FoliA 149

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XIII (2014)

Kaliappan Vijaya, Gangadharan Murugusundaramoorthy,
Murugesan Kasthuri

Starlike functions of complex order involving
$q$-hypergeometric functions with fixed point

Abstract. Recently Kanas and Ronning introduced the classes of starlike and convex functions, which are normalized with $f(\xi) = f'(\xi) - 1 = 0$, $\xi$ ($|\xi| = d$) is a fixed point in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$. In this paper we define a new subclass of starlike functions of complex order based on $q$-hypergeometric functions and continue to obtain coefficient estimates, extreme points, inclusion properties and neighbourhood results for the function class $TS_{\xi}(\alpha, \beta, \gamma)$. Further, we obtain integral means inequalities for the function $f \in TS_{\xi}(\alpha, \beta, \gamma)$.

1. Introduction

Let $\xi$ ($|\xi| = d$) be a fixed point in the unit disc $U := \{z \in \mathbb{C} : |z| < 1\}$. Denote by $A(\xi)$ the class of functions which are regular and normalized by $f(\xi) = f'(\xi) - 1 = 0$ consisting of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n(z - \xi)^n, \quad (z - \xi) \in U. \quad (1)$$

Also denote by $S_{\xi} = \{f \in A(\xi) : f \text{ is univalent in } U\}$, the subclass of $A(\xi)$. Denote by $T_{\xi}$ the subclass of $S_{\xi}$ consisting of the functions of the form

$$f(z) = (z - \xi) - \sum_{n=2}^{\infty} a_n(z - \xi)^n, \quad a_n \geq 0. \quad (2)$$

Note that $S_0 = S$ and $T_0 = T$ be the subclasses of $A = A(0)$ consisting of univalent functions in $U$. By $S_{\xi}^*(\beta)$ and $K_{\xi}(\beta)$ respectively, we mean the classes of analytic

AMS (2010) Subject Classification: 30C45
* Corresponding Author.
functions that satisfy the analytic conditions
\[ \Re \left\{ \frac{(z - \xi)f'(z)}{f(z)} \right\} > \beta, \quad \Re \left\{ 1 + \frac{(z - \xi)f''(z)}{f'(z)} \right\} > \beta \quad \text{and} \quad (z - \xi) \in \mathbb{U} \]
for \( 0 \leq \beta < 1 \) introduced and studied by Kanas and Ronning [11]. The class \( S^*_\xi(0) \) is defined by geometric property that the image of any circular arc centered at \( \xi \) is starlike with respect to \( f(\xi) \) and the corresponding class \( K^*_{\xi}(0) \) is defined by the property that the image of any circular arc centered at \( \xi \) is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [8] for uniformly starlike and convex functions, except that in this case the point \( \xi \) is fixed. In particular, \( K = K_0(0) \) and \( S^*_0 = S^*(0) \) respectively, are the well-known standard classes of convex and starlike functions [10, 19].

We recall a generalized \( q \)-Taylors formula in fractional \( q \)-calculus and certain \( q \)-generating functions for \( q \)-hypergeometric functions studied more recently by Purohit and Raina [15] and further by Mohammed Aabed and Maslina Darus [1]. For complex parameters \( a_1, \ldots, a_l \) and \( b_1, \ldots, b_m \) (\( b_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m \)) the \( q \)-hypergeometric function \( \psi_m(z) \) is defined by
\[ _l\psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z) := \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_l; q)_n}{(b_1; q)_n \cdots (b_m; q)_n} \left[ (-1)^n q^{\binom{n}{2}} \right]^{1+m-l} z^n \tag{3} \]
with \( \binom{n}{2} = \frac{n(n-1)}{2} \), where \( q \neq 0 \) when \( l > m + 1 \) (\( l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ z \in \mathbb{U} \)).

The \( q \)-shifted factorial is defined for \( a, q \in \mathbb{C} \) as a product of \( n \) factors by
\[ (a; q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq) \cdots (1-aq^{n-1}), & n \in \mathbb{N} \end{cases} \]
and in terms of basic analogue of the gamma function
\[ (q^n; q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0. \tag{4} \]
It is interest to note that \( \lim_{q \to 1^{-}} \frac{(q^n; q)_n}{(q; q)_n^n} = (a)_n = a(a+1) \cdots (a+n-1) \) the familiar Pochhammer symbol.

Now for \( z \in \mathbb{U}, 0 < |q| < 1 \) and \( l = m + 1 \), the basic \( q \)-hypergeometric function defined in [3] takes the form
\[ _l\psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_l; q)_n}{(b_1; q)_n \cdots (b_m; q)_n} z^n \]
which converges absolutely in the open unit disk \( \mathbb{U} \). Let
\[ I(a_l, b_m; q, z) = z \psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z) = \sum_{n=0}^{\infty} \gamma_n^{l,m}[a_1, q] z^{n+1}, \]
Starlike functions of complex order involving q-hypergeometric functions

where for convenience,

\[ \Upsilon_n^{l,m}[a_1, q] = \frac{(a_1; q)_n \ldots (a_1; q)_n}{(q; q)_n (b_1; q)_n \ldots (b_m; q)_n}. \]

The operator \( I(a_1, b_m; q) f(z) \) was studied recently by Aabed and Darus \[1\].

In this paper we define a new linear operator for \((z - \xi) \in \mathbb{U}, |q| < 1\) and \(l = m + 1\) as follows:

\[ I(a_1, b_m; q, z - \xi) = (z - \xi) \psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z - \xi) \]

\[ = \sum_{n=0}^{\infty} \Upsilon_n^{l,m}[a_1, q](z - \xi)^{n+1}. \]

Using the above, we let

\[ I(a_1, b_m; q, z - \xi) \ast f(z) = I_m^l f(z) = (z - \xi) + \sum_{n=2}^{\infty} \Upsilon_n^{l,m}[a_1, q] a_n(z - \xi)^n, \quad (5) \]

where

\[ \Upsilon_m^l(a) = \Upsilon_n^{l,m}[a_1, q] = \frac{(a_1; q)_{n-1} \ldots (a_1; q)_{n-1}}{(q; q)_{n-1} (b_1; q)_{n-1} \ldots (b_m; q)_{n-1}} \]

unless otherwise stated.

For \(a_i = q^{a_i}, b_j = q^{b_j}, \alpha_i, \beta_j \in \mathbb{C}, \) and \( \beta_j \neq 0, -1, -2, \ldots, (i = 1, \ldots, l, j = 1, \ldots, m) \) and \( q \rightarrow 1, \) we obtain the well-known Dziok-Srivastava linear operator \[7 \] \[6\] (for \( l = m + 1 \)). For \( l = 1, m = 0, a_1 = q, \) and further specializing the parameters, it gives many (well known and new) integral and differential operators introduced and studied in \[4 \] \[5 \] \[10 \] \[13 \] \[16 \].

Making use of the operator \( I_m^l \) and motivated by the results discussed by Altintas et al. \[2\], (see \[12\] and references stated therein) and Aouf et al. \[3\], in this paper we introduce a new subclass \( S_\xi(\alpha, \beta, \gamma) \) of analytic functions of complex order associated with q-hypergeometric functions as given below.

For \(-1 \leq \alpha < 1, \beta \geq 0\) and \( \gamma \in \mathbb{C} \setminus \{0\}, \) we let \( S_\xi(\alpha, \beta, \gamma) \) be the subclass of \( \mathcal{A}(\xi) \) consisting of functions of the form \[1\] and satisfying the analytic criterion

\[ \Re\left(1 + \frac{1}{\gamma} \left[\frac{(z - \xi) I_m^l f(z)'}{I_m^l f(z)} - \alpha\right]\right) > \beta' \Re\left[\frac{1}{\gamma} \left[\frac{(z - \xi) I_m^l f(z)'}{I_m^l f(z)} - 1\right]\right] \]

for every \( z \in \mathbb{U}, \) where \( I_m^l f(z) \) is given by \[5\]. We also let \( T S_\xi(\alpha, \beta, \gamma) = S_\xi(\alpha, \beta, \gamma) \cap I_\xi. \)

**Example 1**

We note that \( S_\xi(1, 0, \gamma) \equiv S_\xi^\gamma(\gamma), \) the class of starlike functions of complex order \( \gamma (\gamma \in \mathbb{C} \setminus \{0\}) \), satisfying the following conditions

\[ \frac{f(z)}{z - \xi} \neq 0 \quad \text{and} \quad \Re\left(1 + \frac{1}{\gamma} \left[\frac{(z - \xi) I_m^l f(z)'}{I_m^l f(z)} - 1\right]\right) > 0. \]
Further,
\[ S_\xi^*((1 - \delta) \cos \lambda e^{-i\lambda}) = S_\xi^*(\delta, \lambda), \quad |\lambda| < \frac{\pi}{2}; \quad 0 \leq \delta \leq 1 \]
and
\[ S_\xi^*(\cos \lambda e^{-i\lambda}) = S_\xi^*(\lambda), \quad |\lambda| < \frac{\pi}{2}, \]
where \( S_\xi^*(\delta, \lambda) \) denotes the subclass of \( \lambda \)-spiral-like function of order \( \delta \) and \( S_\xi^*(\lambda) \) denotes spiral-like functions with fixed point analogous to the classes introduced and investigated by Libera [11] and Spacek [18].

The main object of this paper is to study some usual properties such as the coefficient bounds, extreme points, radii of close to convexity, starlikeness and integral means inequalities for the class \( T S_\xi(\alpha, \beta, \gamma) \). Further, we obtain neighborhood results and integral means inequalities for aforementioned class.

**2. Coefficient bounds**

In this section we obtain a necessary and sufficient condition for functions \( f \in T S_\xi(\alpha, \beta, \gamma) \).

**Theorem 2.1**

A necessary and sufficient condition for \( f \) of the form (2) to be in the class \( T S_\xi(\alpha, \beta, \gamma) \) is

\[
\sum_{n=2}^{\infty} \left[ (n + |\gamma|)(1 - \beta) - (\alpha - \beta)(r + d)^n - |\gamma|(1 - \beta) \right] \leq (1 - \alpha) \sum_{n=2}^{\infty} \frac{Y_m(n)a_n(z - \xi)^n}{(z - \xi) - \sum_{n=2}^{\infty} Y_m(n)a_n(z - \xi)^n}, \tag{6}
\]

where \( -1 \leq \alpha < 1, \beta \geq 0 \) and \( \gamma \in \mathbb{C} \setminus \{0\} \).

**Proof.** Assume that \( f \in T S_\xi(\alpha, \beta, \gamma) \), then

\[
\Re\left\{ 1 + \frac{1}{\gamma} \left[ \frac{(z - \xi)(1 - \alpha) - \sum_{n=2}^{\infty} (n - \alpha) Y_m(n)a_n(z - \xi)^n}{(z - \xi) - \sum_{n=2}^{\infty} Y_m(n)a_n(z - \xi)^n} \right] \right\} > \beta \left\{ 1 - \frac{1}{\gamma} \left[ \sum_{n=2}^{\infty} (n - 1) Y_m(n)a_n(z - \xi)^n \right] \right\}.
\]

On choosing the values of \( (z - \xi) \) on the positive real axis, where \( 0 < |z - \xi| \leq r + d < 1 \), we have

\[
\left\{ 1 + \frac{1}{|\gamma|} \left[ \frac{(1 - \alpha) - \sum_{n=2}^{\infty} (n - \alpha) Y_m(n)a_n(r + d)^{n-1}}{1 - \sum_{n=2}^{\infty} Y_m(n)a_n(r + d)^{n-1}} \right] \right\} > \beta \left\{ 1 - \frac{1}{|\gamma|} \left[ \sum_{n=2}^{\infty} (n - 1) Y_m(n)a_n(r + d)^{n-1} \right] \right\}.
\]
The simple computation leads the desired inequality
\[
\sum_{n=2}^{\infty} [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] T_m(n) a_n (r + d)^{n-1} \leq (1 - \alpha) + |\gamma|(1 - \beta).
\]

Conversely, suppose that \([6]\) is true for \((z - \xi) \in U\), then
\[
\mathbb{R} \left( 1 + \frac{1}{\gamma} \left[ \frac{(z - \xi)(T_m f(z))'}{T_m f(z)} - \alpha \right] \right) - \beta \left[ 1 + \frac{1}{\gamma} \left[ \frac{(z - \xi)(T_m f(z))'}{T_m f(z)} - 1 \right] \right] > 0.
\]
If
\[
1 + \frac{1}{|\gamma|} \left( \frac{(1 - \alpha) - \sum_{n=2}^{\infty} (n - \alpha) T_m(n) a_n |z - \xi|^{n-1}}{1 - \sum_{n=2}^{\infty} T_m(n) a_n |z - \xi|^{n-1}} \right)
\]
\[
- \beta \left[ 1 - \frac{1}{|\gamma|} \left( \frac{\sum_{n=2}^{\infty} (n - 1) T_m(n) a_n |z - \xi|^{n-1}}{1 - \sum_{n=2}^{\infty} T_m(n) a_n |z - \xi|^{n-1}} \right) \right] \geq 0.
\]
That is if
\[
\sum_{n=2}^{\infty} [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] T_m(n) a_n (r + d)^{n-1} \leq (1 - \alpha) + |\gamma|(1 - \beta),
\]
which completes the proof.

**Corollary 2.2**

Let the function \(f\) defined by \([5]\) belongs \(TS_\xi(\alpha, \beta, \gamma)\). Then
\[
a_n \leq \frac{[(1 - \alpha) + |\gamma|(1 - \beta)]}{(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] T_m(n) (r + d)^{n-1},
\]
n \(\geq 2\), \(-1 \leq \alpha < 1\), \(\beta \geq 0\) and \(\gamma \in \mathbb{C} \setminus \{0\}\), with equality for
\[
f(z) = (z - \xi) - \frac{[(1 - \alpha) + |\gamma|(1 - \beta)]}{(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] T_m(n) (z - \xi)^n.
\]

For the sake of brevity we let
\[
\Theta_d(n, \alpha, \beta, \gamma) = [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] T_m(n) (r + d)^{n-1},
\]
\[
\Theta_d(2, \alpha, \beta, \gamma) = [(2 - \alpha - \beta) + |\gamma|(1 - \beta)] (r + d) \quad (7)
\]
throughout our study.

In the next theorem we state extreme points for the functions of the class \(TS_\xi(\alpha, \beta, \gamma)\).

**Theorem 2.3 (Extreme points)**

Let
\[
f_1(z) = (z - \xi),
\]
\[
f_n(x) = (z - \xi) - \frac{[(1 - \alpha) + |\gamma|(1 - \beta)]}{(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] T_m(n) (z - \xi)^n, \quad n = 2, 3, \ldots \quad (8)
\]
Then \(f \in TS_\xi(\alpha, \beta, \gamma)\) if and only if \(f\) can be expressed in the form \(f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)\), where \(\omega_n \geq 0\) and \(\sum_{n=1}^{\infty} \omega_n = 1\).
The proof of the Theorem 2.3 follows on lines similar to the proof of the theorem on extreme points given in Silverman [19].

3. Close-to-convexity, starlikeness and convexity

In this section we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS_\xi(\alpha, \beta, \gamma)$.

Theorem 3.1
Let $f \in TS_\xi(\alpha, \beta, \gamma)$. Then $f$ is close-to-convex of order $\delta$ ($0 \leq \delta < 1$) in the disc $|z - \xi| < R_1$, that is, $\Re(f'(z)) > \delta$, where

$$R_1 = \inf_{n \geq 2} \left[ \frac{(1 - \delta)\Theta_d(n, \alpha, \beta, \gamma)}{n(1 - \alpha) + |\gamma|(1 - \beta)} \gamma^l_m(n) \right]^{\frac{1}{n-1}}.$$

Proof. Given $f \in T_\xi$ and $f$ is close-to-convex of order $\delta$, we have

$$|f'(z) - 1| < 1 - \delta.$$  (9)

For the left hand side of (9) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n R_1^{n-1}.$$  

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n a_n R_1^{n-1}}{1 - \delta} < 1.$$  

Using the fact, that $f \in TS_\xi(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \beta, \gamma)}{(1 - \alpha) + |\gamma|(1 - \beta)} \gamma^l_m(n) a_n < 1.$$  

We can say (9) is true if

$$\frac{n}{1 - \delta} R_1^{n-1} \leq \frac{\Theta_d(n, \alpha, \beta, \gamma)}{(1 - \alpha) + |\gamma|(1 - \beta)} \gamma^l_m(n).$$  

Or equivalently,

$$R_1 \leq \left[ \frac{(1 - \delta)\Theta_d(n, \alpha, \beta, \gamma)}{n(1 - \alpha) + |\gamma|(1 - \beta)} \gamma^l_m(n) \right]^{\frac{1}{n-1}}.$$  

Which completes the proof.

Theorem 3.2
Let $f \in TS_\xi(\alpha, \beta, \gamma)$. Then

1. $f$ is starlike of order $\delta$ ($0 \leq \delta < 1$) in the disc $|z - \xi| < R_2$; that is, $\Re(\frac{(z-\xi)f'(z)}{f(z)}) > \delta$, where

$$R_2 = \inf_{n \geq 2} \left\{ \frac{(1 - \delta)\Theta_d(n, \alpha, \beta, \gamma)}{(n - \delta)\left[n(1 - \alpha) + |\gamma|(1 - \beta)\right]} \gamma^l_m(n) \right\}^{\frac{1}{n-1}},$$  

where

$$(n-\delta)\left[n(1 - \alpha) + |\gamma|(1 - \beta)\right] \gamma^l_m(n)$$  

and

$$(1 - \delta)\Theta_d(n, \alpha, \beta, \gamma)$$  

are defined as above.
2. $f$ is convex of order $\delta$ ($0 \leq \delta < 1$) in the unit disc $|z - \xi| < R_3$, that is $\Re\left(1 + \frac{(z - \xi)f''(z)}{f'(z)}\right) > \delta$, where

$$R_3 = \inf_{n \geq 2} \left\{ \frac{(1 - \delta)}{n(n - \delta)} \frac{\Theta_d(n, \alpha, \beta, \gamma)}{[(1 - \alpha) + |\gamma|(1 - \beta)]} \Upsilon^l_m(n) \right\}^{\frac{1}{n - 1}}.$$

These results are sharp for the extremal function $f$ given by (8).

**Proof.** For the case 1, notice that for given $f \in T_\xi$ and $f$ is starlike of order $\delta$, we have

$$\left| \frac{(z - \xi)f'(z)}{f(z)} - 1 \right| < 1 - \delta.$$  \hfill (10)

For the left hand side of (10) we obtain

$$\left| \frac{(z - \xi)f'(z)}{f(z)} - 1 \right| \leq \sum_{n=2}^{\infty} \frac{(n - 1)a_n|z - \xi|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z - \xi|^{n-1}}.$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n - \delta}{1 - \delta} a_n |z - \xi|^{n-1} < 1.$$

Using the fact, that $f \in TS_\xi(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \beta, \gamma)}{(1 - \alpha) + |\gamma|(1 - \beta)} \Upsilon^l_m(n) a_n < 1.$$

We can say (10) is true if

$$\frac{n - \delta}{1 - \delta} |z - \xi|^{n-1} < \frac{\Theta_d(n, \alpha, \beta, \gamma)}{(1 - \alpha) + |\gamma|(1 - \beta)} \Upsilon^l_m(n).$$

Or equivalently,

$$R_3^{n-1} < \frac{(1 - \delta)\Theta_d(n, \alpha, \beta, \gamma)}{(n - \delta)((1 - \alpha) + |\gamma|(1 - \beta))} \Upsilon^l_m(n)$$

which yields the starlikeness of the family.

Notice that we can prove case 2, on lines similar the proof of case 1, it is sufficient to use the fact that $f$ is convex if and only if $(z - \xi)f'$ is starlike.

4. **Modified Hadamard products**

For functions of the form

$$f_j(z) = (z - \xi) - \sum_{n=2}^{\infty} a_{n,j}(z - \xi)^n, \quad j = 1, 2$$

we define the modified Hadamard product as

$$(f_1* f_2)(z) = (z - \xi) - \sum_{n=2}^{\infty} a_{n,1} a_{n,2}(z - \xi)^n.$$
Theorem 4.1
If $f_j \in TS_{\xi}(\alpha, \beta, \gamma)$, $j = 1, 2$, then $(f_1 \ast f_2)(z) \in TS_{\xi}(\alpha, \beta, \gamma)$, where

$$\xi = \frac{(2 - \beta) \Theta_d(2, \alpha, \beta, \gamma) Y_m^l(2) - 2(1 - \beta)(1 - \alpha) + |\gamma|(1 - \beta)}{(2 - \beta) \Theta_d(2, \alpha, \beta, \gamma) Y_m^l(2) - (1 - \beta)(1 - \alpha) + |\gamma|(1 - \beta)},$$

with $Y_m^l(2)$ be defined as in (7).

Proof. Since $f_j \in TS_{\xi}(\alpha, \beta, \gamma)$, $j = 1, 2$, we have

$$\sum_{n=2}^{\infty} \Theta_d(n, \alpha, \beta, \gamma) Y_m^l(n) a_{n,j} \leq (1 - \alpha) + |\gamma|(1 - \beta), \quad j = 1, 2.$$

The Cauchy-Schwartz inequality leads to

$$\sum_{n=2}^{\infty} \Theta_d(n, \alpha, \beta, \gamma) Y_m^l(n) \frac{a_{n,1} a_{n,2}}{(1 - \alpha) + |\gamma|(1 - \beta)} \sqrt{a_{n,1} a_{n,2}} \leq 1. \quad (11)$$

Note that we need to find the largest $\rho$ such that

$$\sum_{n=2}^{\infty} \Theta_d(n, \alpha, \rho, \gamma) Y_m^l(n) \frac{a_{n,1} a_{n,2}}{(1 - \alpha) + |\gamma|(1 - \rho)} \leq 1. \quad (12)$$

Therefore, in view of (11) and (12), whenever

$$\frac{n - \xi}{1 - \xi} \sqrt{a_{n,1} a_{n,2}} \leq \frac{n - \beta}{1 - \beta}, \quad n \geq 2,$$

holds, then (12) is satisfied. We have, from (11),

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(n, \alpha, \beta, \gamma) Y_m^l(n)}, \quad n \geq 2. \quad (13)$$

Thus, if

$$\left(\frac{n - \xi}{1 - \xi}\right) \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(n, \alpha, \beta, \gamma) Y_m^l(n)} \leq \frac{n - \beta}{1 - \beta}, \quad n \geq 2,$$

or, if

$$\xi = \frac{(n - \beta) \Theta_d(n, \alpha, \beta, \gamma) Y_m^l(n) - n(1 - \beta)(1 - \alpha) + |\gamma|(1 - \beta)}{(n - \beta) \Theta_d(n, \alpha, \beta, \gamma) Y_m^l(n) - (1 - \beta)(1 - \alpha) + |\gamma|(1 - \beta)}, \quad n \geq 2,$$

then (11) is satisfied. Note that the right hand side of the above expression is an increasing function on $n$. Hence, setting $n = 2$ in the above inequality gives the required result. Finally, by taking the function

$$f(z) = (z - \xi) - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{(2 - \beta) \Theta_d(2, \alpha, \beta, \gamma) Y_m^l(2)} (z - \xi)^2,$$

we see that the result is sharp.
5. Integral means

In order to find the integral means inequality and to verify the Silverman Conjecture \([20]\) for \(f \in TS_\xi(\alpha, \beta, \gamma)\) we need the following subordination result due to Littlewood \([12]\).

**Lemma 5.1** \([12]\)

If the functions \(f\) and \(g\) are analytic in \(U\) with \(g \prec f\), then

\[
\frac{2\pi}{0} |g(re^{i\theta})|^\eta d\theta \leq \frac{2\pi}{0} |f(re^{i\theta})|^\eta d\theta, \quad \eta > 0, \quad z = re^{i\theta} \text{ and } 0 < r < 1.
\]

Applying Theorem 2.1 with extremal function given by (8) and Lemma 5.1, we prove the following theorem.

**Theorem 5.2**

Let \(\eta > 0\). If \(f \in TS_\xi(\alpha, \beta, \gamma)\) and \(\{\Phi(\alpha, \beta, \gamma, n)\}_{n=2}^\infty\) is non-decreasing sequence, then for \((z - \xi) = re^{i\theta}\) and \(0 < r + d < 1\) we have

\[
\frac{2\pi}{0} |f(re^{i\theta})|^\eta d\theta \leq \frac{2\pi}{0} |f_2(re^{i\theta})|^\eta d\theta,
\]

where

\[
f_2(z) = (z - \xi) - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)Y_m(2)} (z - \xi)^2.
\]

**Proof.** Let \(f(z)\) of the form (2) and

\[
f_2(z) = (z - \xi) - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)Y_m(2)} (z - \xi)^2,
\]

then we must show that

\[
\frac{2\pi}{0} \left|1 - \sum_{n=2}^\infty a_n(z - \xi)^{n-1}\right|^\eta d\theta \leq \frac{2\pi}{0} \left|1 - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)Y_m(2)} (z - \xi)\right|^\eta d\theta.
\]

By Lemma 5.1 it suffices to show that

\[
1 - \sum_{n=2}^\infty a_n(z - \xi)^{n-1} \prec 1 - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)Y_m(2)} (z - \xi).
\]

Setting

\[
1 - \sum_{n=2}^\infty a_n(z - \xi)^{n-1} = 1 - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)Y_m(2)} w(z).
\]
From (14) and (6) we obtain
\[ |w(z)| = \left| \sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \beta, \gamma) T_m'(n)}{(1 - \alpha) + |\gamma|(1 - \beta)} a_n (z - \xi)^{n-1} \right| \]
\[ \leq |z - \xi| \sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \beta, \gamma) T_m'(n)}{(1 - \alpha) + |\gamma|(1 - \beta)} a_n \]
\[ \leq |z - \xi| \left( 1 - \alpha \right) + |\gamma| (1 - \beta) \]
\[ < 1. \]

This completes the proof of the Theorem 5.2.

6. Inclusion relations involving \( N_{\delta}(e) \)

In this section following [14, 17], we define the \( n, \delta \) neighborhood of function \( f \in T_\xi \) and discuss the inclusion relations involving \( N_{\delta}(e) \).

\[ N_{\delta}(f) = \left\{ g \in T_\xi : g(z) = (z - \xi) - \sum_{n=2}^{\infty} b_n (z - \xi)^n \text{ and } \sum_{n=2}^{\infty} n |a_n - b_n| \leq \delta \right\}. \]

In particular, for the identity function \( e(z) = z \) we have

\[ N_{\delta}(e) = \left\{ g \in T_\xi : g(z) = (z - \xi) - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n |b_n| \leq \delta \right\}. \]

**Theorem 6.1**

Let
\[ \delta = \frac{2[(1 - \alpha) + |\gamma|(1 - \beta)]}{\Theta_d(2, \alpha, \beta, \gamma) T_m'(2)}, \]
where \(-1 \leq \alpha < 1, \beta \geq 0 \) and \( \gamma \in \mathbb{C} \setminus \{0\} \). Then \( TS_{\xi}(\alpha, \beta, \gamma) \subset N_{\delta}(e) \).

**Proof.** For \( f \in TS_{\xi}(\alpha, \beta, \gamma) \) Theorem 2.1 yields
\[ \Theta_d(2, \alpha, \beta, \gamma) T_m'(2) \sum_{n=2}^{\infty} a_n \leq (1 - \alpha) + |\gamma|(1 - \beta) \]
so that
\[ \sum_{n=2}^{\infty} a_n \leq \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma) T_m'(2)}. \] (15)

On the other hand, from (16) and (15) we have
Starlike functions of complex order involving \(q\)-hypergeometric functions . . .

\[
(1 - \beta)(r + d)\Upsilon_m^l(2) \sum_{n=2}^{\infty} na_n
\]

\[
\leq (1 - \alpha) + |\gamma|(1 - \beta) + [(\alpha - \beta) - |\gamma|(1 - \beta)](r + d)\Upsilon_m^l(2) \sum_{n=2}^{\infty} a_n
\]

\[
\leq (1 - \alpha) + |\gamma|(1 - \beta) \times \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{(2 - \alpha + \beta) + |\gamma|(1 - \beta)(r + d)\Upsilon_m^l(2)}
\]

\[
\leq \frac{[(1 - \alpha) + |\gamma|(1 - \beta)]2(1 - \beta)}{(2 - \alpha + \beta) + |\gamma|(1 - \beta)}. \]

Hence

\[
\sum_{n=2}^{\infty} na_n \leq \frac{2[(1 - \alpha) + |\gamma|(1 - \beta)]}{(2 - \alpha + \beta) + |\gamma|(1 - \beta)(r + d)\Upsilon_m^l(2)}
\]

and

\[
\sum_{n=2}^{\infty} na_n \leq \frac{2[(1 - \alpha) + |\gamma|(1 - \beta)]}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)} = \delta. \quad (16)
\]

Now we determine the neighborhood for each of the function class \(T_S\xi(\alpha, \beta, \gamma)\) which we define as follows:

A function \(f \in T_S\xi\) is said to be in the class \(T_S\xi(\alpha, \beta, \gamma, \eta)\) if there exists a function \(g \in T_S\xi(\alpha, \beta, \gamma)\) such that

\[
\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \eta, \quad (z - \xi) \in \mathbb{U}, \ 0 \leq \eta < 1. \quad (17)
\]

**Theorem 6.2**

If \(g \in T_S\xi(\alpha, \beta, \gamma)\) and

\[
\eta = 1 - \frac{2(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)} - 2[(1 - \alpha) + |\gamma|(1 - \beta)]. \quad (18)
\]

Then \(N_\delta(g) \subset T_S\xi(\alpha, \beta, \gamma, \eta)\).

**Proof.** Suppose that \(f \in N_\delta(g)\), then we find from (16) that

\[
\sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta,
\]

which implies the coefficient inequality

\[
\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2}.
\]

Next, since \(g \in T_S\xi(\alpha, \beta, \gamma)\), we have

\[
\sum_{n=2}^{\infty} b_n \leq \frac{2[(1 - \alpha) + |\gamma|(1 - \beta)]}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)}.
\]
Kaliappan Vijaya, Gangadharan Murugusundaramoorthy, Murugesan Kasthuri

So that
\[
\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} b_n} \\
\leq \frac{\delta}{2} \times \frac{\Theta_d(2, \alpha, \beta, \gamma) \Upsilon_m(2)}{\Theta_d(2, \alpha, \beta, \gamma) \Upsilon_m(2) - 2(1 - \alpha) + |\gamma|(1 - \beta)} \\
\leq 1 - \eta,
\]

provided that \( \eta \) is given precisely by (18). Thus by definition, \( f \in T S_{\xi}(\alpha, \beta, \gamma, \eta) \) for \( \eta \) given by (18), which completes the proof.

Concluding Remarks: By suitably specializing the various parameters involved in Theorem 6 to Theorem 6.2 we can state the corresponding results for the new subclasses defined in Example 1 and also for many relatively more familiar function classes.

Acknowledgement: The authors thank the referee for his insightful suggestions to improve this paper in the present form.

References


Starlike functions of complex order involving $q$-hypergeometric functions . . .


School of Advanced Sciences  
VIT University  
Vellore - 632014  
India  
E-mail: kvijaya@vit.ac.in  
gmsmoorthy@yahoo.com

Received: February 27, 2014; final version: June 3, 2014;  
available online: July 3, 2014.