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On type-2 m-topological spaces

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Abstract. In the present paper, we define a notion of an m2-topological space by introducing a count of openness of a multiset (mset in short) and study the properties of m2-subspaces, mgp-maps etc. Decomposition theorems involving m-topologies and m2-topologies are established. The behaviour of the functional image and functional preimage of an m2-topologies, the continuity of the identity mapping and a constant mapping in m2-topologies are also examined.

1. Introduction

A classical set is a collection of objects where an object can occur only once. But there are a number of situations in science and real life where the repetition of an object is significant. Allowing repetition of elements, N. G. de Bruijn [4] first suggested to generalize classical sets to multisets (msets in short) in a private communication to D. E. Knuth. These sets are very useful structures arising in many areas of mathematics and computer science such as in prime factorization of integers, invariants of matrices in canonical form, zeros and poles of meromorphic functions, multicriteria decision making, knowledge representation in data based systems, biological systems membrane computing etc. Several researchers have worked in variety of terms viz. list, heap, bunch, bag, sample, weighted set, occurrence set and fireset used in different contexts but conveying synonymity with mset. Many authors like Yager [23], Miyamoto [17], Hickman [13], Blizard [3], Girish and John [6, 7], Hallez et al. [10] etc., have studied the set theoretic properties of msets. Some hybridizations of msets may be found in [1, 2, 12, 16]. Structural study, such as topological, are found in [8, 9, 20, 22], algebraic in

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[18, 19]. Note that the m-topology defined on msets by Girish and John [8, 9] as actually an ordinary set τ of some msets. In this paper, an attempt has been made in allowing the repetition of members of m-topology τ. A definition of type-2 m-topology is introduced which will be called m2-topology. The relevance of this approach in fuzzy setting have been done by A. Šostak [21], M. S. Ying [24], U. Höhle and A. Šostak [14], T. Kubiak [15], and Hazra, Chattopadhyay and Samanta [5, 11]. In brief, in this paper, we have defined a notion of an m2-topological space by introducing a count of openness of an mset, m2-cotopological space by introducing a count of closedness of an mset. Moreover, m2-subspaces, mgp-mappings and some of their important properties are studied. Decomposition theorems involving m-topologies and m2-topologies are established. The behaviour of functional image and functional preimage of an m2-topology, the continuity of the identity mapping and a constant mapping in m2-topologies are also examined.

2. Preliminaries

This section consists of some definitions and results of msets and m-topologies which will be used in the main works of the paper. Unless otherwise stated, X will be assumed to be an initial universal set and $\mathbb{N}$ represents the set of all non negative integers.

2.1. Multi sets (or msets)

Definition 2.1 ([7])
An mset M drawn from the universal set X is represented by a count function C_M defined as $C_M\colon X \to \mathbb{N}$, where $\mathbb{N}$ represents the set of non negative integers. Here $C_M(x)$ is the number of occurrences of the element x in the mset M. The presentation of the mset M drawn from $X = \{x_1, x_2, \dots, x_n\}$ will be as $M = \{x_1/m_1, x_2/m_2, \dots, x_n/m_n\}$, where m_i is the number of occurrences of the element x_i,$i = 1, 2, \dots, n$ in the mset M. Also here for any positive integer w, $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times and $[X]^\infty$ is the set of all msets whose elements are in X such that there is no limit on the number of occurrences of an element in an mset. As in [7], $[X]^w$ and $[X]^\infty$ will be referred to as mset spaces. $MS(X)$ denotes the set of all msets drawn from X.

Definition 2.2 ([7])
Let M_1 and M_2 be two msets drawn from a set X. Then M_1 is said to be *submset* of M_2 if $C_{M_1}(x) \le C_{M_2}(x)$ for all $x \in X$. This relation is denoted by $M_1 \subseteq M_2$. Set M_1 is said to be *equal* to M_2 if $C_{M_1}(x) = C_{M_2}(x)$ for all $x \in X$, which will be denoted by $M_1 = M_2$.

Definition 2.3 ([7])
Let w be a positive integer and $\{M_i; i \in I\}$ be a non-empty family of msets in $[X]^w$. Then

(a) the *intersection* of the sets M_i, is a set denoted by $\bigcap_{i\in I} M_i$, such that

$$C_{\bigcap_{i\in I} M_i}(x) = \bigwedge_{i\in I} C_{M_i}(x) \quad \text{for all } x \in X;$$

(b) the *union* of the sets M_i, is a set denoted by $\bigcup_{i\in I} M_i$, such that

$$C_{\bigcup_{i\in I} M_i}(x) = \bigvee_{i\in I} C_{M_i}(x) \quad \text{for all } x \in X;$$

(c) the *complement* of any mset M_i in $[X]^w$ is a set denoted by M_i^c, such that $C_{M_i^c}(x) = w - C_{M_i}(x)$ for all $x \in X$.

Definition 2.4 ([18])
Let X and Y be two non-empty sets and $f\colon X \to Y$ be a mapping. Then

(i) the *image* of an mset $M \in [X]^w$ under the mapping f is a set denoted by $f(M)$, such that

$$C_{f(M)}(y) = \begin{cases} \bigvee_{f(x)=y} C_M(x), & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{otherwise;} \end{cases}$$

(ii) the *inverse image* of an mset $N \in [Y]^w$ under the mapping f is a set denoted by $f^{-1}(N)$, such that $C_{f^{-1}(N)}(x) = C_N[f(x)]$.

Proposition 2.5 ([18])
Let X, Y and Z be three non-empty sets and $f\colon X \to Y$, $g\colon Y \to Z$ be two mappings. If $M_i \in [X]^w$, $N_i \in [Y]^w$, $i \in I$ then

(i) $M_1 \subseteq M_2 \Rightarrow f(M_1) \subseteq f(M_2)$;

(ii) $f\big[\bigcup_{i\in I} M_i\big] = \bigcup_{i\in I} f[M_i]$;

(iii) $N_1 \subseteq N_2 \Rightarrow f^{-1}(N_1) \subseteq f^{-1}(N_2)$;

(iv) $f^{-1}\big[\bigcup_{i\in I} M_i\big] = \bigcup_{i\in I} f^{-1}[M_i]$;

(v) $f^{-1}\big[\bigcap_{i\in I} M_i\big] = \bigcap_{i\in I} f^{-1}[M_i]$;

(vi) $f(M_i) \subseteq N_j \Rightarrow M_i \subseteq f^{-1}[N_j]$;

(vii) $g[f(M_i)] = [gf](M_i)$ *and* $f^{-1}[g^{-1}(N_j)] = [gf]^{-1}(N_j)$.

Proposition 2.6 ([18])
Let X and Y be two non-empty sets and $f\colon X \to Y$ be a mapping. If $M \in [X]^w$ and $N \in [Y]^w$, then

(i) $M \subseteq f^{-1}[f(M)]$;

(ii) $f^{-1}[f(M)] = M$, *if f is injective;*

(iii) $f[f^{-1}(N)] \subseteq N$;

(iv) $f[f^{-1}(N)] = N$, *if f is surjective.*

Definition 2.7 ([18])
Let $P \subseteq X$. Then for each $n \in \mathbb{N}$, we define an mset ${}_nP$ over X, where $C_{{}_nP}(x) = n$ for all $x \in P$. This msets are called *level msets*.

2.2. Msets topology

Definition 2.8 ([8])
Let $M \in [X]^w$ be a multiset and $P^*(M)$ be the collection of all submsets of M. A subcollection τ of $P^*(M)$ is said to be a *multiset topology* (*m-topology* in short) on M if

(i) $M, \emptyset \in \tau$;

(ii) the intersection of any two msets in τ belongs to τ;

(iii) the union of any number of msets in τ belongs to τ.

The pair (M, τ) is called an *m-topological space* on M.

Definition 2.9 ([9])
Let (M, τ) be an m-topological space and N be a submset of M. The collection $\tau_N = \{N \cap U : U \in \tau\}$ is an m-topology on N, called a *subspace m-topology*.

Definition 2.10 ([9])
Let M and N be two m-topological spaces. The mset function $f\colon M \to N$ is said to be continuous if for each open submset V of N, the mset $f^{-1}(V)$ is an open submset of M, where $f^{-1}(V)$ is the mset of all points x/m in M for which $f(x/m) \in {}_nV$ for some n.

3. m2-topological spaces

In this section, we introduce a count of openness, a count of closedness, m2-topological spaces, m2-cotopological spaces, m2-subspaces, mgp-maps and some of their important properties are studied. Decomposition theorems involving m-topologies and m2-topologies are established. The behaviour of the functional image and the functional preimage of an m2-topology, the continuity of the identity mapping and a constant mapping in m2-topologies are also examined.

Unless otherwise stated, X denotes a non-empty set, w is a positive integer, $\mathbb{N}$ denotes the set of all non negative integers, $\mathbb{N}_w$ is the set of all non negative integers not greater than w and $[X]^w$ is the collection of all those msets whose elements are in X such that no element in the mset occurs more than w times.

Definition 3.1
A mapping $\tau\colon [X]^w \to \mathbb{N}_w$ is called a *count of openness* (CO) or an *m2-topology* on $[X]^w$ if it satisfies the following conditions:

(O1) $\tau({}_0X) = \tau({}_wX) = w$;

(O2) $\tau(M_1 \cap M_2) \geq \tau(M_1) \wedge \tau(M_2)$ for $M_1, M_2 \in [X]^w$,

(O3) $\tau\big[\bigcup_{i\in\Delta} M_i\big] \geq \wedge_{i\in\Delta}\tau(M_i)$ for any $M_i \in [X]^w$, $i \in \Delta$.

The pair $([X]^w, \tau)$ is called an *m2-topological space* (m2ts).

Example 3.2
Let $\tau_0\colon [X]^w \to \mathbb{N}_w$, $\tau_w\colon [X]^w \to \mathbb{N}_w$ be two mappings defined by $\tau_0({}_0X) = \tau_0({}_wX) = w$, $\tau_0(M) = 0$ for all $M \in [[X]^w - \{{}_0X, {}_wX\}]$ and $\tau_w(M) = w$ for all $M \in [X]^w$. Then τ_0 and τ_w are two m2-topologies on $[X]^w$.

Definition 3.3
A mapping $\mathcal{F}\colon [X]^w \to \mathbb{N}_w$ is called a *count of closedness* (CC) on $[X]^w$ if it satisfies the following conditions:

(C1) $\mathcal{F}({}_0X) = \mathcal{F}({}_wX) = w$;

(C2) $\mathcal{F}(M_1 \cup M_2) \geq \mathcal{F}(M_1) \wedge \mathcal{F}(M_2)$ for $M_1, M_2 \in [X]^w$;

(C3) $\mathcal{F}\big[\bigcap_{i\in\Delta} M_i\big] \geq \wedge_{i\in\Delta}\mathcal{F}(M_i)$ for any $M_i \in [X]^w$, $i \in \Delta$.

The pair $([X]^w, \mathcal{F})$ is called an *m2-cotopological space.*

Proposition 3.4
Let τ and $\mathcal{F}$ be a count of openness and a count of closedness of $[X]^w$, respectively. Then the mapping $\mathcal{F}_\tau\colon [X]^w \to \mathbb{N}_w$, defined by $\mathcal{F}_\tau(M) = \tau(M^c)$ is a count of closedness on $[X]^w$.

Proof. Let τ, $\mathcal{F}$ be a count of openness and a count of closedness of $[X]^w$ respectively and $\mathcal{F}_\tau\colon [X]^w \to \mathbb{N}_w$ be a mapping defined by $\mathcal{F}_\tau(M) = \tau(M^c)$. Since $({}_0X)^c = {}_wX$ and $({}_wX)^c = {}_0X$, it follows that $\mathcal{F}_\tau({}_0X) = \tau({}_wX) = w$ and $\mathcal{F}_\tau({}_wX) = \tau({}_0X) = w$.

Next let M_1, M_2 be any two members of $[X]^w$. Then

$$\begin{aligned}\mathcal{F}_\tau(M_1 \cup M_2) &= \tau([M_1 \cup M_2]^c) = \tau(M_1^c \cap M_2^c)\\ &\geq \tau(M_1^c) \wedge \tau(M_2^c) = \mathcal{F}_\tau(M_1) \wedge \mathcal{F}_\tau(M_2).\end{aligned}$$

Again let M_i, $i \in \Delta$ be any collection of members of $[X]^w$. Then

$$\begin{aligned}\mathcal{F}_\tau\big(\textstyle\bigcap_{i\in\Delta} M_i\big) &= \tau\big([\textstyle\bigcap_{i\in\Delta} M_i]^c\big) = \tau\big(\textstyle\bigcup_{i\in\Delta} M_i^c\big)\\ &\geq \wedge_{i\in\Delta}\tau(M_i^c) = \wedge_{i\in\Delta}\mathcal{F}_\tau(M_i).\end{aligned}$$

Therefore, the mapping $\mathcal{F}_\tau\colon [X]^w \to \mathbb{N}_w$, defined by $\mathcal{F}_\tau(M) = \tau(M^c)$, is a count of closedness on $[X]^w$.

Proposition 3.5
Let τ and $\mathcal{F}$ be a count of openness and a count of closedness of $[X]^w$, respectively. Then the mapping $\tau_{\mathcal{F}}\colon [X]^w \to \mathbb{N}_w$ defined by $\tau_{\mathcal{F}}(M) = \mathcal{F}(M^c)$ is a count of openness on $[X]^w$.

Proof. Proof is similar to that of Proposition 3.4.

Proposition 3.6
Let τ and $\mathcal{F}$ be a count of openness and a count of closedness on $[X]^w$, respectively. Then $\tau_{\mathcal{F}_\tau} = \tau$ and $\mathcal{F}_{\tau_{\mathcal{F}}} = \mathcal{F}$.

Proof. Proof is straightforward.

Proposition 3.7
Let τ_1 and τ_2 be two counts of openness on $[X]^w$. Then $\tau = \tau_1 \cap \tau_2$ defined by $\tau(M) = \tau_1(M) \wedge \tau_2(M)$ is a count of openness on $[X]^w$.

Proof. Clearly $\tau({}_0X) = \tau({}_wX) = w$. Next, let M_1, M_2 be any two members of $[X]^w$. Then

$$\begin{aligned}
\tau(M_1 \cap M_2) &= \tau_1(M_1 \cap M_2) \wedge \tau_2(M_1 \cap M_2)\\
&\geq [\tau_1(M_1) \wedge \tau_1(M_2)] \wedge [\tau_2(M_1) \wedge \tau_2(M_2)]\\
&= [\tau_1 \cap \tau_2](M_1) \wedge [\tau_1 \cap \tau_2](M_2)\\
&= \tau(M_1) \wedge \tau(M_2).
\end{aligned}$$

Again let M_i, $i \in \Delta$ be any collection of members of $[X]^w$. Then

$$\begin{aligned}
\tau\big(\textstyle\bigcup_{i\in\Delta} M_i\big) &= \tau_1\big(\textstyle\bigcap_{i\in\Delta} M_i\big) \wedge \tau_2\big(\textstyle\bigcap_{i\in\Delta} M_i\big)\\
&\geq [\wedge_{i\in\Delta}\tau_1(M_i)] \wedge [\wedge_{i\in\Delta}\tau_2(M_i)]\\
&= \wedge_{i\in\Delta}[\tau_1 \cap \tau_2](M_i)\\
&= \wedge_{i\in\Delta}\tau(M_i).
\end{aligned}$$

Therefore, $\tau = \tau_1 \cap \tau_2$, defined by $\tau(M) = \tau_1(M) \wedge \tau_2(M)$, is a count of openness on $[X]^w$.

Remark 3.8
If $\{\tau_i,\ i \in \Delta\}$ is any arbitrary family of counts of openness on $[X]^w$, then their intersection $\tau = \bigcap_{i\in\Delta} \tau_i$, defined by $\tau(M) = \wedge_{i\in\Delta}\tau_i(M)$ for all $M \in [X]^w$, is a count of openness on $[X]^w$.

Definition 3.9
Let τ_1 and τ_2 be two counts of openness on $[X]^w$. Define

$$\tau_1 \leq \tau_2 \text{ iff } \tau_1(M) \leq \tau_2(M) \quad \text{for all } M \in [X]^w.$$

If $\tau_1 \leq \tau_2$ then we say that τ_1 *is coarser or weaker or smaller than* τ_2 and τ_2 is *finer or stronger or larger than* τ_1.

Proposition 3.10
Let $\mathcal{T}$ be the collection of all counts of openness on $[X]^w$. Then $(\mathcal{T}, \leq)$ is a complete lattice.

Proof. Let τ_0, τ_w be two counts of openness on $[X]^w$ defined in Example 3.2. Then $\tau_0 \leq \tau \leq \tau_w$ for all $\tau \in \mathcal{T}$. Note that $\tau_1 \cap \tau_2$ is the greatest lower bound (glb) of τ_1 and τ_2 for all $\tau_1, \tau_2 \in \mathcal{T}$.

Moreover, $\bigcap\{\tau \in \mathcal{T} : \ \tau_1 \leq \tau \text{ and } \tau_2 \leq \tau\}$ is the least upper bound (lub) of τ_1 and τ_2 for all $\tau_1, \tau_2 \in \mathcal{T}$ (we note that there exists at least one count of openness viz. τ_w, which is finer than both τ_1 and τ_2). Therefore, $(\mathcal{T}, \leq)$ is a complete lattice.

Proposition 3.11 (First Decomposition Theorem)
Let $([X]^w, \tau)$ be an m-topological space, where τ is a count of openness on $[X]^w$. Then for each $r (\in \mathbb{N}_w) \leq w$, $\tau^r = \{M \in [X]^w : \ \tau(M) \geq r\}$ is a multiset topology on ${}_wX$.

Proof. Since $\tau({}_0X) = \tau({}_wX) = w \geq r$, it follows that ${}_0X, {}_wX \in \tau_r$. Next let $M_1, M_2 \in [X]^w$ be any two members of τ^r. Then $\tau^r(M_1) \geq r$ and $\tau^r(M_2) \geq r$. Since τ is a count of openness on $[X]^w$, it follows that $\tau(M_1 \cap M_2) \geq [\tau(M_1) \wedge \tau(M_2)] \geq r$. Hence $M_1 \cap M_2 \in \tau^r$.

Furthermore let, $\{M_i \in [X]^w, \ i \in \Delta\}$ be any collection of members of τ^r. Then $\tau(M_i) \geq r$ for all $i \in \Delta$. So, $\tau\big(\bigcup_{i\in\Delta} M_i\big) \geq \wedge_{i\in\Delta}\tau(M_i) \geq r$. Thus, $\bigcup_{i\in\Delta} M_i \in \tau^r$.

Therefore, τ_r is an m-topology on ${}_wX$.

Definition 3.12
For each $r (\in \mathbb{N}_w) \leq w$, the family τ^r, defined in Proposition 3.11, is called the *r-level m-topology on ${}_wX$ with respect to the count of openness τ.*

Proposition 3.13
Let $([X]^w, \tau)$ be an m-topological space and $\{\tau^r : \ r \leq w\}$ be the family of all r-level m-topologies with respect to τ. Then this family is a descending family of m-topologies.

Proof. Let $r \geq s$ and $M \in \tau^r$. Then $\tau(M) \geq r \geq s$, hence $M \in \tau^s$. Thus $\tau^r \subseteq \tau^s$ and hence the family $\{\tau^r : \ r \leq w\}$ is descending family of m-topologies.

Definition 3.14
Let τ be a count of openness on $[X]^w$. Then *supp*$(\tau) = \{M \in [X]^w : \ \tau(M) > 0\}$ is called the *support set of τ*.

It is clear that *supp*(τ) is an m-topology on ${}_wX$.

Definition 3.15
Let T be an m-topology on ${}_wX$. Then a count of openness τ on $[X]^w$ is said to be compatible with T if *supp*$(\tau) = T$.

Proposition 3.16
Let T be an m-topology on ${}_wX$. Then for each $r \leq w$ there exists a count of openness T^r on $[X]^w$ compatible with T.

Proof. For each $r \leq w$ we define a mapping $T^r \colon [X]^w \to \mathbb{N}_w$ by

$$T^r(M) = \begin{cases} w, & \text{if } M \in \{{}_0X, {}_wX\}, \\ r, & \text{if } M \in T - \{{}_0X, {}_wX\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, clearly T^r is a count of openness on $[X]^w$ compatible with T.

Proposition 3.17 (Second Decomposition Theorem)
Let $\{T_r : \ r \leq w\}$ be a non-empty descending family of m-topologies on ${}_wX$. Then the mapping $\tau \colon [X]^w \to \mathbb{N}_w$ defined by $\tau(M) = \vee\{r \leq w : \ M \in T_r\}$ is a count of openness on $[X]^w$ and $T_r = \tau^r$ holds for all $r \leq w$.

Proof. Since ${}_0X, {}_wX \in T_r$ for all $r \leq w$, it follows that $\tau({}_0X) = \tau({}_wX) = w$. Let M_1, M_2 be any two members of $[X]^w$ and let $\tau(M_i) =: k_i$ for $i = 1, 2$. If $k_i = 0$ for some i, then obviously $\tau(M_1 \cap M_2) \geq \tau(M_1) \wedge \tau(M_2)$.

Assume now that $k_1 \neq 0$, $k_1 \neq 0$ and $k = k_1 \wedge k_2$. Since $\{T_r : \ r \leq W\}$ is a descending family of m-topologies, it follows that $M_1, M_2 \in T_k$ and hence $M_1 \cap M_2 \in T_k$. Thus

$$\begin{aligned} \tau(M_1 \cap M_2) &= \vee\{r \leq w : \ (M_1 \cap M_2) \in T_r\} \geq k \\ &= k_1 \wedge k_2 = \tau(M_1) \wedge \tau(M_2). \end{aligned}$$

Moreover, let M_i for $i \in \Delta$ be any collection of members of $[X]^w$ and let $\tau(Mi) =: l_i$ for $i \in \Delta$. If $l_i = 0$ for some $i \in \Delta$, then obviously

$$\tau\big(\textstyle\bigcup_{i \in \Delta} M_i\big) \geq \wedge_{i \in \Delta} \tau(M_i).$$

Now let $l_i \neq 0$ for all $i \in \Delta$ and $l = \wedge_{i \in \Delta} l_i$. Since $\{T_r : \ r \leq w\}$ is a descending family of m-topologies, it follows that $M_i \in T_l$, $i \in \Delta$ and hence $\bigcup_{i \in \Delta} M_i \in T_l$. Thus

$$\tau\big(\textstyle\bigcup_{i \in \Delta} M_i\big) = \vee\big\{r \leq w : \ \bigcup_{i \in \Delta} M_i \in T_r\big\} \geq l = \wedge_{i \in \Delta} l_i = \wedge_{i \in \Delta} \tau(M_i).$$

Therefore, the mapping $\tau \colon [X]^w \to \mathbb{N}_w$ defined by $\tau(M) = \vee\{r \leq w : \ M \in T_r\}$ is a count of openness on $[X]^w$.

For second part, let us assume first that $M \in T_r$. Then $\tau(M) \geq r$ and hence $M \in \tau^r$. Thus

$$T_r \subseteq \tau^r. \tag{1}$$

Next let $M \in \tau^r$. Then $\tau(M) \geq r$ this implies that there exists $s \geq r$ such that $M \in T_s$. Since $\{T_r : \ r \leq w\}$ is a descending family of m-topologies and $s \geq r$, it follows that $M \in T_r$. Thus

$$\tau^r \subseteq T_r. \tag{2}$$

From (1) and (2), we have $T_r = \tau^r$.

Proposition 3.18
Let τ_1 and τ_2 be two counts of openness on $[X]^w$. Then $\tau_1 = \tau_2$ if and only if, $\tau_1^r = \tau_2^r$ for all $r \leq w$.

Proof. First let $\tau_1 = \tau_2$. Then $\tau_1(M) = \tau_2(M)$ for all $M \in [X]^w$, so for each $r \le w$, $\tau_1^r = \{M \in [X]^w : \tau_1(M) \ge r\} = \{M \in [X]^w : \tau_2(M) \ge r\}$. Thus $\tau_1^r = \tau_2^r$ for all $r \le w$.

Next let $\tau_1^r = \tau_2^r$ for all $r \le w$. If $\tau_1 \ne \tau_2$, then there exists an $M \in [X]^w$ such that $\tau_1(M) \ne \tau_2(M)$. Let $\tau_1(M) = s_1$, $\tau_2(M) = s_2$ and $s_1 < s_2$. Then $M \in \tau_2^{s_1+1}$ but $M \notin \tau_1^{s_1+1}$, which contradicts our assumption that $\tau_1^r = \tau_2^r$ for all $r \le w$. Therefore $\tau_1 = \tau_2$.

Proposition 3.19
Let T be an m-topology on ${}_wX$. For each $r \le w$ define a mapping $T^r : [X]^w \to \mathbb{N}_w$ by

$$T^r(M) = \begin{cases} w, & \text{if } M \in \{{}_0X, {}_wX\}, \\ r, & \text{if } M \in T - \{{}_0X, {}_wX\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then T^r a count of openness on $[X]^w$ such that $(T^r)_r = T$.

Proof. Proof follows from Proposition 3.16.

Definition 3.20
Let T be an m-topology on ${}_wX$. Then T^r, defined in Proposition 3.19, is called an *r-th count on ${}_wX$* and $([X]^w, T^r)$ is called an *r-th count m2-topological spaces*.

Definition 3.21
Let $M \in [X]^w$ and let τ be a CO on $[X]^w$. Let $\tau_M^r := \{M \cap P : P \in \tau^r\}$ for $r \le w$. Then $\{\tau_M^r : r \le w\}$ is a descending family of subspace m-topology on M.

Definition 3.22
Let X be a non-empty set and $M(\ne {}_0X) \in [X]^w$. A mapping $\tau^M : [X]^w \to \mathbb{N}_w$ is called a *subspace m2-topology* or a *subspace count of openness* (briefly SCO) on M if it satisfies the following conditions:

(i) $\tau^M(M \cap {}_0X) = \tau^M(M \cap {}_wX) = w$;

(ii) if $M_1, M_2, \ldots, M_n \subseteq M$, then $\tau^M(\bigcap_{i=1}^n M_i) \ge \wedge_{i=1}^n \tau^M(M_i)$;

(iii) if $M_i \subseteq M$, $i \in \Delta$, then $\tau^M(\bigcup_{i\in\Delta}^n M_i) \ge \wedge_{i\in\Delta} \tau^M(M_i)$.

The pair (M, τ^M) is called an *m2-subspace* of $([X]^w, \tau)$.

Proposition 3.23
Let X be a non-empty set, τ be a CO on X and $M(\ne {}_0X) \in [X]^w$. A mapping $\tau^M : [X]^w \to \mathbb{N}_w$ defined by

$$\tau^M(P) = \begin{cases} \vee\{\tau(Q) : Q \cap M = P,\ Q \in [X]^w\}, & \text{if } P \subseteq M, \\ 0, & \text{if } P \not\subseteq M \end{cases}$$

is an SCO on M.

Proof. From the fact that ${}_0X, {}_wX \in [X]^w$ and $\tau({}_0X) = \tau({}_wX) = w$, it follows that $\tau^M({}_0X \cap M) = \tau({}_0X) = w$ and $\tau^M({}_wX \cap M) \geq \tau({}_wX) = w$. Also $\tau^M({}_wX) \leq w$. Hence $\tau^M({}_wX \cap M) = w$.

Next let $M_1, M_2, \ldots, M_n \subseteq M$ and $A = \bigcap_{i=1}^n M_i$. Let N_i be an arbitrary member of $[X]^w$ such that $N_i \cap M = M_i$. Then

$$(\bigcap_{i=1}^n N_i) \cap M = \bigcap_{i=1}^n (N_i \cap M) = \bigcap_{i=1}^n M_i = A.$$

Thus,

$$\tau^M(A) \geq \tau(\bigcap_{i=1}^n N_i) \geq \wedge_{i=1}^n \tau(N_i)$$

and hence

$$\begin{aligned}\tau^M(A) &\geq \vee_{\{N_i \in [X]^w:\ N_i \cap M = M_i\}} \wedge_{i=1}^n \tau(N_i) \\ &= \wedge_{i=1}^n \vee_{\{N_i \in [X]^w:\ N_i \cap M = M_i\}} \tau(N_i) \\ &= \wedge_{i=1}^n \tau^M(M_i).\end{aligned}$$

Assume now that $M_i \subseteq M$ for $i \in \Delta$ and $A = \bigcup_{i \in \Delta} M_i$. Let $\beta_i = \{N \in [X]^w : N \cap M = M_i\}$, $i \in \Delta$. For any $N_i \in \beta_i$, $i \in \Delta$, we have

$$(\bigcup_{i \in \Delta} N_i) \cap M = \bigcup_{i \in \Delta} (N_i \cap M) = \bigcup_{i \in \Delta} M_i = A,$$

we also have

$$\tau^M(M_i) = \vee\{\tau(N) :\ N \in \beta_i\}, \quad i \in \Delta.$$

Thus $\tau^M(\bigcup_{i \in \Delta} M_i) \geq \tau(\bigcup_{i \in \Delta} N_i) \geq \wedge_{i \in \Delta} \tau(N_i)$. Therefore $\tau^M(\bigcup_{i \in \Delta} M_i) \geq \wedge_{i \in \Delta} \tau^M(M_i)$ (similarly as above). Hence τ^M is an SCO on M.

Proposition 3.24
Let X, Y be two non-empty sets, $f\colon X \to Y$ be a mapping and τ be a CO on $[X]^w$. Then $f(\tau)\colon [Y]^w \to \mathbb{N}_w$ defined by $[f(\tau)](N) = \tau(f^{-1}(N))$, $N \in [Y]^w$ is a CO on $[Y]^w$.

Proof. Since $f^{-1}({}_0Y) = {}_0X$ and $f^{-1}({}_wY) = {}_wX$, it follows that

$$[f(\tau)]({}_0Y) = [f(\tau)]({}_wY) = w.$$

Now let $N_1, N_2, \ldots, N_n \in [Y]^w$ and $N = \bigcap_{i=1}^n N_i$. Then

$$\begin{aligned}[f(\tau)](N) &= \tau[f^{-1}(N)] = \tau[\bigcap_{i=1}^n (f^{-1}(N_i))] \\ &\geq \wedge_{i=1}^n \tau[f^{-1}(N_i)] = \wedge_{i=1}^n [f(\tau)](N_i).\end{aligned}$$

Finally, let $N_i \in [Y]^w$, $i \in \Delta$ and $N = \bigcup_{i \in \Delta} N_i$. Then

$$\begin{aligned}[f(\tau)](N) &= \tau[f^{-1}(N)] = \tau[\bigcup_{i \in \Delta} (f^{-1}(N_i))] \\ &\geq \wedge_{i \in \Delta} \tau[f^{-1}(N_i)] = \wedge_{i \in \Delta} [f(\tau)](N_i).\end{aligned}$$

Therefore $f(\tau)$ is a CO on $[Y]^w$.

Proposition 3.25
Let X, Y be two non-empty sets, $f\colon X \to Y$ be an onto mapping and ν be a CO on $[Y]^w$. Then $f^{-1}(\nu)\colon [X]^w \to \mathbb{N}_w$ defined by $[f^{-1}(\nu)](M) = \nu[f(M)]$, $M \in [X]^w$ is a CO on $[X]^w$.

Proof. From the fact $f({}_0X) = {}_0Y$, it follows that $[f^{-1}(\nu)]({}_0X) = w$. AS f is onto, we have $f^{-1}({}_wY) = {}_wX$ and hence $[f^{-1}(\nu)]({}_wX) = w$. Assume that $M_1, M_2, \dots, M_n \in [X]^w$ and $M = \bigcap_{i=1}^n M_i$. Then

$$\begin{aligned}[f^{-1}(\nu)](M) &= \nu[f(N)] = \nu[\textstyle\bigcap_{i=1}^n (f(M_i))]\\ &\geq \wedge_{i=1}^n \nu[f(M_i)] = \wedge_{i=1}^n [f^{-1}(\nu)](M_i).\end{aligned}$$

Now let $M_i \in [Y]^w$, $i \in \Delta$ and $M = \bigcup_{i\in\Delta} M_i$. Then

$$\begin{aligned}[f^{-1}(\nu)](M) &= \nu[f(M)] = \nu[\textstyle\bigcup_{i\in\Delta} (f(M_i))]\\ &\geq \wedge_{i\in\Delta} \nu[f(M_i)] = \wedge_{i\in\Delta} [f^{-1}(\nu)](M_i).\end{aligned}$$

Therefore $f^{-1}(\nu)$ is a CO on $[X]^w$.

Proposition 3.26
Let X, Y be two non-empty sets, $f\colon X \to Y$ be an onto mapping and τ be a CO on $[X]^w$. Then $[f(\tau)]^N(P) = \tau^{f^{-1}(N)}[f^{-1}(P)]$ for all $N \in [Y]^w$ and $P \subseteq N$.

Proof. Let $N \in [Y]^w$ and $P \subseteq N$. Then

$$\begin{aligned}[f(\tau)]^N(P) &= \vee\{[f(\tau)](Q) :\ Q \cap N = P\} = \vee\{\tau[f^{-1}(Q)] :\ Q \cap N = P\}\\ &= \vee\{\tau[f^{-1}(Q)] :\ f^{-1}(Q \cap N) = f^{-1}(P)\} \text{ (as } f \text{ is onto)}\\ &= \vee\{\tau[f^{-1}(Q)] :\ f^{-1}(Q) \cap f^{-1}(N) = f^{-1}(P)\}\\ &= \vee\{\tau[M] :\ M \in [X]^w \text{ and } M \cap f^{-1}(N) = f^{-1}(P)\} \text{ (as } f \text{ is onto)}\\ &= \tau^{f^{-1}(N)}[f^{-1}(P)].\end{aligned}$$

Proposition 3.27
Let X, Y be two non-empty sets, $f\colon X \to Y$ be a one-one mapping and ν be a CO on $[Y]^w$. Then $[f^{-1}(\nu)]^M(P) \leq \nu^{f(M)}[f(P)]$ for all $M \in [X]^w$ and $P \subseteq M$.

Proof. Let $M \in [X]^w$ and $P \subseteq M$. Then

$$\begin{aligned}[f^{-1}(\nu)]^M(P) &= \vee\{[f^{-1}(\nu)](Q) :\ Q \cap M = P\} = \vee\{\nu[f(Q)] :\ Q \cap M = P\}\\ &= \vee\{\nu[f(Q)] :\ f(Q \cap M) = f(P)\} \text{ (since } f \text{ is one-one)}\\ &= \vee\{\nu[f(Q)] :\ f(Q) \cap f(M) = f(P)\}\\ &\leq \vee\{\nu[N] :\ N \in [Y]^w \text{ such that } N \cap f(M) = f(P)\}\\ &= \nu^{f(M)}[f(P)].\end{aligned}$$

Proposition 3.28

Let τ be a count of openness on $[X]^w$ and $A \subseteq X$. If $\tau_A \colon [A]^w \to \mathbb{N}_w$ is a mapping defined by

$$\tau_A(N) = \vee\{\tau(M) : \ M \in [X]^w, \ M \cap {}_wA = N\}, \quad N \in [A]^w.$$

Then τ_A is a count of openness on $[A]^w$.

Proof. Since ${}_0A = {}_wA \cap {}_0X$, ${}_wA = {}_wA \cap {}_wX$ and $\tau({}_0X) = \tau({}_wX) = w$, it follows that $\tau_A({}_0A) = \tau_A({}_wA) = w$.

Let N_1, N_2 be any two members of $[A]^w$. Then we find $M_1, M_2 \in [X]^w$ such that $N_1 = M_1 \cap_w Y$ and $N_2 = M_2 \cap {}_wA$. Hence $N_1 \cap N_2 = (M_1 \cap M_2) \cap {}_wA$ and $\tau_A(N_1 \cap N_2) \geq \tau(M_1 \cap M_2) \geq \tau(M_1) \wedge \tau(M_2)$.

Thus

$$\begin{aligned} \tau_A(N_1 \cap N_2) &\geq \vee\{\tau(M_1) \wedge \tau(M_2) : \ M_1, M_2 \in [X]^w \\ &\qquad \text{such that } N_1 = M_1 \cap {}_wA, N_2 = M_2 \cap {}_wA\} \\ &\geq \vee\{\vee\{\tau(M_1) \wedge \tau(M_2) : \ M_2 \cap {}_wA = N_2\} : \ M_1 \cap {}_wA = N_1\} \\ &= \vee\{\tau(M_1) \wedge \tau_A(N_2) : \ M_1 \cap {}_wA = N_1\} \\ &= \tau_A(N_1) \wedge \tau_A(N_2). \end{aligned}$$

Now let N_i, $i \in \Delta$ be any collection of members of $[A]^w$. Then we find $M_i \in [X]^w$, $i \in \Delta$ such that $N_i = M_i \cap {}_wA$, $i \in \Delta$. It follows that $\bigcup_{i\in\Delta} N_i = (\bigcup_{i\in\Delta} M_i) \cap {}_wA$ and similarly as above we have

$$\tau_A(\textstyle\bigcup_{i\in\Delta} N_i) \geq \tau(\bigcup_{i\in\Delta} M_i) \geq \wedge_{i\in\Delta}\tau(M_i) \geq \wedge_{i\in\Delta}\tau_A(N_i).$$

Therefore, τ_A is a count of openness on $[A]^w$.

Proposition 3.29

Let $([A]^w, \tau_A)$ be an m-subspace of the m-topological space $([X]^w, \tau)$ and $N \in [A]^w$. Then

(i) $\mathcal{F}_{\tau_A}(N) = \{\mathcal{F}_\tau(M) : \ M \in [X]^w, \ M \cap {}_wA = N\}$*;*

(ii) if $B \subseteq A \subseteq X$, then $\tau_B = (\tau_A)_B$.

Proof. (i) Let $([A]^w, \tau_A)$ be an m-subspace of the m-topological space $([X]^w, \tau)$ and $N \in [A]^w$. Then

$$\begin{aligned} \mathcal{F}_{\tau_A}(N)\tau_A(N^c) &= \vee\{\tau(M) : \ M \in [X]^w, \ M \cap {}_wA = N^c\} \\ &= \vee\{\tau(M) : \ M^c \in [X]^w, \ M^c \cap {}_wA = N\} \\ &= \vee\{\mathcal{F}_\tau(M^c) : \ M^c \in [X]^w, \ M^c \cap {}_wA = N\} \\ &= \vee\{\mathcal{F}_\tau(M) : \ M \in [X]^w, \ M \cap {}_wA = N\}. \end{aligned}$$

(ii) Let $P \in [B]^w$. Then

$$\begin{aligned} (\tau_A)_B(P) &= \vee\{\tau_A(N) : \ N \in [A]^w, \ N \cap {}_wB = P\} \\ &= \vee\{\vee\{\tau(M) : \ M \in [X]^w, \ M \cap {}_wA = N\} : \ N \in [A]^w, \ N \cap {}_wB = P\} \\ &= \vee\{\tau(M) : \ M \in [X]^w, \ M \cap {}_wB = P\} = \tau_B(P). \end{aligned}$$

Therefore, if $B \subseteq A \subseteq X$, then $\tau_B = (\tau_A)_B$.

Definition 3.30
Let $([X]^w, \tau)$ and $([Y]^w, \nu)$ be two m2-topological spaces and $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ be a mapping. Then f is called a *count preserving map* or an *mgp-map* if $\tau(f^{-1}(N)) \geq \nu(N)$ for each $N \in [Y]^w$.

Proposition 3.31
Let $([X]^w, \tau)$ be an m-topological space. Then the identity mapping $f\colon ([X]^w, \tau) \to ([X]^w, \tau)$ is an mgp-map.

Proof. Since f is the identity mapping, it follows that $f^{-1}(N) = N$ for all $N \in [X]^w$ and hence $\tau(f^{-1}(N)) = \tau(N)$.

Remark 3.32
Let $([X]^w, \tau)$ and $([Y]^w, \nu)$ be two m2-topological spaces and $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ be a constant mapping. Then f is not an mgp-map in general, which shows the following example.

Example 3.33
Let $X = \{x, y, z\}$, $Y = \{a, b, c, d\}$ and $w = 3$. Let $\tau\colon [X]^w \to \mathbb{N}_w$, $\nu\colon [Y]^w \to \mathbb{N}_w$ be two mappings defined by $\tau({}_0X) = \tau({}_wX) = w$, $\tau(M) = 0$ for all $M \in [[X]^w - \{{}_0X, {}_wX\}]$ and $\nu(N) = w$ for all $N \in [Y]^w$. Then τ and ν are two m2-topologies on $[X]^w$, $[Y]^w$, respectively.

Moreover, let $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ be a constant mapping, defined by $f(x) = a$ for all $x \in X$. If $N = \{a, a, b, c\} \in [Y]^3$, then $f^{-1}(N) = {}_2X$ and

$$\tau[f^{-1}(N)] = \tau({}_2X) = 0 \ngeq 3 = w = \nu(N).$$

Therefore, the mapping f is not an mgp-map.

Proposition 3.34
Let $([X]^w, \tau)$ and $([Y]^w, \nu)$ be two m2-topological spaces. If $\tau({}_kX) = w$ for all $k(\leq w) \in \mathbb{N}$, then the constant mapping $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ is an mgp-map.

Proof. Let f be a constant mapping and assume that there exists $y_0 \in Y$ such that $f(x) = y_0$ for all $x \in X$. Then for any $N \in [Y]^w$, $f^{-1}(N) = {}_kX$ for some $k(\leq w) \in \mathbb{N}$. Hence $\tau(f^{-1}(N)) = \tau({}_kX) = w \geq \nu(N)$. Therefore, the constant mapping $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ is an mgp-map.

Proposition 3.35
Let $([X]^w, \tau)$ and $([Y]^w, \nu)$ be two m2-topological spaces and let $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ be a mapping. Then f is a mgp-map iff $f\colon ({}_wX, \tau_r) \to ({}_wY, \nu_r)$ is m-continuous for all $r \leq w$.

Proof. First let $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ be an mgp-map, $r \leq w$ and $N \in \nu_r$. Then $N \in [Y]^w$ and $\nu(N) \geq r$. Since f is an mgp-map, it follows that

$$\tau(f^{-1}(N)) \geq \nu(N) \geq r$$

and hence $f^{-1}(N) \in \tau_r$. Therefore, $f\colon ({}_wX, \tau_r) \to ({}_wY, \nu_r)$ is m-continuous.

Conversely, let $f\colon ({}_wX, \tau_r) \to ({}_wY, \nu_r)$ for all $r \leq w$ be m-continuous and $N \in [Y]^w$. Let moreover $\nu(N) = r$. If $r = 0$, then obviously $\tau(f^{-1}(N)) \geq \nu(N)$, otherwise $N \in \nu_r$ and hence $f^{-1}(N) \in \tau_r$. Thus, $\tau(f^{-1}(N)) \geq r = \nu(N)$. Therefore, $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ is an mgp-map.

Proposition 3.36
Let $({}_wX, T)$ and $({}_wY, T')$ be two m-topological spaces. Then $f\colon ({}_wX, T) \to ({}_wY, T')$ is m-continuous iff $f\colon ([X]^w, T^r) \to ([Y]^w, (T')^r)$ is an mgp-map for each $r \leq w$, where T^r, $(T')^r$ are determined as in Proposition 3.19.

Proof. First let $f\colon ({}_wX, T) \to ({}_wY, T')$ be m-continuous and $N \in [Y]^w$. Then we have the following three possibilities:

(i) $N = {}_0Y$ or $N = {}_wY$;

(ii) $N \in T'$;

(iii) $N \notin T'$.

In the case (i), $f^{-1}({}_0Y) = {}_0X$ and $f^{-1}({}_wY) = {}_wX$. Hence,

$$T^r(f^{-1}({}_0Y)) = T^r({}_0X) = w \geq (T')^r({}_0Y)$$

and

$$T^r(f^{-1}({}_wY)) = T^r({}_wX) = w \geq (T')^r({}_wY).$$

In the case (ii), $N \in T' \Rightarrow (T')^r(N) = r$. Since $f\colon ({}_wX, T) \to ({}_wY, T')$ be m-continuous, it implies that $f^{-1}(N) \in T$ and hence $T^r(f^{-1}(N)) = r$. So, $T^r(f^{-1}(N)) \geq (T')^r(N)$.

In the case (iii), $N \in T' \to (T')^r(N) = 0$ and hence $T^r(f^{-1}(N)) \geq 0 = (T')^r(N)$. Therefore, $f\colon ([X]^w, T^r) \to ([Y]^w, (T')^r)$ is an mgp-map for each $r \leq w$. Converse part follows from Proposition 3.19 and Proposition 3.35.

Proposition 3.37
Let $([X]^w, \tau)$, $([Y]^w, \tau')$ and $([Z]^w, \tau'')$ be m2-topological spaces, where τ, τ' and τ'' are m-gradtions of openness on $[X]^w$, $[Y]^w$ and $[Z]^w$, respectively. If $f\colon ([X]^w, \tau) \to ([Y]^w, \tau')$ and $g\colon ([Y]^w, \tau') \to ([Z]^w, \tau'')$ are mgp-maps, then $g \circ f\colon ([X]^w, \tau) \to ([Z]^w, \tau'')$ is an mgp-map.

Proof. Proof is straightforward.

Definition 3.38
Let $([X]^w, \tau)$ and $([Y]^w, \nu)$ be two m2-topological spaces where τ and ν are counts of openness on $[X]^w$ and $[Y]^w$, respectively. Let $M \in [X]^w$, $N \in [Y]^w$ and τ^M, ν^N be m-subspace gradations of openness on M and N, respectively. Then $f\colon (M, \tau^M) \to (N, \nu^N)$ is said to be an *mgp-map* if

$$\tau^M(f^{-1}(P) \cap M) \geq \nu^N(P) \quad \text{for any } P \subseteq N.$$

Proposition 3.39
Let $([X]^w, \tau)$ and $([Y]^w, \nu)$ be two m2-topological spaces, where τ and ν are counts of openness on $[X]^w$ and $[Y]^w$, respectively. Let $M \in [X]^w$, $N \in [Y]^w$ and τ^M, ν^N be m-subspace gradations of openness on M and N, respectively. If $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ is an mgp-map and $f(M) \subseteq N$, then $f\colon (M, \tau^M) \to (N, \nu^N)$ is an mgp-map.

Proof. Let $P \subseteq N$ and A be any member of $[Y]^w$ such that $P = A \cap N$. Since $f\colon ([X]^w, \tau) \to ([Y]^w, \nu)$ is an mgp-map, it follows that

$$\tau(f^{-1}(A)) \geq \nu(A). \tag{3}$$

Now $f^{-1}(P) = f^{-1}(A \cap N) = f^{-1}(A) \cap f^{-1}(N)$. Thus, $f^{-1}(P) \cap M = f^{-1}(A) \cap f^{-1}(N) \cap M = f^{-1}(A) \cap M$, since $M \subseteq f^{-1}(f(M)) \subseteq f^{-1}(N)$. So, by (3), $\tau^M(f^{-1}(P) \cap M) = \tau^M(f^{-1}(A) \cap M) \geq \tau(f^{-1}(A)) \geq \nu(A)$. Hence,

$$\tau^M(f^{-1}(P) \cap M) \geq \vee\{\nu(A) : \ A \cap N = P\} = \nu^N(P).$$

Therefore $f\colon (M, \tau^M) \to (N, \nu^N)$ is an mgp-map.

Proposition 3.40
Let $([X]^w, \tau)$, $([Y]^w, \nu)$ and $([Z]^w, \omega)$ be three m2-topological spaces where τ, ν and ω are counts of openness on $[X]^w$, $[Y]^w$ and $[Z]^w$, respectively. Let $M \in [X]^w$, $N \in [Y]^w$, $P \in [Z]^w$ and τ^M, ν^N, ω^P be m-subspace gradations of openness on M, N and P, respectively. If $f\colon (M, \tau^M) \to (N, \nu^N)$ and $g\colon (N, \nu^N) \to (P, \omega^P)$ are mgp-maps and $f(M) \subseteq N$, then the composition mapping $g \circ f\colon (M, \tau^M) \to (P, \omega^P)$ is an mgp-map.

Proof. Let $A \subseteq P$. Since g is an mgp-map, it follows that

$$\nu^N(g^{-1}(A) \cap N) \geq \omega^P(A). \tag{4}$$

Again since $[g^{-1}(A) \cap N] \subseteq N$, f is an mgp-map, we have

$$\tau^M M \cap f^{-1}(g^{-1}(A) \cap N)) \geq \nu^N(g^{-1}(A) \cap N).$$

This in view of (4) gives

$$\tau^M(M \cap f^{-1}(g^{-1}(A)) \cap f^{-1}(N)) \geq \omega^P(A)$$

and, since $M \subseteq f^{-1}(N)$,

$$\tau^M(M \cap (g \circ f)^{-1}(A)) \geq \omega^P(A).$$

Therefore, $g \circ f$ is an mgp-map.

4. Conclusion and future work

In this paper, the concepts of a count of openness, a subspace count of openness, an mgp-maps are introduced. We define a generalized m-topological space which is called m2-topological space. We have shown that such a count is generated by a descending family of m-topologies and vice versa. The behaviour of the functional image and the functional preimage of an m2-topology, the continuity of the identity mapping and a constant mapping in m2-topologies are also examined. The concepts of topological structures and their generalizations are one of the most powerful notions in branches of science and information systems. It is the generalized methods for measuring the similarity and dissimilarity between the objects in msets as universe. In this sense, this work has a great importance. There is a wide scope for further research to extend it in topological groups theory, which has many applications in abstract integration theory viz. Haar measure, Haar integral etc. and also in manifolds theory through the development of Lie groups.

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