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On the Lebesgue property in metric spaces

1. We are concerned in this paper with the following properties which a metric space (X, d) might enjoy:

(i) (X, d) is compact;

(ii) For every pair of disjoint, non-empty, closed sets E and F in X, there exist an x in E and a y in F for which $D(E, F) = d(x, y)$, where $D(E, F)$ is the distance between E and F;

(iii) $D(E, F) > 0$ for every pair of disjoint, non-empty, closed sets E and F in X;

(iv) For an open cover Φ of X, there exists an $\eta > 0$ such that for A a subset of X with diam $A < \eta$, then $A \subset O$ for some $O \in \Phi$ (we will call (X, d) an *L-space* (for Lebesgue) when (iv) holds);

(v) Every continuous function f from (X, d) to any space (X^*, d^*) is uniformly continuous;

(vi) (X, d) is complete.

In the sequel, frequent use will be made of

THEOREM 1. (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\leftrightarrow$ (iv) $\leftrightarrow$ (v) $\rightarrow$ (vi).

Proofs of the equivalence of (iii), (iv), and (v) may be found in both [1] and [7]. That (i) implies (ii) is well known and that (ii) implies (iii) is trivial. It is proved in [5] that (v) implies (vi).

EXAMPLE 1. Let (X, d) be any infinite discrete metric space, i.e., let $d(x, y) = 1$ if $x \neq y$. Then (X, d) has property (ii), but not property (i).

EXAMPLE 2. Let $X = \{(n, 0)\colon\ n = 1, 2, \ldots\} \cup \{(n, 1+1/n)\colon\ n = 1, 2, \ldots\}$ in the plane and let d be the usual metric. (X, d) has property (iii), but not property (ii) as is seen by taking $E = \{(n, 0)\colon n = 1, 2, \ldots\}$ and $F = \{(n, 1+1/n)\colon n = 1, 2, \ldots\}$.

The reals with the usual metric is an example of a metric space which has property (vi), but not property (v) (or (iv) or (iii)).

2. In this section we will give another characterization of metric spaces (X, d) which have property L. We begin with

LEMMA 1. *Let (X, d) be a metric space and let Y be a subspace of X. If Y has property* L *(relative to d), then Y is closed.*

Proof. This follows from the fact that L-spaces are complete (Theorem 1).

LEMMA 2. *If (X, d) is an L-space and Y a closed subset of X, then Y is an L-space (relative to d).*

Proof. Let E and F be non-empty, disjoint, closed subsets of Y. Then E and F are closed in X and hence by Theorem 1, $D(E, F) > 0$. Thus (Y, d) has property (iii) and hence is an L-space.

LEMMA 3. *Let (X, d) be a metric space and let E and F be non-empty, closed, and disjoint in X. If $X = E \cup F$, then (X, d) is an L-space iff (1) (E, d) and (F, d) are L-spaces and (2) $D(E, F) > 0$.*

Proof. The necessity follows from Lemma 2 and Theorem 1. To show the sufficiency, let H and J be non-empty, disjoint, closed subsets of X. We will consider only the case in which $E \cap H$, $E \cap J$, $F \cap H$, and $F \cap J$ are each non empty (the remaining cases that can arise may be treated similarly). Now $D(E \cap H, E \cap J)$ and $D(F \cap H, F \cap J)$ are each positive since E and F are presumed to be L-spaces (see Theorem 1). Also $D(E \cap H, F \cap J)$ and $D(E \cap J, F \cap H)$ are each greater than or equal to $D(E, F)$ which is presumed positive. But $D(H, J) = \min\{D(E \cap H, E \cap J), D(E \cap H, F \cap J), D(F \cap H, E \cap J), D(F \cap H, F \cap J)\}$ and thus $D(H, J) > 0$. Thus (X, d) is an L-space by Theorem 1.

DEFINITION 1. A subset A of a metric space (X, d) is termed *uniformly isolated* iff there exists an $\eta > 0$ such that $d(a, a') \geqslant \eta$ whenever $a \neq a'$ in A.

THEOREM 2. *An unbounded metric space (X, d) is an L-space iff there exist non empty sets A and B in X such that (1) $X = A \cup B$, (2) A is bounded and is an L-space, (3) B is uniformly isolated, and (4) $D(A, B) > 0$.*

Proof. The sufficiency follows from Lemmas 1, 2, and 3 and from the fact that a uniformly isolated set is closed and has property (iii) (and therefore is an L-space). We now prove the necessity. Pick x^* arbitrarily in X. It suffices to show for some positive integer n^* that $\mathscr{C}\bar{S}_{n^*}(x^*)$ is uniformly isolated where $\mathscr{C}$ denotes the complement operator and $\bar{S}_{n^*}(x^*)$ denotes the closed ball of radius n^* and center x^*. Suppose then that no such n^* exists. Then $\mathscr{C}\bar{S}_1(x^*)$ is not uniformly isolated and we pick $x_1 \neq y_1$ in $\mathscr{C}\bar{S}_1(x^*)$ such that $d(x_1, y_1) < 1$. Suppose now that $x_1, \ldots, x_k, y_1, \ldots, y_k$ have been chosen, so that (1) $d(x_i, y_i) < 1/i$ for $1 \leqslant i \leqslant k$, (2) $x_i \neq y_j$ for $1 \leqslant i, j \leqslant k$ and (3) $d(x^*, x_i) > i$, $d(x^*, y_i) > i$ for $1 \leqslant i \leqslant k$. Choose $n' = \max\{k+1, d(x^*, x_i), d(x^*, y_i), 1 \leqslant i \leqslant k\}$. Then $\mathscr{C}\bar{S}_{n'}(x^*)$ is not uniformly isolated and we choose $x_{k+1} \neq y_{k+1}$ in $\mathscr{C}\bar{S}_{n'}(x^*)$ such that $d(x_{k+1}, y_{k+1}) < 1/k+1$. Then (1), (2), and (3) hold when k is replaced by $k+1$. Let $E = \{x_n: n \geqslant 1\}$ and $F = \{y_n: n \geqslant 1\}$. It is clear that $D(E, F) = 0$ and that $E \cap F = \emptyset$. We shall arrive at a contradiction when we show that E (and by symmetry F) is closed. Actually, we shall show that E is free of limit points. To this end, suppose that y is a limit point of E. Then $d(x_n, y) < 1$ for an infinite number of n and thus $n < d(x_n, x^*) \leqslant d(x_n, y) + d(y, x^*) < 1 + d(y, x^*)$ for all such n.

COROLLARY 1. *An unbounded metric space (X, d) which is an L-space is disconnected.*

3. In this section we shall develop some theorems involving equivalent metrics. In general if (X, d) is an L-space and d^* is a metric on X which is equivalent to d, then (X, d^*) need not be an L-space as shown by

EXAMPLE 3. Let $X = \{1, 2, \ldots, n, \ldots\}$ and let d be the usual metric. Let $d^*(n, m) = |1/n - 1/m|$. Then (X, d) is an L-space, but (X, d^*) is not. Clearly d and d^* are equivalent.

THEOREM 3. *Let (X, d) be a metric space, $d^* = d/(1+d)$ and $d^{**} = \min\{1, d\}$. Then (X, d) is an L-space iff (X, d^*) is an L-space and iff (X, d^{**}) is an L-space.*

Proof. Note firstly that $d(x, y) \geqslant d^{**}(x, y) \geqslant d^*(x, y)$ for x, y in X and thus $D(A, B) \geqslant D^{**}(A, B) \geqslant D^*(A, B)$ for all A, B in X. Applying Theorem 1, we see that if (X, d^*) is an L-space, then (X, d^{**}) is also; if (X, d^{**}) is an L-space, then so is (X, d). Finally, suppose that (X, d) is an L-space and that E and F are two non-empty, disjoint, closed subsets of X. Then $D(E, F) = \eta > 0$ and hence $D^*(E, F) = \eta/(1+\eta) > 0$ as the reader can easily check. Thus (X, d^*) is an L-space.

THEOREM 4. *Let (X, d) be a metric space which has property* L, *but is not compact. Then there exists a metric d^* for X such that* (1) (X, d^*) *has property* L, (2) (X, d^*) *is unbounded and* (3) d^* *is equivalent to d.*

Proof. Since (X, d) is not compact, there exists a real continuous unbounded function f^* on X. Let $d^*(x, y) = d(x, y) + |f^*(x) - f^*(y)|$ for all x, y in X. Now d^* is a metric for X which is equivalent to d (see [6]) and clearly $d^* \geqslant d$. If E and F are two disjoint, non-empty, closed subsets of X, then $D^*(E, F) \geqslant D(E, F) > 0$ by Theorem 1. Thus (X, d^*) is an L-space.

COROLLARY 2. *Let (X, d) be an L-space which is not compact. Then there exists a metric d^{**} for X which is equivalent to d and such that (X, d^{**}) is not an L-space.*

Proof. By Theorem 4 there exists a metric d^* for X which is equivalent to d, is unbounded and such that (X, d^*) has property L. Then by Theorem 2 there exist two non-empty, disjoint sets A and B such that A is bounded (relative to d^*) and B is uniformly isolated (again relative to d^*). Since B is an infinite set we can choose $\{x_n\}$ an infinite sequence of distinct points in B and let $C = \{x_n\colon n \geqslant 1\}$. Clearly C is closed in X. Let $d^{**}(x_n, x_m) = |1/n - 1/m|$. Now d^{**} is equivalent to d^* on C and by a well known theorem of Bing (see [2]), d^{**} can be extended to all of X. But (X, d^{**}) does not have property L; for let $E = \{x_{2n}\colon n \geqslant 1\}$ and $F = \{x_{2n-1}\colon n \geqslant 1\}$. E and F are closed, non empty and disjoint, but $D^{**}(E, F) = 0$.

Corollary 3. *Let (X, d) be a non compact metric space which has property* L. *Then (X, d) is disconnected.*

Proof. This follows from Theorem 4 and Corollary 1. (This is Theorem 2 in [3], IV.)

Theorem 5. *Let (X, d) be a metric space and let E and F be two non-empty, disjoint subspaces each of which is an L-space (and hence closed by Lemma* 1). *Suppose further that $X = E \cup F$ and that $D(E, F) = 0$. Then* (a) *(X, d) is not an L-space but* (b) *there exists a metric d^* on X such that* (1) *d^* is equivalent to d and* (2) *(X, d^*) is an L-space.*

Proof. (a) follows from Lemma 3. To prove (b), we define

$$d^*(x, y) = \begin{cases} d(x, y) \text{ if } x \text{ and } y \text{ are in } E, \\ d(x, y) \text{ if } x \text{ and } y \text{ are in } F, \\ d(x, y) + 1 \text{ otherwise.} \end{cases}$$

We leave it to the reader to verify that d^* is a metric on X which is equivalent to d. Since E and F are still L-spaces relative to d^* and since $D^*(E, F) \geqslant 1$, it follows from Lemma 3 that (X, d^*) is an L-space.

References

[1] Masahiko Atsuji, *Uniform continuity of continuous functions of metric spaces*, Pacific Journ. Math. 8 (1958), pp. 11-16.

[2] R. H. Bing, *Extending a metric*, Duke Math. Journ. 14 (1947), pp. 511-519.

[3] K. Iseki, *On the property of Lebesgue in uniform spaces I, II, III, IV*, Proc. Japan Acad. 31 (1955), 32 (1956).

[4] Shoura Kasahara, *On the Lebesgue property in uniform spaces*, Math. Japon. 13 (1953-1955).

[5] N. Levine and W. G. Saunders, *Uniformly continuous sets in metric spaces*, Amer. Math. Monthly 27 (2) (1960).

[6] N. Levine, *A characterization of compact metric spaces*, Amer. Math. Monthly 68 (7) (1961).

[7] A. A. Monteiro et M. M. Peixoto, *Le nombre de Lebesgue et la continuité uniforme*, Portugal. Math. 10 (1951).

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