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Perturbation of Toeplitz operators and reflexivity

Abstract. It was shown that the space of Toeplitz operators perturbated by finite rank operators is 2-hyperreflexive.

1. Introduction

In [6] it was shown that the rank one perturbation preserves 2-hyperreflexivity of Toeplitz operators. In this paper we will generalise this result for a finite rank perturbation.

Let us start with basic notations and definitions. For a Hilbert space $\mathcal{H}$ we will write $\mathcal{B}(\mathcal{H})$ for the algebra of all bounded linear operators on $\mathcal{H}$.

By τc denote the space of trace class operators (which is predual to $\mathcal{B}(\mathcal{H})$ with the dual action $\langle S, t\rangle = \mathrm{tr}(St)$ for $S \in \mathcal{B}(\mathcal{H})$ and $t \in \tau c$) equipped with the trace norm $\|\cdot\|_1$. Let $F_k = \{t \in \tau c : \ \mathrm{rank}(t) \le k\}$. Each rank one operator can be written as $x \otimes y$, for $x, y \in \mathcal{H}$, and $(x \otimes y)z = \langle z, y\rangle x$ for $z \in \mathcal{H}$. Moreover, $\mathrm{tr}(S(x \otimes y)) = \langle Sx, y\rangle$.

Let us now recall the definition of reflexivity. *The reflexive closure* of a subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is given by the formula

$$\mathrm{ref}\,\mathcal{M} = \{A \in \mathcal{B}(\mathcal{H}) : \ Ah \in [\mathcal{M}h] \text{ for all } h \in \mathcal{H}\},$$

here $[\cdot]$ denotes the norm-closure. If $\mathcal{M} = \mathrm{ref}\,\mathcal{M}$ then $\mathcal{M}$ is said to be *reflexive*. It is known (see [10]) that if subspace $\mathcal{M}$ is a weak* closed, then $\mathcal{M}$ is reflexive if and only if operators of rank one are linearly dense in $\mathcal{M}_\perp$ (i.e., $\mathcal{M}_\perp = [\mathcal{M}_\perp \cap F_1]$), where $\mathcal{M}_\perp$ is the preannihilator of $\mathcal{M}$.

A subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called *k-reflexive* if $\mathcal{M}^{(k)} = \{T^{(k)} : \ T \in \mathcal{M}\}$ is reflexive in $B(\mathcal{H}^{(k)})$, where $T^{(k)} = T \oplus \cdots \oplus T$ and $\mathcal{H}^{(k)} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Similarly as before, in case of weak* closed subspaces we have an equivalent condition to k-reflexivity proved by Kraus and Larson [9, Theorem 2.1]. Namely, a weak* closed subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is k-reflexive if and only if $\mathcal{M}_\perp = [\mathcal{M}_\perp \cap F_k]$.

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For a closed subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ denote by $d(A, \mathcal{M})$ the usual distance from an operator A to a subspace $\mathcal{M}$, i.e., $d(A, \mathcal{M}) = \inf\{\|A - T\| : T \in \mathcal{M}\}$. When $\mathcal{M}$ is weak* closed then $d(A, \mathcal{M}) = \sup\{|\operatorname{tr}(At)| : t \in \mathcal{M}_\perp, \|t\|_1 \leq 1\}$.

Hyperreflexivity was introduced by Arveson in [2] for operator algebras. In [8] his definition was generalized to the operator subspaces. Namely, a subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is said to be *hyperreflexive* if there is a constant c such that

$$d(A, \mathcal{M}) \leq c \sup\{\|Q^\perp AP\| : P, Q \text{ are projections such that } Q^\perp \mathcal{M} P = 0\}$$

for all $A \in \mathcal{B}(\mathcal{H})$. In [9] it was shown that the supremum on the right hand side is equal to $\sup\{|\langle A, x \otimes y\rangle| : x \otimes y \in \mathcal{M}_\perp, \|x \otimes y\|_1 \leq 1\}$.

Let us recall the definition of k-hyperreflexivity from [7]. For a subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and an operator $A \in \mathcal{B}(\mathcal{H})$ denote by

$$\alpha_k(A, \mathcal{M}) = \sup\{|\operatorname{tr}(At)| : t \in \mathcal{M}_\perp \cap F_k, \|t\|_1 \leq 1\}.$$

A subspace $\mathcal{M}$ is *k-hyperreflexive* if there is a constant $c > 0$ such that

$$d(A, \mathcal{M}) \leq c\alpha_k(A, \mathcal{M}) \tag{1}$$

for any $A \in \mathcal{B}(\mathcal{H})$. The constant of k-hyperreflexivity is the infimum of all constants c such that (1) holds and is denoted by $\kappa_k(\mathcal{M})$.

2. Finite rank perturbation of Toeplitz operators

Denote by H^2 the classical Hardy space on the unit circle $\mathbb{T}$ and let $P_{H^2} : L^2 \to H^2$ be the orthogonal projection. The *Toeplitz operator* with the symbol $\varphi \in L^\infty$ is defined as follows $T_\varphi : H^2 \to H^2$ and $T_\varphi f = P_{H^2}(\varphi f)$ for $f \in H^2$. Let $\mathcal{T}$ denote the space of all Toeplitz operators.

It is well known that $\mathcal{T} = \{T_\varphi : \varphi \in L^\infty\} = \{A : T_z^* A T_z = A\}$ (see [5, Corollary 1 to Problem 194]). Therefore $\mathcal{T}$ is closed in weak* topology.

Let $\{e_j\}_{j \in \mathbb{N}}$ be the usual basis in H^2. Let J be a finite subset of $\mathbb{N} \times \mathbb{N}$. Denote by $\mathcal{S}_J = \operatorname{span}\{e_i \otimes e_j : (i, j) \in J\}$ and consider the subspace

$$\mathcal{S} = \mathcal{T} + \mathcal{S}_J = \operatorname{span}\{T_\varphi + g : \varphi \in L^\infty, g \in \mathcal{S}_J\}.$$

Notice that $\mathcal{S}$ is weak* closed. It was shown in [3, Theorem 3.1] that $\mathcal{T}$ is not reflexive but it is 2-reflexive. In [6] similar result was obtained for Toeplitz operators perturbated by rank one operator. In this paper we will prove the same for the subspace $\mathcal{S}$.

Proposition 1
The subspace $\mathcal{S} = \mathcal{T} + \mathcal{S}_J$ is not reflexive but it is 2-reflexive.

Proof. It is easy to see that $(\mathcal{S})_\perp = \mathcal{T}_\perp \cap (\mathcal{S}_J)_\perp$. Because there is no rank one operator in $\mathcal{T}_\perp$, hence $\mathcal{S}$ cannot be reflexive.

On the other hand, $\mathcal{T}_\perp = \operatorname{span}\{e_i \otimes e_j - Se_i \otimes Se_j : \ i, j = 1, 2, \ldots\}$, where S denotes the unilateral shift. Hence

$$(\mathcal{S})_\perp = \operatorname{span}\{e_i \otimes e_j - Se_i \otimes Se_j : \ i, j = 1, 2, \ldots, \\ (i, j) \neq J \text{ and } (i+1, j+1) \neq J\}.$$

That implies 2-reflexivity of $\mathcal{S}$.

In [4] Davidson proved hyperreflexivity of the algebra of all analytic Toeplitz operators. Since the space $\mathcal{T}$ is not reflexive it cannot be hyperreflexive, but we know due to [7, 11] that $\mathcal{T}$ is 2-hyperreflexive with $\kappa_2(\mathcal{T}) \leq 2$. Now we will prove that the finite rank perturbation preserves 2-hyperreflexivity of $\mathcal{T}$. The projection $\pi \colon \mathcal{B}(H^2) \to \mathcal{T}$ given by Arveson in [1] will be a useful tool in the proof.

Proposition 2
The subspace $\mathcal{S} = \mathcal{T} + \mathcal{S}_J$ is 2-hyperreflexive with constant $\kappa_2(\mathcal{S}) \leq 2$.

Proof. Let $\pi \colon \mathcal{B}(H^2) \to \mathcal{T}$ be the projection defined in [1, Proposition 5.2]. This projection has the property that for any $B \in \mathcal{B}(H^2)$ the operator $\pi(B)$ belongs to the weak* closed convex hull of $\{T_{z^n}^* B T_{z^n} : \ n \in \mathbb{N}\}$.

Let $A \in \mathcal{B}(H^2) \setminus \mathcal{S}$ and $A = (a_{ij})_{i,j \in \mathbb{N}}$. Since J is a finite set, there is $r \in \mathbb{N}$ such that for every $(i, j) \in J$ we have $(i+r, j+r) \notin J$. For each $(i, j) \in J$ we define $\lambda_{ij} = a_{ij} - a_{i+r,j+r}$ and put $\tilde{A} = A - \sum_{(i,j) \in J} \lambda_{ij} e_i \otimes e_j$. Notice that $\pi(A) = \pi(\tilde{A})$.

Observe that for any $\lambda \in \mathbb{C}$,

$$d(A, \mathcal{S}) \leq \Big\| A - \pi(A) - \sum_{(i,j) \in J} \lambda_{ij} e_i \otimes e_j \Big\| = \|\tilde{A} - \pi(\tilde{A})\|.$$

In [7] it was shown that the space of Toeplitz operators $\mathcal{T}$ is 2-hyperreflexive with constant at most 2. Using similar calculations as in [7] we obtain that

$$d(\tilde{A}, \mathcal{T}) \leq \|\tilde{A} - \pi(\tilde{A})\| \leq 2\alpha_2(\tilde{A}, \mathcal{T}).$$

Now we will show that

$$\alpha_2(\tilde{A}, \mathcal{T}) = \alpha_2(A, \mathcal{S}). \tag{2}$$

Firstly, note that $\alpha_2(\tilde{A}, \mathcal{T}) \geq \alpha_2(A, \mathcal{S})$ and

$$\alpha_2(\tilde{A}, \mathcal{T}) = \sup\{|\operatorname{tr}(\tilde{A}t)| : \ 2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k},\, k \geq 1,\, i, j = 0, 1, 2, \ldots\}.$$

If the supremum above is realized by $2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k}$ for $(i, j) \notin J$ and $(i+k, j+k) \notin J$, then we have the equality (2).

Now consider the case, when $2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k}$ and $(i, j) \in J$ and $(i+k, j+k) \notin J$. Then

$$|\operatorname{tr}(\tilde{A}t)| = \frac{1}{2}|a_{ij} - \lambda_{ij} e_i \otimes e_j - a_{i+k,j+k}| = \frac{1}{2}|a_{i+r,j+r} - a_{i+k,j+k}| \leq \alpha_2(A, \mathcal{S})$$

(since $e_{i+r} \otimes e_{j+r} - e_{i+k} \otimes e_{j+k} \in \mathcal{S}_\perp$).

Similarly, if $2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k}$ and $(i,j) \notin J$ and $(i+k, j+k) \in J$, then

$$|\operatorname{tr}(\tilde{A}t)| = \frac{1}{2}|a_{ij} - a_{i+k+r,j+k+r}| \leq \alpha_2(A, \mathcal{S}).$$

Finally, if $2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k}$ and $(i,j) \in J$ and $(i+k, j+k) \in J$, then

$$|\operatorname{tr}(\tilde{A}t)| = \frac{1}{2}|a_{i+k+r,j+k+r}| \leq \alpha_2(A, \mathcal{S}).$$

We obtained that $\alpha_2(\tilde{A}, \mathcal{T}) = \alpha_2(A, \mathcal{S})$ and the proof is completed.

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