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Three mathematical papers of Adam Adamandy Kochański – annotated English translation

Abstract The paper includes English translations (together with original Latin versions) of three papers of Adam Adamandy Kochański (1631-1700), originally published in Acta Eruditorum in 1682 („Solutio theorematum"), 1685 („Observationes cyclometricae"), and 1686 („Considerationes quaedam"). The translation is as faithful as possible, often literal, and it is mainly intended to be of help to those who wish to study the original Latin text. The papers have been annotated in a number of places, giving more information about names, places and terms which may not be familiar to the contemporary reader. Misprints present in the original edition are indicated, with corrections included in footnotes. A brief introduction outlining the contect of the papers is included. „Solutio theorematum" is followed by an appendix containing all quotes from „Elements" necessary to understand proofs given by Kochański.

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Key words and phrases: rectification of a circle, magic squares, Adam Kochański.

Contents

Introduction. Adam Adamandy Kochański SJ (1631–1700) was one of the most prominent members of the intellectual elite of the 17-th century Europe. A polymath of diverse interests, he pursued problems in an impressive variety of fields, including philosophy, mathematics, astronomy, philology, design and construction of mechanical clocks and mechanical computers, and many others. He published relatively little, and today his writings remain rather unknown, even among historians of science. Fortunately, through efforts of B. Lisiak SJ, all surviving works of

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Kochański have been recently reprinted [10–12]. Moreover, all of his currently known correspondence, consisting of 163 surviving letters, has been published in 2005 in a volume edited by B. Lisiak SJ and L. Grzebień SJ [13]. At the same time, a comprehensive monograph of Kochański's appeared, also authored by B. Lisiak SJ [14], giving detailed account of Kochański's life and work, and including an extensive bibliography of the relevant literature.

The most important and original mathematical works of Kochański appeared in *Acta Eruditorum* between 1682 and 1696. Among those, three are particularly interesting, namely papers published in *Acta* in years 1682 [7], 1685 [8], and 1686 [9] (all reprinted in [10]). The author decided to translate these three papers into English, which today plays the same role in the academic world as Latin used to play in times of Kochański. One can modestly hope that such a translation makes his works accessible to a wider audience and, perhaps, sparks some renewed interest in Kochański outside of his home country.

An excellent discussion of these papers (as well as other papers of Kochański) can be found in [16], thus we will not repeat it here. We will only briefly describe their content, then leaving the texts to speaks for themselves.

The first of the three, *Solutio Theorematum Ab illustri Viro in Actis hujus Anni Mense Januario, pag. 28. propositorum* [7], is a response to a question posed by an anonymous author in an earlier issue of *Acta*.

Kochański first proves a modified version of the theorem proposed by the anonymous author, namely that *in every triangular pyramid in which three edges meeting at the apex are mutually perpendicular, the sum of squares of these edges is equal to the square of the main diagonal of the rectangular parallelepiped obtained by complementing this pyramid.* The proof is quite rigorous, relying on sections of *Elements* and Clavius' commentaries to *Elements*. The proof is followed by two corollaries. The first one is the theorem proposed by the anonymous author, *in every triangular pyramid in which three edges meeting at the apex are mutually perpendicular, the sum of squares of these edges is equal to square of the diameter of the sphere circumscribing the prism.* The second corollary is not a corollary in contemporary sense, but rather an observation of an analogy between the main diagonal of a rectangular prism and a diagonal of a rectangle.

Inspired by another question posed by the anonymous author, Kochański then proves one more geometric theorem, rather unrelated to the first one. It could be stated as follows. *In a semicircle ADC with diameter AC let two lines be drawn, AD and DC. From D, let a line be drawn perpendicular to AC and interesting it at E. Similarly, from E, let a line be drawn perpendicular to AD and intersecting it at at F. Then $AF : AE = AD : AC$.* The proof is again heavily based on *Elements*.

In the last part of the paper Kochański discusses a possible generalization of the above theorem to three dimensions, and then presents an idea of two devices made of connected rods for drawing proportional line segments inside a circle. In order to help the reader to understand the proofs contained in the paper, appropriate fragments of *Elements* and Clavius's commentaries are quoted in the Appendix following the translation, and are given in both Latin and English.

The second paper, *Observationes cyclometricae ad facilitandam praxin accomodatae* [8], is the most famous of Kochański's works. In its first part, he presents an approximate ruler-and-compass construction for the rectification of the circle, which became known as *Kochański's construction*. This simple and elegant construction, followed by calculations illustrating the quality of the approximation, earned Kochański's some fame and permanent recognition in many works devoted to history of mathematics (cf. [16] and references therein).

What is generally less known is the second part of this paper which included an interesting sequence of rational approximations of π, somewhat similar to continuous fractions. Detailed discussion of this sequence and its properties can be found in a recent paper of the author [4, 5], thus we will not dwell on it here. We will only add that the sequence of integers on which the aforementioned approximation of π is based is now included in the Online Encyclopedia of Integer Sequences as A191642 [15]. The author proposed to call it *Kochański sequence*.

The final paper of the trio is *Considerationes quaedam circa Quadrata & Cubos Magicos* [9]. In the first part of this work Kochański presents some previously unknown magic squares using the method of construction of magic squares developed by A. Kircher. The second part is more original and interesting. Kochański introduces there a new type of magic squares, to be called *quadrata subtractionis* or *squares of subtraction*. An $n \times n$ square of subtraction is an arrangement of consecutive integers from 1 to n^2 in such a way that in every row, column and diagonal, if the entries are sorted in increasing order, the difference between the sum of even entries and the sum of odd entries is constant. Several examples of squares of subtraction are given in the paper, but the method of their construction is not revealed. Nevertheless, the method which Kochański used is most likely somewhat similar to Kircher's method for building magic squares, as discussed in [16]. At the end of the paper, examples of both magic squares and squares of subtraction with the sum (resp. difference) equal to the year of publication (1686) are given. One of them, however, is not quite correct, meaning that not all sums are the same. A possible origin of this mistake (different than proposed in [16]) is presented in author's note following the paper.

The above three papers are clearly of different quality and importance. The first one presents very simple, almost trivial by today's standards, geometric theorems. It is of mainly historical interest, as one can get from it a glimpse of how proofs were written in the 17-th century, how Euclid's *opus magnum* was used and cited, and what terminology and notation was common among European mathematicians of the time. The second and most widely known paper of Kochański had some lasting impact, and although the proof of transcendence of π put it mostly into oblivion, its second part still has some potential to inspire further research, as demonstrated in [4]. Yet the third paper holds perhaps the greatest potential to put Kochański's name back into a wider circulation. His *quadrata subtractionis*, although very interesting, have been totally forgotten, and nobody has studied them in the last three centuries. It is not unlikely that one day they are rediscovered and Kochański's idea finds followers in what we call "recreational mathematics", or even elsewhere. If the translation presented here helps make it happen, the author will be very pleased indeed.

As a concluding remark, let us add that the following parallel Latin and English versions of Kochański's three papers have been annotated in a number of places, giving more information about names, places and terms which may not be familiar to the contemporary reader. In the process of translation, some obvious misprints have been discovered in the original Latin text. In faithfulness to the original, they have not been corrected – instead, footnotes indicating correct version (by translator's judgment) have been inserted in appropriate places. Figures in *Solutio theorematum* are reproduced exactly as in the original, while the illustration of squaring of the circle in *Observationes cyclometricae* has been faithfully redrawn.

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Solutio Theorematum by Adam Adamandy Kochański – Latin text[1] with annotated English translation

SOLUTIO THEOREMATUM Ab illustri Viro in Actis hujus Anni Mense Januario, pag. 28. propositorum, data ab Adamo Adamando Kochanski S.J. quondam Pragensi Mathematico.

DUPLICATIONEM Trigoni Isogoni, citra Proportiones demonstrandam P. Sigismundus Hartman[2] e Soc. Jesu, publico Programmate proposuerat, istudque Schediasma mihi pro veteri necessitudine transmiserat e Bohemia in Poloniam. Reposui humanissimo Authori solutiones fere vicenas, ab eo sic probatas, ut una cum aliis, aliunde ad se missis, in lucem daturus fuisset, si Parcae Viro tanto pepercissent.

Cum vero his primum diebus in manus meas venerint Acta Eruditorum Lipsiensia, & in his Solutio problematis istius Hartmanniani, a quodam illustri viro data, & ad P. Coppilium[3], defuncti in Mathematici munere successorem, quodammodo directa, quasi is editioni posthumae operum Hartmanni, maximeque *Protei Geometrici*, ab eo nuper promissi incumberet; cum tamen ab obitu Authoris elaboratum nihil, sed prima solum Operis lineamenta reperta fuisse mihi ab Amicis nunciatum fuerit; Eam ob rem non aegre laturum spero P.

SOLUTION OF THE THEOREM proposed by an illustrous man in this year's January issue of Acta on p. 28, given by Adam Adamandy Kochanski SJ, once mathematician of Prague.

Fr. Sigismundus Hartman from Soc. of Jesus proposed in a public program the problem of DUPLICATION of an equilateral triangle without the use of proportions, and conveyed this question to me from Bohemia to Poland, on the account of old friendship. I returned to the kindest author perhaps twenty solutions, by him thus examined, so that one of these, together with others, send to him from elsewhere, was to be put to light, if only Fate had spared the man so great.

When, however, those days Acta Eruditorum of Leipzig came into my hands, and in them solution of this problem of Hartman, given by a certain illustrous man, directed to Fr. Copillus, successor of the deceased in the mathematical office, and arranged in a certain way, as if he was leaning towards posthumous edition of Hartman's works, especially *Protei Geometrici*, recently promised by him; when still from the death of the Author nothing has been worked out, but friends announced to me that only the preliminary outline of the Works has

[1] Originally published in [7].

[2] Sigismundus Ferdinandus Hartmann SJ (1632–1681) – Bohemian Jesuit and mathematician, professsor of the University of Prague.

[3] Matthaeus Coppilius SJ (1642–1682) – Bohemian Jesuit and mathematician, author of books on mechanics.

Coppilium, si dum illum alio in opere fructuose versari intelligo, hic ejus partes occupare ausus fuero: non enim it temere, aut praesidenter egisse videbor, sed veluti jure quodam antiquitatis; quod videlicet ante illum in eodem Matheseos Pragensis pulvere quendam Professor fuerim versatus; adeoque prior tempore, licet non eruditione.

Dilatis autem in aliud tempus meis illis Duplicationum Particularium, & Universalium Solutionibus, una cum Pythagoricae nova, ac multiplici Demonstratione, ceterisque meis considerationibus Geometricis, usque dum ultroneam Typographi, vel Bibliopolae cujuspiam humanitatem invenerint; suffecerit hoc loco strictim ea persequi, quae pertinent ad geminum illud Theorema Geometricum, quod Anonymus ille Problematis Hartmanniani δείχτης loco sup. memorato proposuit.

been produced; therefore I hope that Fr. Copillus is not offended if, while he, as I understand, is busy with other fruitful works, I dared to assume his role: this will not, indeed, seem to be acting rashly, or daringly, but by a certain old right; because evidently I used to dwell in the same dust of Prague as Professor of Mathematics[4]; indeed, preceding [Fr. Hartman] in time, but not in erudition.

Leaving for another time my particular and general solutions of the Duplication, one with a novel Pythagorean proof, others based on my Geometrical considerations, until they find a willing printer or a kind publisher; it will suffice in this place to pursue ony what pertains to these two Geometrical Theorems, which the Anonymus exhibitor put forward in the aforementioned place.

ARTICULUS I.

Circa primum illorum Theorematum consideranda veniunt sequentia. Inprimis dissentire me in eo ab *Illustri Viro*, quod is existimet Theorema Pythagoreum continuari in Sphaera, Pyramidem Rectangulam circumscribente; cum nec Pythagoras suum illud Orthogonium, tanquam circulo inscriptum consideraverit, nec, si universaliter agamus de potentiis cujusvis Trianguli, haec consideratio proprie ad Circulum pertinere videatur, sed potius ad Parallelogrammum universim ; quando videlicet istud Diametro sua sectum concipitur in duo Triangula aequalia, eaque Orthogonia, Amblygonia, vel Oxygonia, pro diversitate parallelogrammi; Harum enim Diametrorum Potentiae cum suis lateribus comparantur 47. *Primi*, nec non 13, &

ARTICLE I.

The following considerations revolve around the first of these theorems. First of all I disagree with the *Illustrious Man* in his opinion that the Pythagorean Theorem extends to a sphere circumscribed around a right-angled pyramid; when neither Pythagoras thought of his right triangle as if it was inscribed in a circle, nor, if we talk in general about properties of an arbitrary triangle, does such considerations seem to pertain specifically to a circle, but rather to parallelogram in general; when clearly this [parallelogram] is understood to be cut by its diameter [i.e., diagonal] into two equal triangles, either right-angled or obtuse-angled or acute-angled, according to diversity of parallelograms; Powers [squares] of these diameters [diagonals] are compared with powers of

[4]Kochański stayed in Prague from 1670 to 1672, lecturing in mathematics and (probably) moral philosophy.

13. *Secundi Elem.* Universalius autem hoc ipsum consideratur 31 *Sexti* saltem quoad orthogonium Triangulum; nam quoad reliqua, videri possunt ea, quae demonstrat Clavius e Pappo, *in Scholio ad 47. Primi, ad finem.*

Quamobrem non ineleganter fluente Analogia, nobis dicendum videtur, Pyramides Triangulares Orthogonias, Amblygonias, & Oxygonias, cum suis laterum, ac diametrorum Potentiis, immediate quidem reduci oportere ad Prismata sua, bases triangulares habentis, quorum Pyramides illae sunt partes tertiae *per Prop. 7. Duodecimi Elem.* Haec ipsa vero Prismata, tanquam partes revocantur ad totum Parallelepipedum, cujus sunt medietates. Istas porro relationes partium ad sua Tota intelligi volumus de ordine *Doctrinae* potius, quam *Naturae*, constat enim, Triangulum esse prius, ac simplicius Parallelogrammo, non minus ac Pyramidem Tetrahedram Prismate, vel Parallelepipedo: Hinc δογματιχῶς sperando, & ipsae Linearum Potentiae, non Triangulis aequilateris, sed Quadratis, omnium consensu taxantur, licet illa sint istis priora, magisque simplicia.

Quamvis autem Pyramidum Polyhedrarum aliae inscribi possint Sphaeris, aliae vero Sphaeroidibus, ac tum Potentiae laterum conferri cum Diametro corporis circumscribentis: eadem tamen Pyramides adhuc secari poterunt in Tetrahedras, atque ita Prismatibus suis, ac tum Parallelepipedis veluti postliminio quodam restitui. Si quis nihilominus omnia Trilatera a Quadrilateris, omnia Tetrahedra a Pentahedris & Hexahedris emancipare contenderit cum eo nequaquam cruento Marte dimicabimus.

their [parallelograms] sides in prop. 47 of the *first book of Elements* as well as in and prop. 13 and prop. 13[5] of the *second book.* This is considered more generally in prop. 31 of the *sixth book*, as long as the right triangle is considered. On the other hand, cases which remains can be seen as those demonstrated by Clavius following Pappus *at the end of commentary to prop. 47.*[6]

It seems we need to state an elegantly flowing analogy, that it is right to put triangular prisms, right-angled as well as obtuse-angled and acute-angled, with powers [squares] of their sides and diameters, into their respective prisms, having triangular bases, of which these prisms constitute third parts [by volume], by *prop. 7 of the twelfth book.* Such prism are in truth recalled just as if they were parts of a complete parallelepiped, of which they are halves. Hereafter we want these relations of parts to their wholes to be understood from *Principles* rather than from *Nature*, as it is evident that the triangle is prior to and simpler than a parallelogram, not less as the tetrahedral pyramid is prior to and simpler than a prism, or parallelepiped: from there observing the thing dogmatically, powers of own lines taken together are valued not with equilateral triangles, but squares, although those [triangles] are prior to these [squares], and much simpler.

Although some polyhedral pyramids could be inscribed in spheres, and others in spheroids [i.e., ellipsoids of revolution], and then powers of their sides could be matched against diameters of circumscribed bodies: besides, the same pyramids could be divided into tetrahedrons, and therefore also

[5]Obviously, this should be "prop. 12. and prop. 13".

[6]Pappus builds parallelograms on sides of an arbitrary triangle, cf. p. 366 of [?].

brought into their own prisms, and then parallelepipeds, as if by the right to return home. None the less, if somebody wanted to alienate all trilaterals from quadrilaterals, all tetrahedra from pentahedra and hexahedra, we will by no means spill blood in a fight with him.

PROPOSITIO I. Theorema.

In omni Pyramide rectangula, tria Quadrata laterum, Angulum rectum in vertice comprehendentium, aequalia sunt Quadrato Diametri totius Parallelepipedi aeque alti, Pyramidem illam complectentis.

Sit Pyramis ABC rectangula ad verticem D. sive jam sit *Aequilatera*, prout in Cubo *Figura* I. sive *Isosceles*, ut est in Fig. II Parallelepipedi, supra basin quadratam ADCF assurgentis; sive demum *Scalena*, qualem exhibet in *Fig.* III. solidum rectangulum, supra basin ADCF altera parte longiorem, erectum.

Dico, tria Quadrata DA. DB, DC aequari Quadrato Diametri AE, per oppositos solidi angulos incedentis. Nam primam in Triangulo ADB angulus D rectus est, ex hypoth. Igitur Qu. lateris AB aequatur (*47. I. Elem.*) Quadratis AD. DB duorum laterum datae Pyramidis. Deinde Triangulum pariter ABE rectangulum est ad B. (id ostendi potest *per 4. Undecimi*) Quocirca Quadratum AE aequabitur Quadrato AB, hoc est duobus DA. DB, & insuper Quadrato BE, hoc est ipsi aequali DC, quod est tertium latus datae Pyramidis ABCD.

Tria igitur omnis Pyramidis rectangulae latera, potentia aequantur Diametro Parallelepipedi Pyramidem continentis, q. e. d.

PROPOSITION I. Theorem.

In every right-angled pyramid, sum of squares of three sides coming from the vertex embracing the right angle is equal to the square of the diameter of the complete parallelepiped of equal height, encompassing this pyramid.

Let the pyramid ABC be right-angled at the vertex D, whether *equilateral*, as in the cube of Fig. I, or *isosceles*, as in the parallelepiped of Fig. II, rising over the square basis ADCF, or finally *sceles*[7], as in the rectangular solid erected over the basin ADCF elongated in the other direction, as displayed in Fig. III.

I say that three squares of DA, DB, and DC are equal in sum to the square of the diameter AE, stretched between opposite angles of the solid. For, first of all, in the triangle ADB, the angle D is right, by hypothesis. Therefore the square of AB is equal (by *prop. 47 of Elem. I*) to the sum of squares of two sides AD and DB of the given pyramid. Next also triangle ABE is right-angled at B (this can be shown by *prop. 4 of the eleventh book*), on account of which square of AE is equal to the square of AB, that is sum of squares of DA and DB, plus square of BE, which itself is equal to DC, the third side of the given pyramid ABCD.[8]

Therefore three sides of any rectangular pyramid are equal in power[9] to the

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[7] Having three equal sides.

[8] $AB^2 = AD^2 + DB^2$, $AE^2 = AB^2 + BE^2 = AD^2 + DB^2 + BE^2 = AD^2 + DB^2 + DC^2$.

[9] i.e., the sum of their squares is equal to the square of...

diameter of the parallelepiped enclosing the pyramid, Q.E.D.

Corollarium I.

Colligitur hinc, in omni Prismate rectangulo ACDBGA, cui basis est Orthogonium Triangulum ADC cujus Parallelogrammi rectanguli AGEC, quod angulos D. & B. rectos subtendit, Diametrum AE. Potentia aequari iisdem tribus lateribus Pyramidis rectangulae ABCD: eo quod ipsa DE sit eadem omnino cum Diametro totius Solidi GC.

Corollary I.

One obtains from there that in all rectangular prisms ACDBGA, whose base is a right triangle ADC, square of the diameter AE of the right-angled parallelogram AGEC, which extends below right angles D and B, is equal to sum of squares of three sides of the right-angled pyramid ABCD: consequently DE[10] itself would be entirely the same as the diameter GC of the whole solid.

Corollarium II.

Hinc quoque manifestum est, Diametrum Sphaerae, quae Pyramidi rectanguli ABCD circumscripta est, aequari potentia tribus lateribus ejusdem Pyramidis, rectos angulos constituentibus in vertice D. Nam cum omni solido rectangulo Sphaera circumscribi possit, non minus ac Circulus ejus basi rectangulae, erit Diameter solidi eadem omnino quae circumscriptae Sphaerae, per ea, quae Pappus demonstrat *Lemmate 4. apud Clavium ad calcem Lib. 16. Elem.*

Si cui placuerit in simili materia ingenium exercere circa Pyramides Triangulares, tam Acutangulas, quam Obtusangulas, itemque mixtis in vertice angulis contentas in illis Potentias laterum cum Diametro totius Solidi obliquanguli comparando; id vero non difficulter poterit expedire ope duarum Propositionum, videlicet penultimae, & antepenultimae *Lib. Secundi Elem.*

Corollary II.

From this it is evident that the square of the diameter of the sphere circumscribed around a right-angled pyramid ABCD, is equal to the sum of squares of three sides of this pyramid forming the right angle at the vertex D. For while every rectangular solid can be circumscribed by a sphere, as much as its rectangular base [can be circumscribed] by a circle, the diameter of this solid will be entirely the same as the diameter of the circumscribing sphere, as Pappus demonstrated in *Lemma 4 in Clavius' commentary at the end of book 16*[11] *of Elements.*

If somebody would please to exercise his talent in similar matters regarding triangular pyramids, either acute-angled or obtuse-angled, and likewise stretching over mixed angles, comparing sums of squares of the sides with the diameter of the whole solid; this will certainly not be difficult to obtain by the power of two propositions, namely the second last and the third from the end proposition of the *second book of Elements.*

[10] Clearly a misprint. Should be *AE*.

[11] Book XVI was a medieval addenum to *Elements.*

ARTICULUS II.

Circa alterum Theorema percontatur *Illustris Vir* AN sicut in Circulo unica Media Proportionalis, ita etiam insistendo Analogiae, in Sphaera duae Mediae inveniri possint ? Ad hanc quaestionem Respondeo I. Nec Antecedens Analogiae hujus tam ratum esse, ut absolute loquendo, Circulus duas Medias excludere pronunciari possit, aut debeat: Fieri namque potest, ut in Semicirculo ACD (*inspice Fig. 4.*) continuae sint AB. BC. BD. DA. cujus Problematis Geometricam constructionem Mathematum peritis propono, interim vero Arithmeticum sequentibus numeris expono. Ponatur enim

ARTICLE II.

The Anonymous *Illustrious Man* inquires about another theorem, as if there exist one geometric mean in a circle, could two means me found in a sphere? I make first response to this question, that the first part of this analogy is not so strongly established that, absolutely speaking, it would exclude a possibility that a circle having two means. For in fact, it can happen that in a semicircle ACD (*see Fig. 4*) there are successive lines AB, BC, BD, DA. I leave Geometric Construction of this problem to mathematical experts, and in the meanwhile I explain Arithmetic one with the following numbers. It is namely assumed that diameter will be

Diameter	AD.	- -	2 00000	00000	
erit	AB.	- -	63534	43923.	+–
	BC.	- -	93114	24637.	+–
	BD.	- -	1 36465	56077.	+–

Unde *per 19. Septimi Elem.* erunt aequalia Rectangula

From there by *Prop. 19 of the seventh book of Elements* [the following] rectangles will be equal

DAB.	- -	1 17068	87846	00000	00000.
CBD.	- -	1 17068	87846	55798	69049.

Gg.

Et *per 20. ejusdem*, Quadrata mediarum aequabuntur Rectanguli sub earundem extremis.

And *by Prop. 20*[12] *of the same*, squares of means will be equal to [areas of] rectangles under their outer segments.[13]

[12]Prop. 20 is often omitted in modern editions of *Elements*, as it is considered a later addition, and a direct consequence of proposition 19.

[13]"Outer" means neighbours in the sequence. Kochanski considers sequence of linear segments AB, BC, BD, DA, where both BC and BD are geometric means of their nearest neighbours in the sequence.

$\square$BC	- - -	86702	62877	05305	81769.
ABD	- - -	86702	62877	72943	70071.
$\square$BD	- - -	1 86288	49276	27056	29929.
ADBC	- - -	1 86228	49274	00000	00000.

Respondeo II. De Analoga illa nihil certi statui posse videtur: Unius enim Mediae inventio, quae Circulo tribuitur, etiam Sphaerae congruit; & inventio duarum Mediarum, hic a nobis demonstranda, aeque ad circulum, sicut & Sphaeram aptari poterit, quemadmodum ex dicendis constabit.

Secondly, I respond than nothing certain seems to possible to state about this analogy. Invention of one mean, assigned to a circle, suits to sphere too. Invention of two means, demonstrated here by us, can be adapted as well to a circle as to a sphere, in what way it will agree with what has been said.

PROPOSITIO II.
Theorema.

In Circulo ADC a Diametro AC descripto ducantur utcunque ad peripheriam duae rectae AD DC: tum ex D cadat ad AC perpendicularis D; similiterque ex E sit ad AD normalis EF.

Dico in Circulo ADC, haberi Quatuor continue Proportionales, AF. AE. AD. AC. Describatur enim Diametro AD Circulus AGDE, quem EF producta secet in G, & connectantur G A. GD.

Demonstr. Recta AD subtendit Angulum rectum DEA. Igitur Circulus AGDE Diametro AD descripsit transit per verticem Anguli recti DEA. *juxta Schol. Clavii ad 31. 3 Elem.* Est autem recta EFG perpendicularis ad Diametrum AD. Ergo *per 3. 3. Elem.* tota EG bifariam secatur in F. Triangula igitur AFG, AFE orthogonia in F, sunt *per 4.1. Elem.* invicem aequalia: Eademque de causa aequantur Triangula DFG. DFE, ac proinde & totum DGA toti DEA aequale. Jam sic. In Orthogonio AED (par ratio de aequali AGD) ab angulo recto E cadit perpendicularis EF in basin AD: Ergo *per Coroll. 8.6. Elem.* Proportionales

PROPOSITION II.
Theorem.

In a cirlce ADC traced out from a diameter AC two straight lines are drawn as far as to the perimeter: then from D a line D[14] perpendicular to AC line is led, and similarly from E a line EF normal to AD.

I say that in the circle ACD four consecutive proportionals exist, AF, AE, AD, AC. Indeed, let a circle AGDE with diameter AD be drawn, which intersects with extension of the line EF at G, and let G connecting lines GA and GD be made.

Proof. Line AD extends beneath the right angle DEA. Therefore the circle AGDE determined by the diameter AD passes through the vertex of the right angle DEA, *according to commentary of Clavius to prop. 31 of book 3 of Elements.* The straight line EFG is in fact perpendicular to the diameter AD. Therefore by *prop. 3, book 3 of Elements*, the entire EG is divided into two equal parts at F. The triangles AFG, AFE having straight angle at F, are by *Prop. 4.1 Elem.* equal to each other. For the same reason triangles

[14] A clear misprint: this should be DE.

sunt tres AF.AE.AD. Sed eadem de causa in Orthogonio ADC duabus postremis e praecedenti serie, videlicet AE. AD. proportionalis est tertia AC. Igitur omnes quatuor AF.AE.AD.AC sunt in continua Proportione intra Circulum ADC, q.e.d.

Corollarium.

Non difficulter hinc elicitur, easdem quatuor continuas in Spaerae quoque concipi posse, non modo praedicta, sed & alia ratione: Fingamus enim Sphaera ADC auferri Segmentum AHDA, cujus basis erit Circulus a Diametro AD, cujus meditas esto AGDA. Ductis autem Orthogonalibus DE. EF. in plano Circuli Spherae maximi ADC, nec non Orthogonali FG, in altero plano Semicirculi AGD, hoc est basi Segmati AHDA, jungatur AG: erunt enim ut antea, quatuor AF. AF, AD.AC. continue proportionales, id patet e praecedenti discursu, qui non difficulter huc applicari poterit, licet plana Circulorum ADG. AGD. sint diversa, & ad rectos invicem collocate.

Notandum vero est, Theorema praecedens, loquendo pressius, non tam Semicirculo, quam Orthogonio cuilibet in similia subdiviso convenire: quia tamen ejus Demonstratio sequentibus inserviet Problematibus, visum est illud hoc loco tantisper indulgere Circulo, sine quo illa absolvi non possunt.

DFG, DFE are equal, and hence the whole [triangle] DGA is equal to the whole [triangle] DEA. Now in the right-angled triangle AED (similarly reasoning applies to AGD) from the right angle E a perpendicular line EF falls onto the base AD: therefore, by *Coroll. 8. 6. Elem.*, there are three proportionals AF, AE, AD. But for the same reason, in the right-angled triangle ADC, AC is proportional to he two last lines from the aforementioned sequence [of proportionals], namely AE and AD. Therefore all four AF, AE, AD, AC are proportional in succession within the circle ADC, Q.E.D.

Corollary.

Form there it is not difficult to elicit that the same four successive proportionals can be devised in a sphere, not by the preceding method, by by a different reasoning: let us namely imagine that a segment[15] AHDA is taken from a sphere ADC, whose basis is a circle with diameter AD, and whose one half is AGDA. Drawing perpendicular lines DE, EF in the plane of the great circle ADC, and perpendicular line GF, in another planar semicircle AGD, that is, in the base of the segment[15] AHDA, let they be joined by AG: there will be, as before, four successive proportionals AF, AG, AD, AC. It stands clear from the previous discourse, which could be applied here without difficulty, that it is permitted that planes of circles ADG and AGD are different, and placed at right angles to each other.

One must observe, however, that the previous theorem, speaking more precisely, is not as tied to a semicircle as to an arbitrary triangle subdivided into similar ones: yet because its proof lends

[15]spherical cup

itself to the following problems, certain leniency toward the circle appeared in this place, without which they could to be brought out.

PROPOSITIO III.
Problema.

Inter duas datas, duas medias in continua ratione, duobus tantum digitis reperire.

Celeberrimum illud Problema Deliacum quot & quanta totius Orbis eruditi exercuerit ingenia, Geometris est notissimum, ut & variae illius absolvendi Praxes Organicae, a compluribus excogitatae, quarum aliae aliis sunt operosiores: Nostra haec videri poterit nonnihil Paradoxa, quod duobus tantum digitis unius manus, absolvatur, cum nonnullae requirant, & occupent utramque.

Datae sint, *in Figura VI.* duae AC. AB. quas inter duae mediae quaeruntur. In communi utriusque termino A figatur Regula AZ, instructa Cursore FY, qui semper insistat ad rectos ipsi regulae AZ, idque firmiter, ubicunque collocetur. In illa fumatur AF, qualis datae AB, minori altera AC. Descripto autem super tota AC Semicirculo ADC in plano quopiam verticali, hoc est ad Horizontem recto, cui aequidistet Diameter AC: applicetur ad Peripheriam ADC Stylus quidam gracilis DS, e quo deorsum propendeat filum subtile cum appenso Pondere X, vel certe hujus loco regula quaedam sub gravis, accurate tamen aequilibrata: Nam si Stylus DS duobus digitis apprehensus pedetentim promoveatur per Circumferentiam AD, usque dum Perpendiculum DE cum Cursore FE sese mutuo intersecent alicubi in recta AC, velut in Puncto E: istud probe notatum offeret quatuor Proportionales, quarum duae AE. AD. inter datas extremas AF hoc est AB

PROPOSITION III.
Problem.

Between two given [quantities], find two means in successive proportions, with only two fingers.

It is very well known to Geometers how many and how great learned [men] exercised [their] talents on this most famous Delian problem, and how various practical methods of its solution utilizing mechanical instruments, devised by many, have advantage one over another. With our method one could these as paradoxes, because it utilizes only two fingers of one hand, while some other [methods] require and occupy both [hands].

Given are, as in *Figure VI*, two [quantities] AC and AB, between which two means are sought. At their common end A a ruler AZ is fixed,equipped with cursor FY, which always firmly stands at the right angle to the ruler AZ, wherever placed. With the ruler the distance AF is taken, equal to the given AB, smaller that AC. Somewhere in a vertical plane, that is, perpendicular to the horizontal line, let a semicircle ADC be drawn over the entire AC, whose diameter is equal to AC. Let at the circumference of ADC a thin stylus DS be placed, from which a fine string is hanging down with a weight X attached, and at which finally the ruler is placed in equilibrium under gravity. For if the stylus DS is carefully held with two fingers and moved through the circumference AD, all the way until the perpendicular DE intersects with he cursor FE somewhere on the straight line AC, nec non AC interponentur.

Demonstratio Problematis hujus, quoad rem, eadem est; cum adducto praecedenti Theoremate.

for instance at point E, this, as it rightly to be noted, produces four proportionals, of which two, AE and AD, are interposed between two given outward [lines] AF (that is AB) and AC.

Demonstration of [correctness of solution of] this problem follows from the preceding proved theorem.

PROPOSITIO IV.
Problema.
Id ipsum aliter, una Circini apertura.

Quoniam praecedens praxis ob situm plani verticalem, & usum Perpendiculi, nonnihil impedita cuipiam videri possit, quin & a Geometriae moribus aliena, dabimus alteram, praedictis incommodis haud obnoxiam.

Positis iisdem, in locum perpendiculi DEX Figurae praecedentis, subrogetur *in hac Figura VII.* Parallelogrammum materiale KLMN, cujus unum Latus KL. fixum sit in plano, alterum vero MN mobile, semper tamen ad rectos ipsi AC. Nam si tota AC divisa bifariam in O Circini pes unus figatur in O, alter autem intervallo OC diductus, tamdiu in Arcu CD provehatur, impellatque binas Regulas AZ. & MN in puncto intersectionis D, usque dum regula MN Cursorem FY secet in puncto E, posito in recta AC, obtinebuntur eaedem mediae AE. AD. longe commodiori ratione, quam fuerit praecedens; quae tamen ipsa, si Figura magnae molis fuerit, in vasto quopiam pariete usui esse poterit Architectis.

Non est cur hoc loco moneam de circino, ejusque Cruribus in regula quapiam mobilibus, nec de acie pedis alterius, quae regulas in puncto D subtiliter impellere debet; nec denique de nisu quodam regularum AZ, MN contra Circinum ; hunc enim vel ipsarum regularum pondere, vel Elatare quopiam, aut

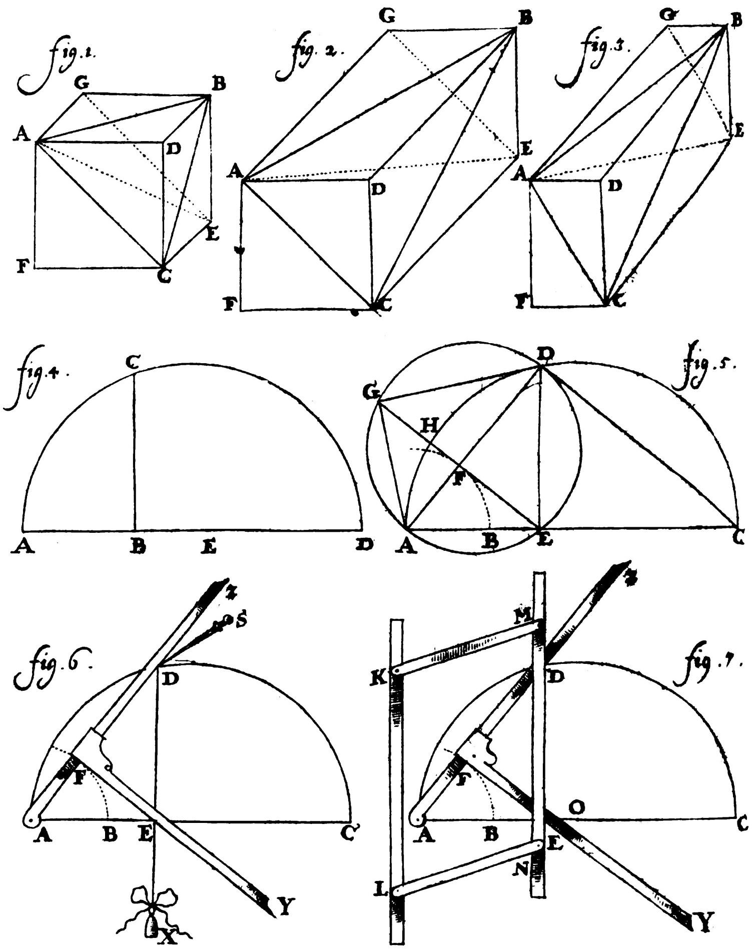
PROPOSITION IV.
Problem.
The same thing differently, with one aperture of the compass.

Because the previous method seems to be hindered by the positioning of the vertical plane and by the use of the plumbline, which is alien to customs of geometers, we will give another [method], not burdened with the aforementioned inconveniences.

Setting up things as before, in place of the perpendicular DEX of the preceding figure, let in *Figure VII* a material parallelogram KLMN be substituted, whose one side KL is fixed in the plane, and another [side] MN is mobile, still always staying perpendicular to AC. For if the whole AC is divided into two [equal] parts at O, and one foot of the compass is placed at O, and another draws a line with [the aperture of] the interval AO, as long as it moves along the arc CD, let two rules AZ and MN, intersecting at the point D, be pushed until the ruler MN intersects the cursor FY at point E, positioned on the line AC. By this the same means AE and AD will be obtained, with a much more convenient method than the preceding one. This method, if the Figure was much larger, [placed] somewhere on a huge wall, could be of use to architects.

It is not a place for me to give advise about the compass and its legs moveable is a certain ruler, or about the sharpness unius digiti impulsu consequetur ingeniosus quivis, ac istis etiam nostris longe meliora excogitabit.

of the other leg, which should delicately push rules at point D, or finally about the pressure of rules AZ and MN against the compass. This indeed someone ingenious will attempt to achieve by the weight of rulers, or by some spring, or by the push of one finger, or will devise something much better than that.



Figures 1 – 7.

Appendix - list of propositions from *Elements* mentioned in the text of *Solutio theorematum*

The first number indicates proposition number, the second one is the book number. Latin text from [3], English translation of propositions from [2]. Pappus generalization of 47.1 and Clavius' scholium for Prop. 31.3 translated by H. F.

Prop. 4.1

Si duo triangula duo latera duobus lateribus aequalia habeant, alterum alteri; habeant autem et angulum angulo aequalem, qui aequalibus rectis lineis continentur: et basim basi aequalem habebunt; et triangulum triangulo aequale erit; et reliqui anguli reliquis angulis aequales, alter alteri, quibus aequalia latera subtenduntur.

Prop. 4.1

If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

Prop. 47.1

In rectangulis triangulis, quod a latere rectum angulum subtendere describitur, quadratum aequale est quadratis quae a lateribus rectum angulum continentibus describuntur.

Prop. 47.1

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Prop. 12.2

In obtusangulis triangulis quadratum ex latere obtusum angulum subtendere, majus est quam quadrata ex lateribus obtusum angulum continentibus, rectangulo contento bis ab uno laterum quae sunt circa obtusum angulum in quod productum perpendicularis cadit, et recta linea intercepta exterius a perpendiculari ad angulum obtusum.

Prop. 12.2

In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.

Prop. 13.2

In omni triangulo, quadratum ex latere acutum angulum subtendere, minus est quam quadrata ex lateribus angulum illum continentibus, rectangulo contento bis ab uno laterum quae sunt circa acutum angulum, in quod productum perpendicularis cadit, et recta linea intercepta a perpendiculari ad angulum acutum.

Prop. 13.2

In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

Prop. 3.3

Si in circulo recta linea per centrum ducta, rectam lineam quandam non ductam

Prop. 3.3

If in a circle a straight line through the centre bisects a straight line not through per centrum bifariam secet; et ad angulos rectos ipsam secabit quod si ad angulos rectos ipsam secet, et bifariam secabit.

Prop. 8.6

Si in triangulo rectangulo, ab angulo recto ad basim perpendicularis ducatur; quae ad perpendicularem sunt triangula, et toti, et inter se sunt similia.

Prop. 31.6

In triangulis rectangulis figura rectilinea quae sit a latere rectum angulum subtendere, aequale est eis quae a lateribus rectum angulum continentibus sunt, similibus et similiter descriptis.

Prop. 19.7

Si quatuor numeri proportionales fuerint, qui ex primo, & quarto fit, numerus, aequalis erit ei, qui ex secundo, & tertio fit, numero. Et si, qui ex primo, & quarto fit, numerus, aequalis fuerit ei, qui ex secundo, & tertio fit, numero; ipsi quatuor numeri proportionales erunt.

Prop. 20.7

Si tres numeri proportionales fuerint; qui sub extremis continetur, aequalis est ei, qui a medio efficitur: Et si, qui sub extremis continetur, aequalis fuerit ei, qui a medio describitur; ipsi tres numeri proportionales erunt.

Prop. 4.11

Si recta linea duabus rectis lineis se invicem secantibus in communi sectione ad rectos angulos insistat, etiam ducto per ipsas plano ad rectos angulos erit.

Prop. 7.12

Omne prisma triangulem habens basim, dividitur in tres pyramides aequales inter se, quae triangulares bases habent.

Pappus generalization of 47.1, as given by Clavius

In omni triangulo, parallelogramma quaecunque super duobus lateribus descripta, aequalia sunt parallelogrammo super reliquo latere constituto, cuius

the centre, it also cuts it at right angles; and if it cut at right angles, it also bisects it.

Prop. 8.6

If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another.

Prop. 31.6

In right-angled triangles the figure of the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Prop. 19.7

If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and the third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

Prop. 20.7 (stated in the commentary to prop. 19.7 in [?])

If three numbers be proportional, the product of the extremes is equal to the square of the mean, and conversely.

Prop. 4.11

If a straight line be set up at right angles to two straight lines which cut one another, at their common point of section, it will also be at right angles to the plane through them.

Prop. 7.12

Any prism which has a triangular base is divided into three pyramids equal to one another which have triangular bases.

Pappus generalization of 47.1, as given by Clavius

In every triangle, any parallelograms built on two sides are equal to the parallelogram built on the remaining side, whose other side is equal and alterum latus aequale sit, & parallelum rectae ductae ab angulo, quae duo illa latera comprehendunt, ad punctum, in quo conveniunt latera parallelogrammorum lateribus trianguli opposita, si ad partes anguli dicti producantur.

Scholium Clavii ad 31.3

Manifestum quoque est conversum huius theorematis. Hoc est, segmentum circuli, in quo angulus constitutus est rectus, semicirculus est. Nam si esset maius, angulus in eo foret acutus; si minus, obtusus.

Corollarium ad Prop. 8.6

Ex hoc manifestum est, perpendicularem quae in rectangulo triangulo ab angulo recto in basin demittitur, esse mediam proportionalem inter duo basis segmenta.

parallel to the straight line drawn from the angle made by the two sides of the triangle to the point of intersection of extensions of the sides of parallelograms opposite to the sides of the triangle.

Clavius' scholium for Prop. 31.3

It is also clear that the converse of this theorem is true. That is, a segment of a circle, in which a right angle is constituted, is a semicircle. For is it was greater, the angle in it would be acute, if lesser, it would be obtuse.

Corollary to Prop. 8.6

From this is clear that, if a right angled triangle a perpendicular be drawn from the right angle to the base, the straight line so drawn is a mean proportional between the segments of the base.

Observationes Cyclometricæ by Adam Adamandy Kochański – Latin text[16] with annotated English translation

ADAMI ADAMANDI E SOCIET. JESU

Kochanski Dobrinniaci, Sereniss. Poloniarum Regis Mathematici & Bibliothecari, OBSERVATIONES Cyclometricæ, ad facilitandam Praxin accomodatæ; ex Epistola ad Actorum Collectores.

Qui Mathemata serio coluerit, nec tamen ad difficillima quæque & adhuc insoluta Problemata vires ingenii sui pertentandas censuerit, vix quenquam repertum esse existimo. Haud equidem diffiteor, me quoque olim eodem morbo laborasse, & ut alia prætereeam, in Circulo quidem quadrando, vel examinandis aliorum in eo conatibus, operæ non nihil collocasse. Non attinet hic enumerare Methodos, quas ea in re secutus fueram: unam tantum, quam fortasse quispiam felicius excolere poterit, commemorabo. Persuaseram mihi conjectura quadam, possibiles esse aliquas Rectarum sectiones, quarum segmenta invicem, & cum aliis rectis Longitudine vel Potentia incommensurabilia essent, Circuli tamen Areæ, vel Peripheriæ partibus Longitudine aut Potentia commensurarentur; ita ut inventa sectione istiusmodi, liceret ex ea Tetragonismum expedire Geometrice, vel saltem rationem Diametri ad Ambitum, in numeris ad lubitum

BY ADAM ADAMANY FROM THE SOCIETY OF JESUS

Kochański of Dobrzyń[17], Mathematician and Librarian of the Most Serene King[18] of Poland, Cyclometric OBSERVATIONS, accommodated for easiness of practical use; from a letter to editors[19] of Acta.

I suppose one could hardy find anyone who would seriously cultivate knowledge[20] and who would nevertheless not think that strengths of his talents are worth trying out on difficult and yet unsolved problems. For my part, I do not deny that I too was once affected by the same weakness, and, to omit other things, I put not a small effort into squaring of a circle and in examination of works of others attempting it. I does not belong here to list methods which I had followed in this matter: I will mention only one, which perhaps somebody luckier will be able to improve. I had convinced myself about a certain conjecture, namely that certain sections of a straight line are possible, whose fragments are incommensurable to each other and to other straight lines in length and square, yet commensurable to parts of area or circumference in length or square; so that by finding the section with this method, one might procure from it a quadrature of the circle geometrically, or at least

[16] Originally published in [8].

[17] Dobrzyń nad Wisłą – Kochański's birthplace, a town in Poland on the Vistula River, with settlement history dating back to 1065.

[18] John III Sobieski (1629 – 1696), from 1674 until his death King of Poland and Grand Duke of Lithuania.

[19] *Collectores Actorum* was an assembly of scholars who contributed to editing of *Acta*.

[20] *Mathemata* could mean both knowledge or mathematics.

maximis supputare.

Ad eam porro cogitationem videbar mihi non temere, sed illius Quadratricis, a Dinostrato inventæ, ductu devenisse. At cum ab istis laboribus ad alia disparata studia animus avocaretur, illum tandem adjeci, & quidem magnorum Virorum exemplis incitatus, ad investiganda compendia quædam Cyclometrica, Praxibus mechanicis utilia, idque tam in Numeris, quam Lineis; quorum nonnulla hoc loco adferre lubet.

compute the ratio of the diameter and circumference with as many digits as one likes.

It seems that I have arrived to this idea not blindly, but guided by a quadratrix[21], invented by Dinostratus[22]. And while my mind was diverted from this work by other separate pursuits, eventually, inspired by examples of great men, I turned to investigation of certain profits pertaining to cyclometry, useful in mechanical practice, as much numerically as geometrically.

DIAMTERI AD PERIPHERIAM CIRCULI
Rationes Arithmeticæ[23]

	Defectivæ		*Excessivæ.*
A	1. ad 3. -+-	Aa	1. ad 4. —
B	8. ad 25. -+-	Bb	7. ad 22. —
Z	1.... 15.... 3.	Zz	1.... 16.... 3.
C	106. ad 333. -+-	Cc	113. ad 355. —
Y	1.... 4697.... 3.	Yy	1.... 4698... 3.
D	530762. ad 1667438 -+-	Dd	530875. ad 1667793. —
X	1.... 5448[24].... 3.	Xx	1.... 5449[25].... 3.
E	Diam. 2945 294501.	Ee	Diam. 2945 825376.
	Periph. 9252 915567 -+-		Periph. 9254 583360. —
V	1.... 14774.... 3.	Vv	1.... 14775.... 3.
F	Dia. 43 521624 105025.	Ff	Dia. 43 524569 930401
	Per. 136 727214 560643 -+-		Per. 136 736469 144003 —

[21]Quadratrix of Hippias is a curve with equation $y = x\cot(\pi x/2a)$. It can be used to solve the problem of squaring the circle, although this is not a pure "ruler and compass" solution.

[22]Dinostratus (ca. 390 B.C. - ca. 320 B.C) was a Greek mathematician and geometer, a disciple of Plato.

[23]*Arithmetic ratios of diameter and circumference of a circle.* Ratios representing lower ("defective") bounds are on the left, upper ("excessive") bounds on the right.

[24]A misprint in the original text, should be 5548.

[25]Another misprint, should be 5549.

Harum qœdam minoribus terminis,[26]

cc	$22\frac{3}{5}$	ad	71 —
d.	265381.	ad	833719 +—
e.	30685681.	ad	96401910 +—

Methodicam prædictorum Numerorum Synthesin in *Cogitatis, & Inventis Polymathematicis*, quæ, si DEUS vitam prorogaverit, utilitati publicæ destinavi, plenius exponam; sufficiet interim ad eorum notitiam insinuasse sequentia. Numeri Characteribus Z. Y. X. V. tam simplicibus, quam geminatis insigniti, sunt *Genitores*, e quorum ductu, Numeri illis subjecti C. D. E. F. simplici, geminoque charactere notati, procreantur hoc modo. Ratio 7. ad 22. Excessiva, ducta in Genitorem Z. 15; & adjecto ad Productum Diametri, numero 1. ad Peripheriæ autem, hoc altero adjacente 3; constituit Rationem C.106. ad 333, Defectivam: Genitor autem major Zz.16, ductus in eosdem terminos Excedentes 7. ad 22, adjectisque ad horum Producta numeris 1 & 3, conficit Rationem CC. 113, ad 355. Excess:

Similiter hi termini Excessivi 113, 355, multiplicati per Genitores Y.Yy. videlicet 4697 & 4698. servata adjectione numerorum 1. & 3 ad Producta Diametri Peripheriæque, offerent terminos Rationum D. &Dd, quæ longe propius accedunt ad Archi-

I will explain the aforementioned method more completely in *Polymathic thoughts and inventions*, which work, if God prolongs my life, I have decided to put out for public benefit. In the meanwhile, for acquaintance with this method, the following introduction will suffice. Numbers denoted by both single and double characters Z, Y, X, V are *Originators*, from which numbers subjected to them, denoted by single and double characters A, B, C, D, are derived this way. Ratio of 7 to 22, excessive, is multipied by the Originator Z. 15, and with added product of the diameter, equal to 1, and 3, close to circumference, yields the defective ratio of 106 to 333.[27] Moreover, the major originator Zz.16, multiplied by the same exceeding bounds 7 and 22, and with numbers 1 and 3 added to the products, makes excessive ratio CC, 113 to 355.[28]

Similarly, those excessive bounds 113, 355, multiplied by originators Y and Yy, that is, 4697 and 4698, keeping addition of numbers 1 and 3, yield bounds[29] on the ratio D and Dd, which come far closer to the Archimedean ratio, expressed by Ludolph[30] and our Grum̃berger[31] by great

[26] *Of these [ratios], some expressed in reduced form.* Here, cc, d, and e are reduced forms of respectively Cc, D, and E, e.g. $\frac{71}{22\frac{3}{5}} = \frac{355}{113}$.

[27] Fraction $\frac{22}{7}$ is transformed into $\frac{22 \cdot 15+3}{7 \cdot 15+1} = \frac{333}{106}$.

[28] This produces $\frac{22 \cdot 16+3}{7 \cdot 16+1} = \frac{355}{113}$.

[29] These bounds are $D = \frac{355 \cdot 4697+3}{113 \cdot 4697+1} = \frac{1667438}{530762}$ and $Dd = \frac{355 \cdot 4698+3}{113 \cdot 4698+1} = \frac{1667793}{530875}$.

[30] Ludolph van Ceulen (1540 – 1610) was a German-Dutch mathematician who calculated 35 digits of π.

[31] Christoph Grienberger SJ (1561 – 1636) was an Austrian Jesuit astronomer, author of a catalog of fixed stars as well as optical and mathematical works.

medeam, a Ludolpho, & Grümbergero nostro vastissimis expressam numeris. Eadem ratione in reliquorum Terminorum genesi proceditur. Ut autem oculis ipsis usurpare liceat, quantum exactitudinis adferant Rationes illæ, visum est hoc loco adjicere Synopsin totius calculi, quo prædictæ Rationes ad Archimedeam, tanquam ad lapidem Lydium examinantur, ut appareat, quantum sit uniuscujusque peccatum, defectu vel excessu Peripheriæ taxato in partibus Diametri totius, in particulas Decimales subdivisæ.

many digits. Remaining bounds are produced by proceeding in the same manner. In order to see accuracy of these ratios with one's own eyes, it seemed fit to add in this place synopsis of all calculations, by which the aforementioned ratios are tested against Archimedean ratio like against the Lydian stone[32], so that it would become evident how big was the error of each of these ratios, with defect or excess of the circumference expressed as parts of the entire diameter, and with digits divided into small groups.

Examen Rationum Cyclometricarum.[33]

Diam.	100000	00000	00000	00000	00000	Archimedis
Periph.	314159	26535	89793	23846	26433	– Ratio.
B.	312500	00000				
	16519	26535	Defectus[34]			
Bb.	314285	71428				
	126	44892	Excessus.			
C.	314150	94339	62264			
	8	32196	27592	Defectus		
Cc.	314159	29203	53982			
		2667	64189	Excessus		
D.	314159	26535	81077	77120		
			8715	46725	Defect.	
Dd.	314159	26536	37862	02024		
			48068	78178	Excess.	
E.	314159	26535	89787	82814		
			5	41031	Defect.	
Ee.	314159	26535	89796	49172		
			3	25326	Excess.	
F.	314159	26535	89793	23833	89913	
				12	36520	Defectus.
Ff.	314159	26535	89793	23855	91866	
				9	65432	Excessus.

[32] Lydian stone (touchstone) – stone used to test gold for purity.

[33] *Examination of cyclometric ratios.*

[34] *Defect*, that is, the value of $\pi - B$. Similarly, *excess* (lat. excessus) is the value of $Bb - \pi$.

Ex hac Tabella colligitur: imprimis quantitas Defectus, vel Excessus cujusvis Rationis, taxata Fractione, cujus Denominator est Diameter supremo loco posita, videlicet 1 cum tot zeris, quot libet assumere: Numerator autem erit is, qui in eodem cum Denominatore gradu Decimali consistit. Sic Rationis C Defectum metitur hæc Fractio $\frac{8}{100000}$ quæ exactior erit, si prolixior Denominator assumatur.

Colligitur ex eadem secundo. Rationes nobis exhibitas, adeo compendiosas esse, ut earum nonnullæ, duplo pluribus notis Archimedeis æquivaleant ; quanquam ipsa Cc duplum earum excedat, quæ proinde brevitate, nec non exactitudine sua, in Praxi cæteris præferenda videatur, cui, dum quid accuratius quæritur, ipsa d succedat. Præter has quidem mihi suppetunt adhuc plures, consimili dote præditæ, sed eas, ne nimius videar, alteri occasioni servandas existimo. Concludam interim singulari quadam, & ut ita dicam, curiosa Ratione, quæ est 991 ad $3113\frac{991}{3113}$, quæ cum Archimedea consentit in octonis notis prioribus, ac tum primum illam incipit excedere, minus quam 23 centesimis.

From this table one infers, first of all, quantities of the defect or excess of any ratio, estimated by a fraction whose denominator is the diameter placed in the initial position, with as many zeros as one wants to take. Numerator, on the other hand, is this one, which takes the same position as the denominator. Thus the defect of the ratio C is estimated by the fraction $\frac{8}{100000}$, which would be more accurate if one took a longer denominator.

From the same table, a second thing is inferred. Ratios exhibited by us are so advantageous, that some of them are equivalent to Archimedean ratio with twice as many digits as others; Yet Cc itself twice exceed others[35], hence in practice by shortness and accuracy it seems to be preferred to others. If one sought a more accurate one, d would be a successor. Besides those, I have at hand indeed even more of them, similar in quality to the mentioned ones, but, in order not to appear excessive, I consider saving them for another occasion. I will, in the meanwhile, conclude with a certain singular, and so to speak curious ratio, which is 991 to $3113\frac{991}{3131}$, which agrees with Archimedean in the first 8 digits, and then it starts to exceed it, by less than 23 hundredths.[36]

GRAMMICÆ RATIONES CYCLOMETRICÆ,
Ad Usus Mechanicos.

Harum quidem complures olim a me repertæ; hoc tamen loco visum mihi est eam tantum proponere, quæ huic Anno præsenti, quo ista scribimus, affinitate quadam conjuncta est.

Oporteat igitur Semiperipheriæ B C D Rectam proxime æqualem reperire. Ducantur Tangentes B G, D

GEOMETRIC CYCLOMETRIC CONSTRUCTIONS,
For Use by Mechanics.

Of which several were once found by me. In this place, nevertheless, it seemed appropriate to present only one, associated with the current year, in which we write this.

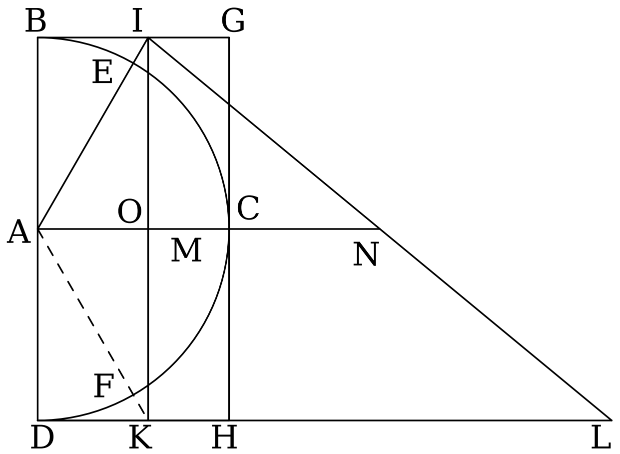
It would be then required to find a straight line nearly equal to the semicircle BCD. Let tangent lines BG, DH be

[35]That is, it exceeds B, Bb, and C in accuracy.

[36]Defining $r = \frac{3113\frac{991}{3131}}{991} = 3.1415192677\ldots$, we have $r - \pi \approx 0.23 \cdot 10^{-7}$.

H, quarum prior Radio AC æqualis, & jungantur GCH. Tum Radio CA secentur ex C arcus utrinque æquales CE & EF: quorum quivis complectetur Gradus 60, reliqui autem BE, DF singuli gr. 30. Agatur per E Secans AI, determinans Tangentem BI. Capiatur tandem HL, æqualis Diametro BD; ac tum ducatur IL.

drawn, equal to the radius AC and connected by GCH. Then from C, let both parts of the arc be cut by CE and EF[37], equal to the radius CA. Each of them will embrace the angle of 60 degrees, while the remaining angles BE, DF will be 30 degrees each. Let a line AI be driven through E, determining the extent of the tangent BI. Finally, let HL be taken equal to the diameter BD; and then let IL be drawn.



Dico Inprimis IL æqualem esse Semiperipheriæ BCD proxime. Demonstratur calculo Trigonometrico. Intelligatur autem ducta esse IK, quæ Tangentes BI, DK conjungat.

Quoniam ad Radium

AB. 100000 00000 00000.

Tangens gr. 30 est

BI. 57735 02691 89626. Erit hujus

Compl. ad Radium, ipsa

IG. 42264 97308 10373. Igitur

Tota KH ✠ HL, sive

KL. 2 42264 97308 10373. Ergo

IK q ✠ XL q.

9 86923 17181 95572 75995 52843

I say in the first place that IL is nearly equal to the semicircle BCD. This is demonstrated by trigonometric calculations. Let us assume that the line IK is drawn, which connects tangents BI, DK. Since, with the radius

AB. 100000 00000 00000,

tangent of 30 degrees is

BI. 57735 02691 89626.

Its complement[38] to the radius, IG itself, will be

IG. 42264 97308 10373. Therefore,

together KH ✠ HL, or

KL. 2 42264 97308 10373. Hence

IK q ✠ XL[39] q.

9 86923 17181 95572 75995 52843

[37] This should likely be CF.

[38] $1 - \tan 30°$.

[39] Obviously a misprint, should be KL instead of XL. IK q ✠ KL q. means $IK^2 + KL^2$.

99129.
Horum Radix est.
IL. 3 14153 33387 05093. Hæc autem
Deficit ab Archimedea.
Z. .... 5 93148 84700.
Continetur in AB. vicib.
X. .... 16859.

Dico deinde, Peripheriam sic inventam, ab Archimedea veræ proxima deficere minori Ratione, ea, quam habet Unitas ad Decuplum currentis Anni 1685 a Christo nato, Æra vulgari numerati; majore autem, quam eadem habeat ad decuplum anni 1686, proxime secuturi. Cum enim Peripheria nostra IL, ab Archimedea (præcedentis Tabulæ) deficiat numero Z, qui totam Diametrum, numero AB taxatam, metitur numero X: manifestum est, Unitatem ad numerum præsentis Anni decuplum, videlicet 16850, majorem habere rationem quam ad X, priore majorem: minorem autem, quam ad hunc 16860, per demonstrata Prop. 8 Quinti Elem. Euclid. E quibus Praxeos istius ἐυπορία καὶ ἀκϱίβεια, cum intellectu comprehendi, tum memoria facile retineri poterit.

Epimetri loco adjungam alteram Praxin Linearem Mechanicorum Circino opportunissimam, quod ea continua Diametri bisectione peragatur, sitque longe exactior præcedente: sic autem instituitur. Dati Circuli Diametrum, Circino bisectioni destinato, divide in partes 32. Talium enim Peripheria erit $100\frac{17}{32}$, hoc est, erit eorum Ratio, 1024. ad 3217. In Praxi igitur, Triplo Diametri, sive partibus 96, adjiciendæ

99129,
of which the root is
IL. 3 14153 33387 05093.[40] But this is
short of Archimedean by
Z. .... 5 93148 84700,[41]
contained in AB approx.
X. .... 16859.[42]

I say then that the circumference found this way differs from the Archimedean ratio by less then the ratio of one to ten times the date of the current year 1685 after Christ, numbered with the common era, and more than one to ten times the date of the next year to come, 1686. When indeed our periphery IL is short of the Archimedean (of preceding table) by the number Z, which measures the whole diameter, expressed by the number AB, with the number X: it is clear that the ratio of the unity to ten times the current year, or 16850, is larger than the ratio of 1 to X, and less than 1 to 16860, as demonstrated by prop. 8 of the 5th book of Euclid[43]. From this exactness but also easiness of the construction, when embraced by the intellect, can easily be retained in memory.

As a supplement, I will add another linear construction suitable for the compass of mechanics, which would be carried out by successive bisections of the diameter, and would be far more exact than the previous one: it is set up as follows. Given the diameter of the circle, intended for the bisection by compass, divide it into 32 parts. Of such kind, the circumference will be $100\frac{17}{32}$, that is, will be the ratio of 1024 and 3217. In practice, therefore,

[40] $IL = \sqrt{4 + \left(3 - \frac{\sqrt{3}}{3}\right)^2} = \frac{1}{3}\sqrt{120 - 18\sqrt{3}}$.

[41] $Z = \pi - IL \approx 0.0000593148847$.

[42] $X = \frac{1}{Z} \approx 16859$.

[43] Prop. 8 of Euclid's book 5 says: *Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.*[?]

erunt $\frac{4}{32}$, sive $\frac{8}{1}$ totius Diametri, & insuper Semis unius Trigesimæ secundæ, cum alterius Semissis particula decima sexta. Hujus ἐγχειρήσεως exactitudinem probat calculus, quo provenit Peripheria P. 314160156.– quæ Archimedeam excedit numero Q ....891, qui minor est Defectu Z, Peripheriæ præcedentis. Quanquam nec istud subticendum sit, istam Praxin in Majoribus Circulis potissimum locum habere, in parvis oculorum effugere, quoad particulam, postremo addendam.

to three diameters, or 96 parts, $\frac{4}{32}$ will be added, or $\frac{8}{1}$ of the total diameter[44], and, moreover, one half of 32nd, with $\frac{1}{16}$ of another particle's half.[45] Calculations prove exactness of this procedure, yielding Circumference P. 3.14160156, which exceeds Archimedean by the number Q ....891, which is smaller than the defect Z, of the preceding circumference[46]. In spite of this, one must not be silent about the fact that this construction has its place principally applied to larger circles, yet in smaller circles it is beyond one's ability to see, especially with respect to the small particle added at the end.

[44]Clearly, the author means 1/8 here.

[45]This amounts to $\frac{96}{32} + \frac{4}{32} + \frac{1}{2} \cdot \frac{1}{32} + \frac{1}{32 \cdot 32} = \frac{3217}{1024} = 3.1416015625$

[46]$Q = \frac{3217}{1024} - \pi \approx 0.00000891.$

Considerationes quaedam circa Quadrata & Cubos Magicos by Adam Adamandy Kochański – Latin text[47] with annotated English translation

ADAMI ADAMANDI KOCHANSKI E SOCIETATE JESU,

Sereniss. Poloniarum Regis Mathematici,

CONSIDERATIONES quaedam circa Quadrata & Cubos Magicos, nec non aliquot Problemata, omnibus Arithmophilis ad investigandum proposita: *In Litteris ad Actiorum Collectores.*

Numeros Progressionum certa quadam ratione in quadrata dispositos, quae ab aliquibus Magica, ab aliis Divina appellantur, magna semper apud Viros ingeniosos in admiratione fuisse probant ea, quae Kircherus, Fermat, Stifelius, Faulhaberus, Remmelinus, Henischius, Rot, Spinola, citante Schwentero in Delic. Phys. Math. aliique complures in hac materia scripta reliquerunt: quin immo, totus ferme Oriens Mahomentis superstitionibus addictus, nescio quae in his Quadratis mysteria, & Amuleta

BY ADAM ADAMANDY KOCHANSKI FROM THE SOCIETY OF JESUS,

Mathematician of the Most Serene King of Poland,

Certain CONSIDERATIONS concerning Magic Squares and Cubes, as well as some Problems, proposed for consideration to all enthusiasts of Arithmetic. *In Letters to editors of Acta.*

That Progressions of numbers arranged in squares in a certain way, which by some are called Magic, by others Divine, was always in admiration among ingenious men, is attested by many writings about those matters, left by Kircher[48], Fermat[49], Stifel[50], Faulhaber[51], Remmelin[52], Henisch[53], Roth[54], Spinola[55], cited by Schwenterius[56] in "Physico-Mathematical Delights" and others. Indeed, I do not know what secrets and prophylactic charms [contained] in those Squares are dreamed of by almost the whole Mahometan East

[47] Originally published in [9].

[48] Athanasius Kircher SJ (ca. 1601–1680), German Jesuit scholar. He discussed magic squares in his book *Arithmologia* (1665).

[49] Pierre de Fermat (ca. 1601–1665), French lawyer and mathematician. Discussion of magic squares and cubes can be found in his correspondence with Marin Mersenne O.M. (1588 –1648).

[50] Michael Stifel O.E.S.A. (1486 or 1487–1567), German Augustinian friar and mathematician, author of *Arithmetica integra.*

[51] Johann Faulhaber (1580–1635), German mathematician, author of *Academia Algebra.*

[52] Johann Remmelin (1585–1632), German physician, mathematician and philosopher.

[53] Georg Henisch (1549–1618), Hungarian-born educator, professor of mathematics in Augsburg, author of *De numeratione multiplici, vetere et recente* (1605).

[54] Peter Roth (1580–1617), author of *Arithmetica philosophica* (1608).

[55] Publio Francesco Spinola (ca. 1520–ca. 1567), author of *De intercalandi ratione corrigenda* (1562).

[56] Daniel Schwenter (1585–1636), German Orientalist and mathematician, author of *Deliciae Physico-Mathematicae* (1636).

prophylactica sibi somniat. Me quoque juvenilis olim curiositas eo adduxit, ut horum numerorum synthesin, in qua Kircherus in sua Arithmologia exiguo fructu magnopere desudavit, faciliorem redderem ; id quod magna parte mihi e voto processit. Nam praeterquam quod istiusmodi Quadratorum constructionem quemque non indocilem adolescentulum intra semiquadrantem horae, una cum mu1tiphci collocationum varietate edocere possim, excogitavi praeterea complura problemata materiam hanc concernentia, inter quae sunt I. Nova species Quadratorum magicorum, quae non jam per Additionem, sed Subtractionem, eundem quendam numerum ubique exhibens. II. Progressiones numerorum (Arithmeticas hic intelligo,) ita dispositae cubica forma, sive trinam dimensionem referente, ut tam per Additionem, quam Subtractionem numerorum, cujuscunque seriei etiam Diagonalium, idem quidam numerus acquiratur. III. Nec reticendum Problema istiusmodi : Data Specie Quadrati, & dato numero universali, hoc est eo, qui tam in columnis, quam trabibus, & Diagoniis Quadrati, per Additionem vel Subtractionem prodire debet, primum terminum progressionis arithmeticae invenire &c. Plura alia in hoc genere volente DEO, in *Cogitatis & Inventis meis Polymathematicis* plenius persequar.

Ut igitur Scriptionis compendium faciam, sequentia Additionis & Subtractioni Quadrata consideranda propono.

addicted to superstitions. Curiosity induced me to this too, once in my youth, so that I rendered easier synthesis of those numbers, on which Kircher in his Arithmologia labored with little fruit. It proceeded for the most part from a vow. For beyond the fact that I would be able to teach, in a minute, a young non-ignorant man the construction of that kind of Squares together with a variety of arrangements, I have thereafter invented many problems concerning these matters, among which are: I. New type of magic Squares, which produce the same number everywhere no longer by Addition, but by Subtraction. II. Progessions of numbers (here I understand them to be Arithmetic) such that when arranged in a form of a cube, or if rendered in three dimensions, every sequence, even diagonal, acquires the same total number by addition as well as by subtraction. III. One is not to be silent of the following problem: Given a type of a square, and given a universal number, that is the one which should come forth in rows, columns and diagonals of the square, find the first term of the arithmetic progression etc. I will pursue, GOD willing, many other [problems] of this kind in my *Polymathic Thoughts and Inventions.*

In order to make a summary of the work, I put forward for consideration the following Squares of Addition and Subtraction.

QUADRATA ADDITIONIS. SQUARES OF ADDITION.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

11	8	5	10
14	1	4	15
2	13	16	3
7	12	9	6

12	6	7	9
1	15	14	4
13	3	2	16
8	10	11	5

16	13	2	3
4	1	14	15
5	8	11	10
9	12	7	6

1: Aggegatum ubique est 34./The sum is everywhere 34.

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

11	17	24	5	8
7	14	20	23	1
4	10	13	16	22
25	3	6	12	19
18	21	2	9	15

12	21	20	9	3
1	15	24	18	7
10	4	13	22	16
19	8	2	11	25
23	17	6	5	14

15	4	7	16	23
24	12	1	8	20
17	21	13	5	9
6	18	25	14	2
3	10	19	22	11

2: Aggregatum ubique conficitur 65./The sum is everywhere 65.

Quaternarii at Quinarii Quadrata primo loco posita sunt ea, quae passim apud Auctores inveniuntur ; quae vero consequuntur, per quandam universalem regulam variata sunt, & multoties adhuc per eandem variari possunt, quarum quidem Variationum numerum, e combinationum doctrina, alias determinabimus. De Quaternario illud interim commemorabo, illius Quadrata quoad substantiam, id est nulla habita situs ratione, variari posse Quadragies Octies, tam in additionis quam subtractionis artificio: e quibus tamen haec paucula protulisse sufficiet hoc loco.

In the first place are positioned those Quaternary and Quintary Squares, which are found here and there among [writings of the aforementioned] Authors; those which follow have been transformed by a certain universal rule, and can still be changed many times by the same rule. We will indeed subsequently determine the number of these variations from the doctrine of combinations. Regarding the Quaternary, I will meanwhile mention that its square can be changed 48 times with respect to the substance, that is, without regard to the structure, as much by the method of addition as by subtraction: of this let it suffice that only a little has been brought forward in this place.

QUADRATA SUBTRACTIONIS. SQUARES OF SUBTRACTION.

1	6	13	2
10	5	14	9
7	12	11	16
8	3	4	15

1	15	8	2
5	11	12	14
10	16	7	9
6	4	3	13

7	10	9	16
8	1	2	15
3	6	13	4
12	5	14	11

11	7	16	12
13	1	2	6
4	8	15	3
14	10	9	5

3: Residuum ubique habetur 8./Residuum is everywhere 8.

11	24	9	16	3
4	12	25	8	20
19	5	13	21	7
6	18	1	14	22
23	10	17	2	15

8	24	5	20	12
22	3	17	15	10
1	19	13	7	25
16	11	9	23	4
14	6	21	2	18

12	21	16	7	3
1	15	24	18	9
6	4	13	22	20
17	8	2	11	25
23	19	10	5	14

15	4	9	20	23
24	12	1	8	16
19	21	13	5	7
10	18	25	14	2
3	6	17	22	11

4: Residuum in omnibus est 18./Residuum in all [squares] is 18.

In his Quadratis Subtractione tractandis ita proceditur. Numerus in assumpta Columna, Trabe, vel Diagono minimus, subducendus est a proxime majore, residuum majoris hujus detrahatur a proxime consequente, atque ita porro : ultimum enim residuum est illud universale, quale in praemissis Quadratis est 8 & 13. *Vel aliter:*

Collocatis assumpta Columnae numeris ordine suo, a minimo ad maximum procedendo, ut in prima Columna Quinarii istis 4. 6. 11. 19. 23. a summa numerorum 4. 11. 23. locis imparibus consistentium, quae est 38. subducatur duorum 6. 19. qui loca paria, nempe secundum & quartum occupans, summa 25. relinquetur universale residuum 13. In Quadratis autem,

In these Squares produced by Subtraction one proceeds as follows. The smallest number in a selected Column, Row, or Diagonal is subtracted from the nearest larger number, the difference is subtracted form a subsequent larger number, and so on: the final result is this universal number, which in the Squares [to be displayed] ahead is equal to 8 and 13. *Or otherwise:*

With the numbers of the selected column arranged by their order, proceeding from the smallest one to the largest one, as in the first Column of the Quintary [Square], 4, 5, 11, 19, 23, from the sum of numbers 4, 11, 23 positioned in odd places, which is 38, one subtracts 25, the sum of two, 6 and 19, which are in even places, namely occupying the second and quorum latus est par numerus; Summa numerorum locis imparibus primo, tertio &c. in ordine illo consistentium, subducenda est a summa reliquorum.

the fourth [position], leaving the universal residuum 13. On the other hand, in Squares whose side is an even number; the Sum of numbers from odd places, first, third, etc., arranged in increasing order, is subtracted from the sum of remaining numbers.

Est autem in Quaternarii Quadrato subtractionem concernente id singulare, quod illud nonagies sexies, hoc est duplo numero variari possit, si residuum Diagonalium non jam par sit aliis columnarum trabiumque residuis, quod diximus esse ubique 8. nempe semissem Quadrati 16. Sed tantum sit lateri quadrati 4. aequale. Quadratorum hujusmodi, qualia sunt 96, Specimen sunto sequentia paucula.

Moreover, in Quaternary Square, concerning subtraction, there is this singular thing, that it can be varied 96 times, twice as many times [as the one with addition], if the residuum of Diagonals were not equal to residua of other columns and rows, which we said to be 8., truly half of Square's 16. But this happens only to squares with side equal to 4. Let a few examples selected from 96 Squares of this type be shown below.

1	6	13	2
8	3	4	15
7	12	11	16
10	5	14	9

1	8	15	2
6	3	4	13
5	12	11	14
10	7	16	9

1	2	13	6
10	9	14	5
7	16	11	12
8	15	4	3

1	8	2	15
6	3	13	4
5	12	14	11
10	7	9	16

5: In Trabibus & Columnis Residuum est 8. in Diagoniis 4./In Rows and Columns the Residuum is 8, in Diagonals 4.

In Quadratis igitur tam pro additione, quam subtractione formatis, a radicibus numerorum cum Imparium, tum Pariter parium, credo me deprehendisse quantum sat est, ad eorum synthesin instituendam. In Quadratis tamen a radicibus impariter paribus, quales sunt 6. 10. 14 &c. ad Subtractionem concinnandis, quae speciales habent difficultates, (a me quidem in Additionis artificio, saepius variabili, feliciter superatas;) visum mihi est provocare homines, nec ingenio nec otio destitutos, ut in hoc latissimo campo, qui plurium culturam admittat, vires suas experiantur. Sit itaque

PROBLEMA I. In Quadrato Senarii, cellulas 36 complectente, numeros progressionis Arithmeticae, ab 1. ad

Consequently, in Squares formed by both addition or subtraction, with odd or doubly even bases, I believe that I have discovered as much as it is sufficient for establishment of the method of their construction. With Squares with singly even bases, such as 6, 10, 14, etc., rendered by subtraction, which have special difficulties, it seems to me good to provoke people who are not devoid of talent and leisure, to test their powers in this spacious field permitting to be cultivated by many. Let therefore be:

PROBLEM I. In a square of base 6, containing 36 cells, to arrange numbers of Arithmetic sequence, proceeding inclusively from 1 to 36, in such a a way that by the method of subtraction explained earlier, the number 18 remains

36. inclusive procedentes, ita disponere, ut subtractionis artificio, prius explicato, in omnibus columnis, trabibus, & utraque Diagonali relinquatur numerus 18.

PROBLEMA II. In cubis numerorum pariter parium, velut Radice 4. 8. 16. Progressiones Arithmeticas illis competentes, ita disponere, & variare, ut in omnibus Columnis, Trabibus, & Diagoniis Cuborum, per Additionem eadem summa confletur, per Subtractionem vero residuum idem relinquatur.

PROBLEMA III. Eadem omnia praestare in Cubis Radicum pariter imparium, ut sunt 6. 10. 14 &c.

De Quadratis & cubis Centralibus, hoc est iis quae post ablationem Zonarum circumjacentium, relinquunt interiora quadrata vel cubos magicos, alias agemus: si cui tamen commodissimam horum & praecedentium problematum synthesin fors obtulerit, ab humanissimo Inventore certior ea de re, privatim vel publice fieri, enixe peto.

Coronidis loco addere lubet quadrata, a Radicibus 4 & 5 suis numeris sic instructa, ut additionis ac subtractionis ope, annum currentem offerant: pro quibus concinnandis, alio tempore locoque, regulas generales, volente Deo tradituri sumus.

in all columns, rows and both Diagonals.

PROBLEM II. In a doubly even cube of numbers, such as the one with base 4, 8, 16 etc., to arrange and vary suitable arithmetic sequences in such a way that in all Columns, Rows and Diagonals of cubes, the same sum is forged by Addition; moreover, by Subtraction, the same residuum remains.

PROBLEM III. To apply all the same in Cubes with singly even bases, such as 6, 10, 14, etc.

We will deal elsewhere with central Squares and Cubes, that is, Squares and Cubes whose interiors remain magic after removing surrounding layers: nevertheless, if the fate presents to someone the most opportune method of solving of this [problem] as well as preceding problems, I earnestly ask to be notified about it by the kindest Inventor, either privately or publicly.

In place of the colophon, it is pleasing to add squares, with bases 4 and 5, equipped with numbers in such a way, that with the help of addition and subtraction, they bestow the current year: we are going, God willing, to deliver general methods for their construction at another time and in another place.

423	421	418	424
428	414	417	427
416	426	429	415
419	425	422	420

336	346	330	341	327
326	339	348	332	343
340	328	338	346	334
331	342	326	335	350
347	333	344	330	338

6: Summa ubique est 1686./Sum everywhere is 1686.

Translator's remark: The second square, which is supposed to be a standard magic square with sum 1686, actually yields row sums 1680, 1688, 1686, 1684, 1692, column sums 1680, 1688, 1686,

$1690\frac{1}{4}$	2744	1901	$847\frac{1}{4}$
215	$2533\frac{1}{4}$	$4\frac{1}{4}$	1058
$3165\frac{2}{4}$	$2111\frac{3}{4}$	$1268\frac{3}{4}$	$2322\frac{2}{4}$
$2954\frac{3}{4}$	$636\frac{2}{4}$	$1479\frac{2}{4}$	$425\frac{3}{4}$

1406	2526	426	3366	986
1126	1826	2106	286	2806
3226	706	1686	2666	146
566	3086	1266	1546	2246
2386	6	2946	846	1966

7: Residuum, aut Differentia ubique relinquitur 1686./Residuum, or Difference everywhere remains 1686.

1684, 1692, and diagonal sums 1686, 1686. It has beens suggested in [16] that this may be a result of typesetter's mistakes, and that the correct square could be as shown below (with modifications in bold font). Since this would require modification of 9 digits, it is quite unlikely for the typesetter

336	346	330	34**7**	327
3**30**	339	348	332	3**37**
340	328	338	346	334
33**3**	342	326	335	350
347	33**1**	344	3**26**	338

to make so many mistakes. Upon closer examination it turns out that the aforementioned square is a half-baked square of *subtraction* with residuum 338. Residua of all rows, columns, and diagonals are the same and equal to 338, but the entries do not form an arithmetic sequence. For example, the number 326 appears in it twice, similarly as numbers 330, 338, and 346. It may be an unfinished square which Kochański mistakenly submitted to the publisher instead of the desired magic square with the sum 1686.

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Trzy matematyczne prace Adama Adamandy Kochańskiego – opatrzone tłumaczeniem na język angielski

Henryk Fukś

Streszczenie Praca zawiera tłumaczenia z łaciny na język angielski trzech najważniejszych prac matematycznych Adama Adamandego Kochańskiego (1631-1700), wraz z oryginalną wersja łacińską, z przypisami i wprowadzeniem. Prace były opublikowane w Acta Eruditorum w r. 1682 („Solutio theorematum"), 1685 („Observationes cyclo- metricae"), oraz 1686 („Considerationes quaedam"). Tłumaczenie jest tak wierne jak to możliwe, często dosłowne. Zamiarem tłumacza było uczynienie go pomocnym dla tych, którzy zechcą zapoznać się z oryginalnym tekstem łacińskim. Prace zostały opatrzone licznymi przypisami zawierającymi informacje o nazwiskach i terminach, które mogą nie być znane współczesnemu czytelnikowi. Błędy w druku, obecne w oryginalnym wydaniu, zostały zaznaczone, a poprawki podano w przypisach. Aby ułatwić zrozumienie dowodów zawartych w pracy „Solutio theorematum", uzupełniono ją dodatkiem zawierającym cytowania z „Elementów" w wersji łacińskiej, używanej przez Kochańskiego.

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