}=\{s\in S\colon A_{s,n} \text{ is of the class } \gamma_m\}.$$

The family $\{A_{s,n}\}_{s\in S_{m,n}}$ is locally finite, hence, by the inductive assumption, the union $\bigcup_{s\in S_{m,n}} A_{s,n}$ is of the additive class $\gamma_m+1\leqslant a$. Since $S=\bigcup_{m=1}^{\infty} S_{m,n}$ for every n, the set

$$\bigcup_{s\in S} A_s=\bigcup_{s\in S}\bigcup_{n=1}^{\infty} A_{s,n}=\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{s\in S_{m,n}} A_{s,n}$$

is of the additive class a.

COROLLARY 1. *The union of a σ-locally finite family of sets of an additive class a is the set of the same class.*

Since the sets with the Baire property can be characterized as unions of a G_δ-set and a set of the first category (see [3], p. 56), the following corollary follows from Theorem 1, from the theorem on the union of sets of the first category (see [3]. p. 49), and from σ-additivity of the Baire property.

COROLLARY 2. *The union of a σ-locally finite family of sets with the Baire property has the Baire property.*

Remark 1. From the above proof of Theorem 1 it follows that this theorem and the corollaries remain valid for perfectly normal paracompact spaces.

THEOREM 2. *The set C of points at which a set A of a space X is locally of an additive class a, or of a multiplicative class $a>0$, is of the same class.*

Proof. Let $\{B_s\}_{s\in S}$ be a σ-locally finite base of X (see [1], p. 127) and let S' be a set of such $s\in S$ that $A\cap B_s$ is of the considered class; we have

$$C=\bigcup_{s\in S'} A\cap B_s.$$

The validity of the theorem for an additive class a follows from Corollary 1. In the case of a multiplicative class we have

$$C = A \cap \bigcup_{s \in S'} B_s = \bigcup_{s \in S'} B_s \setminus \bigcup_{s \in S'} (B_s \setminus A),$$

where $B_s \setminus A = B_s \setminus A \cap B_s$ is of the additive class a. Hence the theorem also follows from Corollary 1.

COROLLARY 3. *If a set A is locally of an additive class $a \geqslant 0$ or of a multiplicative class $a > 0$ at each of its points, then A is itself of the same class.*

We now prove a modification of a lemma from [3] (p. 285).

LEMMA 1. *Let $\{B_s\}_{s \in S}$ be a base of X and let $\{r_s\}_{s \in S}$ be a set of the points in X such that $r_s \in B_s$ for $s \in S$. For every continuous function $g\colon X \to Y$ and a closed set $F \subset Y$ we have*

(i) $$\big(g(x) \in F\big) \equiv \prod_n \sum_s \big[(x \in B_s)\big(\delta(B_s) < 1/n\big)\big(g(r_s) \in S_n\big)\big],$$

where $S_n = \{y \in Y\colon \varrho(y, F) < 1/n\}$.

Proof. Let $\{B_{s_i}\}_{i=1}^{\infty}$ be a base at the point x. We may suppose that $\delta(B_{s_i}) < 1/i$ for $i = 1, 2, \ldots$, and we have

$$\lim r_{s_i} = x \quad \text{and} \quad \lim g(r_{s_i}) = g(x).$$

It follows that $\varrho\big(g(r_{s_i}), g(x)\big) < 1/n$ and $\delta(B_{s_i}) < 1/n$, for a sufficiently large i. If $g(x) \in F$, then $g(r_{s_i}) \in S_n$ and the right-hand side of (i) is satisfied.

Conversely, if for every n there exists an $s_n \in S$ such that $x \in B_{s_n}$, $\delta(B_{s_n}) < 1/n$ and $g(r_{s_n}) \in S_n$, i.e. $\varrho\big(g(r_{s_n}), F\big) < 1/n$, then $\lim r_{s_n} = x$ and $\lim g(r_{s_n}) = g(x)$. By the equality $\lim \varrho\big(g(r_{s_n}), F\big) = 0$ it follows that $\varrho\big(g(x), F\big) = 0$, i.e. $g(x) \in F$.

THEOREM 3. *If the function $f\colon X \times Y \to Z$ is continuous with respect to the variable x and is of a class a with respect to the variable y, then f is of the class $a+1$.*

Proof. Let $\{B_s\}_{s \in S}$ be a σ-locally finite base of X and F an arbitrary closed subset of Z. From Lemma 1 it follows that

$$\big(f(x, y) \in F\big) \equiv \prod_n \sum_s \big[(x \in B_s)\big(\delta(B_s) < 1/n\big)\big(f(r_s, y) \in S_n\big)\big],$$

therefore we have

$$f^{-1}(F) = \bigcap_{n=1}^{\infty} \Big[\bigcup_{\substack{s \in S \\ \delta(B_s) < 1/n}} (B_s \times Y) \cap \big(X \times \{y\colon f(r_s, y) \in S_n\}\big)\Big].$$

By the hypothesis, the set $\{y\colon f(r_s, y) \in S_n\}$ is of the additive class a; it follows from Corollary 1 that the set in the brackets [ ] is of the same class. Thus the set $f^{-1}(F)$ is of the multiplicative class $a+1$.

The following theorem can be obtained in the same manner from Lemma 1 and Corollary 2.

THEOREM 4. *If the function $f\colon X\times Y \to Z$ is continuous with respect to the variable x and has the Baire property with respect to the variable y, then f has the Baire property.*

LEMMA 2. *For an arbitrary function $f\colon X \to Y$ and a base $\{B_s\}_{s\in S}$ of Y there exists such a family $\{B_s^*\}_{s\in S}$ of open subsets of Y that*

$$I = X\times Y\setminus\Big(\bigcup_{s\in S} f^{-1}(B_s^*)\times B_s\Big), \tag{ii}$$

where $I = \{(x, y)\colon y = f(x)\}$ is the graph of f.

Proof. If $(x, y)\notin I$, i.e. if $f(x) \neq y$ there exist open set $B^*(x, y)$ and $s(x, y)\in S$ such that

$$f(x)\in B^*(x, y),\ y\in B_{s(x,y)} \quad \text{and} \quad B^*(x, y) \cap B_{s(x,y)} = 0.$$

Then we have

$$(x, y)\in f^{-1}\big(B^*(x, y)\big)\times B_{s(x,y)} \subset X\times Y\setminus I,$$

and it follows that

$$\bigcup_{(x,y)\notin I} f^{-1}\big(B^*(x, y)\big)\times B_{s(x,y)} = X\times Y\setminus I.$$

Putting $B_s^* = \bigcup_{s(x,y)=s} B^*(x, y)$ we obtain (ii).

THEOREM 5. *The graph of a function $f\colon X \to Y$ of a class a is of the multiplicative class a in $X\times Y$.*

Proof. The theorem follows from Lemma 2 (where $\{B_s\}_{s\in S}$ is a σ-locally finite base of Y) and from Corollary 1.

The following theorem can be obtained in the same manner from Lemma 2 and Corollary 2.

THEOREM 6. *The graph of a function $f\colon X \to Y$ with the Baire property has the Baire property.*

Remark 2. In the case of a separable metric space the proofs of the above theorems can be carried out in a similar way without the use of Corollaries 1 and 2 (i. e. without the use of paracompactness of metric spaces); it suffices to replace the σ-locally finite bases by the countable ones.

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