COLLOQUIUM MATHEMATICUM

VOL. XXXI 1974 FASC. 2

REPRESENTATION THEOREMS AND FATOU THEOREMS FOR PARABOLIC SYSTEMS IN THE SENSE OF PETROVSKIĬ

BY

J. CHABROWSKI (KATOWICE)

Representation theorems and Fatou theorems for non-negative classical solutions of second-order parabolic equations have been considered in several papers. The representation theorem for non-negative solutions of the equation of heat conduction was first proved by Widder [23] and subsequently extended to solutions of second-order parabolic equations with smooth coefficients by Krzyżański [20]. Aronson and Besala [6] and Bodanko [8] obtained Widder representation for parabolic equations with unbounded coefficients. Properties of the non-negative weak solutions of linear equations with discontinuous coefficients were studied by Aronson [2]-[4]. Further extensions of representation theorems for non-negative solutions of a special parabolic system were obtained by the author [11]-[13]. Kato [19] derived Fatou theorem for non-negative solutions of the equation of heat conduction by using Widder representation. These results were extended by Johnson [17] and [18] to the class L^p_β, which includes non-negative solutions. In this paper a representation theorem and the Fatou theorem are proved for the solutions of parabolic systems of arbitrary order.

1. Preliminaries. Consider the parabolic system

$$\frac{\partial u_i}{\partial t} = \sum_{|k|=2b}^{j=1,\ldots,N} A_k^{ij}(t,x)\frac{\partial^{|k|} u_j}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} \qquad (i=1,\ldots,N), \tag{1}$$

where the A_k^{ij} are functions defined on $[0,T]\times R_n$, $k_1,\ldots,k_n$ are non-negative integers, $k=(k_1,\ldots,k_n)$, and $|k|=k_1+\ldots+k_n$.

Throughout the paper it will be assumed that the coefficients satisfy the following conditions:

I. The principal coefficients of (1) are continuous in t uniformly with respect to $(t,x)\in[0,T]\times R_n$ and Hölder continuous (exponent α) in x, uniformly with respect to (t,x) in $[0,T]\times R_n$.

II. The derivatives

$$\frac{\partial^{|h|}}{\partial x_1^{h_1} \dots \partial x_n^{h_n}} A_k^{ij}(t, x) \qquad (0 \leqslant |h| \leqslant |k|)$$

are continuous bounded functions in $[0, T] \times R_n$ and Hölder continuous (exponent a) in x, uniformly with respect to (t, x) in bounded subsets of $[0, T] \times R_n$.

III. The system (1) is uniformly parabolic in the sense of Petrovskiĭ, i. e. the roots λ_j of the polynomial

$$\det_{r,s=1,\dots,N} \Big[\sum^{|k|=2b} A_k^{rs}(t, x)(i\xi_1)^{k_1} \dots (i\xi_n)^{k_n} - \delta_{rs}\lambda \Big]$$

satisfy the inequality

$$\max_{j=1,\dots,N} \sup_{|\xi|=1} \{\mathrm{Re}\,\lambda_j(t, x;\ \xi)\} \leqslant -\delta,$$

where δ is a positive constant.

Under these hypotheses, the fundamental matrix $\{\Gamma_{ij}(t, x;\ \tau, y)\}$ $(i, j = 1, \dots, N)$ for the system (1) exists and satisfies the estimate

$$|\Gamma_{ij}(t, x;\ \tau, y)| \leqslant C(t-\tau)^{-n/2b} \exp\left[-c\left(\frac{|x-y|^{2b}}{t-\tau}\right)^{1/(2b-1)} \right] \tag{2}$$

$$(i, j = 1, \dots, N)$$

for $x, y \in R_n$, $0 \leqslant \tau < t \leqslant T$, where c and C are positive constants ([21], [15], p. 73, [16], chap. 9). Notice that from our assumptions it follows that there exists a fundamental matrix for the adjoint system, so uniqueness theorems for the Cauchy problem can be applied ([15], chap. 3).

The function space that enters in the theorems is the set of all measurable functions f defined on R_n, such that

$$\int_{R_n} |f(x)|^p \exp[-p\beta |x|^{2b/(2b-1)}]\, dx < \infty, \qquad 1 \leqslant p < \infty,$$

where the parameter β is non-negative. We denote this space by L_β^p and define the norm

$$\|f\|_{L_\beta^p} = \Big\{ \int_{R_n} |f(x)|^p \exp[-p\beta |x|^{2b/(2b-1)}]\, dx \Big\}^{1/p}, \qquad 1 \leqslant p < \infty.$$

In the case $p = \infty$, we introduce the norm

$$\|f\|_{L_\beta^\infty} = \operatorname*{ess\,sup}_{R_n} |f(x)| \exp[-\beta |x|^{2b/(2b-1)}]$$

and the set of all measurable functions f for which $\|f\|_{L_\beta^\infty} < \infty$ is denoted by L_β^∞.

2. Representation theorems and Fatou property. Before proving the main theorem, let us establish the following

LEMMA 1. *Let* $\{\mu_j\}$ $(j = 1, \ldots, N)$ *be measures such that*

$$\int_{R_n} \exp(-\beta |x|^{2b/(2b-1)}) |\mu_j|(dx) < \infty \qquad (j = 1, \ldots, N).$$

Then

$$\lim_{t\to 0} \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ 0,\ y)\mu_j(dy) = \mu_i \qquad (i = 1, \ldots, N)$$

in the topology of weak convergence of measures.

In the sequel the word *measure* will always mean a Borel measure on R_n.

Proof. The proof is an adaptation of Johnson's method [17]. Let φ be of class $C^\infty(R_n)$ with compact support. Using the decomposition of Γ_{ij} we get ([16], chapter 9, section 4)

$$\begin{aligned}\int_{R_n} \Big[\int_{R_n} \sum_{j=1}^{N} &\Gamma_{ij}(t, x;\ 0,\ y)\mu_j(dy)\Big] \varphi(x)dx \\ &= \sum_{j=1}^{N} \int_{R_n} \mu_j(dy) \int_{R_n} Z_{ij}(t, x-y;\ 0,\ y)\varphi(y)dx + \\ &\quad + \sum_{j=1}^{N} \int_{R_n} \mu_j(dy) \int_{R_n} Z_{ij}(t, x-y;\ 0, y)[\varphi(x)-\varphi(y)]dx + \\ &\quad + \sum_{j=1}^{N} \int_{R_n} \mu_j(dy) \int_{R_n} R_{ij}(t, x;\ 0, y)\varphi(x)dx = J_1 + J_2 + J_3,\end{aligned}$$

where

$$\begin{aligned}&\Gamma_{ij}(t, x;\ 0,\ y) \\ &= Z_{ij}(t, x-y;\ 0,\ y) + \int_0^t d\sigma \int_{R_n} \sum_{k=1}^{N} Z_{ik}(t, x-z;\ \sigma,\ z)\Phi_{kj}(\sigma, z;\ 0,\ y)dz \\ &= Z_{ij}(t, x-y;\ 0,\ y) + R_{ij}(t, x;\ 0, y)\end{aligned}$$

and $\{Z_{ij}(t, x-z;\ \tau, y)\}$ $(i, j = 1, \ldots, N)$ is the fundamental solution of the system (with y fixed)

$$\frac{\partial u_i}{\partial t} = \sum_{|k|=2b}^{j=1,\ldots,N} A_k^{ij}(t, y) \frac{\partial^{|k|} u_j}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \qquad (i = 1, \ldots, N)$$

and satisfies the estimate

$$(3) \qquad |Z_{ij}(t, x-z;\ \tau, y)| \leqslant C(t-\tau)^{-n/2b} \exp\left[-c\left(\frac{|x-z|^{2b}}{t-\tau}\right)^{1/(2b-1)}\right]$$

for $x, y \in R_n$, $0 \leqslant \tau < t \leqslant T$. From the uniqueness theorem [1] for the Cauchy problem it follows that

$$\int_{R_n} Z_{ij}(t, x-z;\ \tau, y)\,dx = \delta_{ij} \quad (i, j = 1, \ldots, N). \tag{4}$$

To evaluate the integral J_2 we note that for any positive numbers β and $\varepsilon \in (0, c)$ there exist positive numbers β_1 and a_1 such that

$$-\frac{\varepsilon}{a_1^{1/(2b-1)}}\,|x-y|^{2b/(2b-1)} + \beta\,|y|^{2b/(2b-1)} \leqslant \beta_1\,|x|^{2b/(2b-1)}$$

for $x, y \in R_n$. Hence

$$\Big|\exp(\beta\,|y|^{2b/(2b-1)}) \int_{R_n} Z_{ij}(t, x-y;\ 0, y)[\varphi(x)-\varphi(y)]\,dx\Big| \tag{5}$$

$$\leqslant C\,\|\exp(\beta\,|y|^{2b/(2b-1)})\varphi(y)\|_{L^\infty} t^{-n/2b} \int_{R_n} \exp\Big[-c\Big(\frac{|x-y|^{2b}}{t}\Big)^{1/(2b-1)}\Big]\,dx +$$

$$+\,Ct^{-n/2b} \int_{R_n} \exp\Big[-(c-\varepsilon)\Big(\frac{|x-y|^{2b}}{t}\Big)^{1/(2b-1)}\Big]\,dx\,\|\varphi(x)\exp(\beta_1\,|x|^{2b/(2b-1)})\|_{L^\infty}$$

for $0 < t \leqslant a_1$. The estimate (5) and Lebesgue's dominated convergence theorem applied to μ_j show that $\lim_{t\to 0} J_2 = 0$. The integral J_3 can be estimated in a similar way, because

$$|R_{ij}(t, x;\ 0,\ y)| \leqslant Ct^{-(n-\alpha)/2b}\exp\Big[-c\Big(\frac{|x-y|^{2b}}{t}\Big)^{1/(2b-1)}\Big]$$

for $x, y \in R_n$, $0 < t \leqslant T$. Hence $\lim_{t\to 0} J_3 = 0$.

We can now state the main theorem of this paper:

THEOREM 1. *Let $\{u_j(t, x)\}$ $(j = 1, \ldots, N)$ be a solution of the system* (1) *in $(0, T]\times R_n$ such that*

$$\|u_j(t, \cdot)\|_{L^p_\beta} \leqslant M \quad (j = 1, \ldots, N)$$

for $0 < t \leqslant T$, where M is a positive constant. If $1 < p \leqslant \infty$, then there exist unique f_j in L^p_β $(j = 1, \ldots, N)$ and a positive constant T_1 such that

$$u_i(t, x) = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ 0,\ y) f_j(y)\,dy \quad (i = 1, \ldots, N) \tag{6}$$

for $(t, x) \in (0, T_1]\times R_n$. If $p = 1$, then there exist unique measures μ_j $(j = 1, \ldots, N)$ such that

$$\int_{R_n} \exp(-\beta\,|x|^{2b/(2b-1)})\,|\mu_j|(dx) < \infty \quad (j = 1, \ldots, N) \tag{7}$$

and

$$(8)\qquad u_i(t,x) = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t,x;\ 0,\ y)\mu_j(dy) \qquad (i=1,\ldots,N)$$

for $(t,x)\in(0,T_1]\times R_n$.

Proof. By the assumption

$$\left\{u_i\left(\frac{1}{k},x\right)\exp(-\beta|x|^{2b/(2b-1)}),\ k=1,2,\ldots\right\} \qquad (i=1,\ldots,N)$$

is bounded in $L^p(R_n)$ $(p>1)$, and hence there is a subsequence $\{k_s\}$ such that

$$\lim_{s\to\infty}\int_{R_n} u_i\left(\frac{1}{k_s},y\right)\exp(-\beta|y|^{2b/(2b-1)})g_i(y)\,dy$$

$$= \int_{R_n} f_i(y)\exp(-\beta|y|^{2b/(2b-1)})g_i(y)\,dy$$

for each $g_i\in L^{p'}(R_n)$. It follows from the estimate (2) that there exists a T_1 $(T_1\leqslant T)$ such that $\Gamma_{ij}(t,x;\ 0,\ y)\exp(\beta|y|^{2b/(2b-1)})$ is in $L^{p'}$ as a function of y for all $0<t\leqslant T_1$. Therefore

$$(9)\qquad \lim_{s\to\infty}\int_{R_n}\Gamma_{ij}(t,x;\ 0,y)u_j\left(\frac{1}{k_s},y\right)dy = \int_{R_n}\Gamma_{ij}(t,x;\ 0,\ y)f_j(y)dy.$$

By the uniqueness theorem for the Cauchy problem ([15], chapter 3) we have

$$u_i(t,x) = \int_{R_n}\sum_{j=1}^{N}\Gamma_{ij}\left(t,x;\ \frac{1}{k_s},y\right)u_j\left(\frac{1}{k_s},y\right)dy \qquad (i=1,\ldots,N)$$

for $(t,x)\in(1/k_s,T_1]\times R_n$.

Fix $(t,x)\in(0,T_1]\times R_n$. We may assume that $1/k_s\leqslant t/2$. Consider the difference

$$u_i(t,x) - \int_{R_n}\sum_{j=1}^{N}\Gamma_{ij}(t,x;\ 0,\ y)f_j(y)\,dy$$

$$=\left[\int_{R_n}\sum_{j=1}^{N}\Gamma_{ij}\left(t,x;\ \frac{1}{k_s},y\right)u_j\left(\frac{1}{k_s},y\right)dy - \int_{R_n}\sum_{j=1}^{N}\Gamma_{ij}(t,x;\ 0,y)u_j\left(\frac{1}{k_s},y\right)dy\right]+$$

$$+\left[\int_{R_n}\sum_{j=1}^{N}\Gamma_{ij}(t,x;\ 0,y)u_j\left(\frac{1}{k_s},y\right)dy - \int_{R_n}\sum_{j=1}^{N}\Gamma_{ij}(t,x;\ 0,y)f_j(y)\,dy\right]$$

$$= J_1 + J_2.$$

It suffices to prove $\lim_{s\to\infty} J_i = 0$. It follows from (9) that $\lim_{s\to\infty} J_2 = 0$. For fixed $x\in R_n$ and any $\varepsilon > 0$ choose a compact set $K \subset R_n$ such that

$$\int_{R_n-K} |\Gamma_{ij}(t, x;\ 0, y)|^{p'} \exp(p'\beta |y|^{2b/(2b-1)})\,dy \leqslant \varepsilon, \tag{10}$$

$$\int_{R_n-K} \left|\Gamma_{ij}\left(t, x;\ \frac{1}{k_s}, y\right)\right|^{p'} \exp(p'\beta |y|^{2b/(2b-1)})\,dy \tag{11}$$

$$\leqslant C\left(t-\frac{1}{k_s}\right)^{-n/2b} \int_{R_n-K} \exp\left[-c\left(\frac{|x-y|^{2b}}{t-1/k_s}\right)^{1/(2b-1)}\right] \exp(p'\beta |y|^{2b/(2b-1)})\,dy$$

$$\leqslant C\cdot 2^{n/2b} \int_{R_n-K} t^{-n/2b} \exp\left[-c\left(\frac{|x-y|^{2b}}{t}\right)^{1/(2b-1)}\right] \exp(p'\beta |y|^{2b/(2b-1)})\,dy \leqslant \varepsilon.$$

Note that

$$\lim_{s\to\infty} \Gamma_{ij}\left(t, x;\ \frac{1}{k_s}, y\right) = \Gamma_{ij}(t, x;\ 0, y) \tag{12}$$

uniformly for $y\in K$. By Hölder's inequality we obtain

$$|J_1| \leqslant \sum_{j=1}^{N} \left\|u_j\left(\frac{1}{k_s}, \cdot\right)\right\|_{L^p_\beta} \left[\int_{R_n} \left|\Gamma_{ij}\left(t, x;\ \frac{1}{k_s}, y\right) - \Gamma_{ij}(t, x;\ 0, y)\right|^{p'} \times \right.$$
$$\left. \times \exp(p'\beta |y|^{2b/(2b-1)})\,dy\right]^{1/p'}$$

$$\leqslant M \sum_{j=1}^{N} \left[\int_{R_n-K} \left|\Gamma_{ij}\left(t, x;\ \frac{1}{k_s}, y\right) - \Gamma_{ij}(t, x;\ 0, y)\right|^{p'} \exp(p'\beta |y|^{2b/(2b-1)})\,dy + \right.$$
$$\left. + \int_{K} \left|\Gamma_{ij}\left(t, x;\ \frac{1}{k_s}, y\right) - \Gamma_{ij}(t, x;\ 0, y)\right|^{p'} \exp(p'\beta |y|^{2b/(2b-1)})\,dy\right]^{1/p'}.$$

Combining this with (10), (11) and (12) we find that $\lim_{s\to\infty} J_1 = 0$. The uniqueness of f_j follows from Theorem 2. When $p = 1$, the proof is identical, except that now there are measures μ_j satisfying (7). The uniqueness of μ_j is a consequence of Lemma 1.

Remark. Denote by T^* the least upper bound of such T_1 that formula (6) holds true in the set $(0, T_1]\times R_n$. We prove that in the case $T^* < T$ formula (6) is not satisfied for $(t, x) = (T^*, x)$. Suppose the contrary;

formula (6) holds true for $(t, x) = (T^*, x)$. It follows from the uniqueness theorem ([15], chapter 3) that there exists a positive ϱ such that

$$u_i(t, x) = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ T^*, y) u_j(T^*, y) dy \qquad (i = 1, \ldots, N)$$

for $(t, x) \in (T^*, T^* + \varrho] \times R_n$. By Kolmogorov's identity ([15], p. 94) we have

$$\begin{aligned} u_i(t, x) &= \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ T^*, y) \Big[\int_{R_n} \sum_{k=1}^{N} \Gamma_{jk}(T^*, y;\ 0, z) f_k(z) dz \Big] dy \\ &= \int_{R_n} \sum_{k=1}^{N} \Big[\int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ T^*, y) \Gamma_{jk}(T^*, y;\ 0, z) dy \Big] f_k(z) dz \\ &= \int_{R_n} \sum_{k=1}^{N} \Gamma_{ik}(t, x;\ 0, z) f_k(z) dz \end{aligned}$$

for $(t, x) \in (T^*, T^* + \varrho] \times R_n$, which contradicts the definition of T^*

THEOREM 2. *If $f_j \in L^p_\beta(R_n)$ $(j = 1, \ldots, N)$ for $1 < p \leqslant \infty$, then for almost all $x \in R_n$*

$$\lim_{\substack{|z-x| < \gamma t^{1/2b} \\ t \to 0}} u_i(t, z) = f_i(x) \qquad (i = 1, \ldots, N),$$

where

$$u_i(t, x) = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ 0, y) f_j(y) dy \qquad (i = 1, \ldots, N)$$

and γ is a positive constant.

If μ_j $(j = 1, \ldots, N)$ are complex measures satisfying (7), *then*

$$\lim_{\substack{|z-x| < \gamma t^{1/2b} \\ t \to 0}} u_i(t, z) = \mu_i'(x) \qquad (i = 1, \ldots, N),$$

where

$$\mu_i'(x) = \lim_{a \to 0} \frac{\mu_i(w;\ |w - x| < a)}{|(w;\ |w - x| < a)|}$$

and

$$u_i(t, x) = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ 0, y) \mu_j(dy) \qquad (i = 1, \ldots, N)$$

($|(w;\ |w - x| < a)|$ is the Lebesgue measure of the ball $(w;\ |w - x| < a)$).

Proof. It suffices to prove the second part of theorem. Write Lebesgue decomposition of the measure

$$\mu_j(dy) = g_j(y) dy + \nu_j(dy) \qquad (j = 1, \ldots, N),$$

where the functions g_j belong (locally) to L^1 and ν_j is singular with respect to the Lebesgue measure. It is well-known that

$$\lim_{a\to 0}\frac{1}{a^n}\int_{|y-x|<a}[|g_j(y)-g_j(x)|dy+|\nu_j|(dy)]=0\qquad (j=1,\ldots,N)$$

almost everywhere in R_n. Hence for fixed x and for any $\varepsilon>0$ there exists a number $\delta>0$ such that

$$a^{-n}\int_{|y-x|<a}[|g_j(y)-g_j(x)|dy+|\nu_j|(dy)]<\varepsilon\qquad (j=1,\ldots,N)\tag{13}$$

for $0<a\leqslant 2\delta$. For each $0<t<\min(2\delta,T)$ choose a non-negative integer $P(t)$ such that

$$2^{P-1}\gamma t^{1/2b}\leqslant\delta<2^P\gamma t^{1/2b}.$$

It is clear that

$$\Big|u_i(t,z)-\int_{R_n}\sum_{j=1}^N\Gamma_{ij}(t,z;\,0,y)g_j(x)dy\Big|$$
$$\leqslant\sum_{j=1}^N\int_{|y-x|<2\gamma t^{1/2b}}|\Gamma_{ij}(t,z;\,0,y)|[|g_j(y)-g_j(x)|dy+|\nu_j|(dy)]+$$
$$+\sum_{l=2}^P\int_{2^{l-1}\gamma t^{1/2b}\leqslant|y-x|<2^l\gamma t^{1/2b}}\sum_{j=1}^N|\Gamma_{ij}(t,z;\,0,y)|[|g_j(y)-g_j(x)|dy+|\nu_j|(dy)]+$$
$$+\sum_{j=1}^N\int_{|y-x|\geqslant\delta}|\Gamma_{ij}(t,z;\,0,y)||g_j(x)|dy+$$
$$+\sum_{j=1}^N\int_{|y-x|\geqslant\delta}|\Gamma_{ij}(t,z;\,0,y)||\mu_j|(dy)=J_1+J_2+J_3+J_4.$$

In view of (13) we have

$$J_1\leqslant C\int_{|y-x|<2\gamma t^{1/2b}}t^{-n/2b}\exp\Big[-c\Big(\frac{|y-z|^{2b}}{t}\Big)^{1/(2b-1)}\Big]\sum_{j=1}^N|g_j(y)-g_j(x)|dy+|\nu_j|(dy)$$
$$\leqslant NC\varepsilon(2\gamma)^n$$

Notice that the inequalities $|z-x|\leqslant\gamma t^{1/2b}$ and $|x-y|\geqslant 2^{l-1}\gamma t^{1/2b}$ mply $|z-y|\geqslant 2^{(l-2)}\gamma t^{1/2b}$ for $l=2,\ldots,P$, hence

$$J_2=\sum_{l=2}^P NC\varepsilon\cdot 2^{ln}\gamma^n\exp[-c(2^{l-2}\gamma)^{2b/(2b-1)}]$$
$$\leqslant CN\varepsilon\sum_{l=2}^\infty 2^{ln}\gamma^n\exp[-c(2^{l-2}\gamma)^{2b/(2b-1)}].$$

Since $|z-x| < \gamma t^{1/2b}$ and $|y-x| \geqslant \delta$, we can assume that $|z-y| \geqslant \delta/2$ for t sufficiently small, and hence

$$J_3 \leqslant \sum_{j=1}^{N} |g_j(x)| \exp\left[-\frac{c}{2}\left(\frac{\delta^{2b}}{2^{2b}t}\right)^{1/(2b-1)}\right] t^{-n/2b} \int_{R_n} \exp\left[-\frac{c}{2}\left(\frac{|z-y|^{2b}}{t}\right)^{1/(2b-1)}\right] dy.$$

Hence $\lim_{t\to 0} J_3 = 0$. In a similar way we have

$$J_4 \leqslant Ct^{-n/2b} \exp\left[-\frac{c}{2}\left(\frac{\delta^{2b}}{2^{2b}t}\right)^{1/(2b-1)}\right] \int_{R_n} \exp\left[-\frac{c}{2}\left(\frac{|z-y|^{2b}}{t}\right)^{1/(2b-1)} + \right.$$

$$\left. + \beta |y|^{2b/(2b-1)}\right] \exp(-\beta |y|^{2b/(2b-1)}) |\mu_j|(dy).$$

It is easy to show that there exists a positive constant β' such that

$$\exp\left[-\frac{c}{2}\left(\frac{|z-y|^{2b}}{t}\right)^{1/(2b-1)} + \beta |y|^{2b/(2b-1)}\right] \leqslant \exp(\beta' |z|^{2b/(2b-1)})$$

for $z, y \in R_n$ and for t sufficiently small, hence $\lim_{t\to 0} J_4 = 0$. Noting finally that

$$\lim_{t\to 0} \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, z;\ 0, y) g_j(x) dy = \mu_i'(x) = g_i(x),$$

we obtain

$$\lim_{\substack{|z-x|<\gamma t^{1/2b} \\ t\to 0}} u_i(t, z) = u_i'(x).$$

3. Application to special parabolic systems. Consider a parabolic system of the form

$$(14) \quad \sum_{i,j=1}^{n} a_{ij}^k(t, x) \frac{\partial^2 u_k}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^k(t, x) \frac{\partial u_k}{\partial x_i} + \sum_{l=1}^{N} c_l^k(t, x) u_l - \frac{\partial u_k}{\partial t} = 0$$

$$(k = 1, \ldots, N),$$

where the k-th equation contains derivatives of only one unknown function.

Throughout this section we assume

I′. The coefficients a_{ij}^k, b_i^k, c_l^k and derivatives

$$\frac{\partial}{\partial x_i} a_{ij}^k, \quad \frac{\partial^2}{\partial x_i \partial x_j} a_{ij}^k, \quad \frac{\partial}{\partial x_i} b_i^k$$

are continuous bounded functions in $[0, T] \times R_n$ and Hölder continuous in x uniformly with respect to (t, x) in $[0, T] \times R_n$; additionally the principal coefficients are continuous in t uniformly with respect to (t, x) in $[0, T]\times R_n$.

II'. There exists a positive constant $\varkappa$ such that for any real vector $\xi \in R_n$

$$\sum_{i,j=1}^{N} a_{ij}^{k}(t, x)\,\xi_i\,\xi_j \geqslant \varkappa|\xi|^2 \qquad (k = 1, \ldots, N)$$

for all $(t, x)\in [0, T]\times R_n$.

III'. $c_k^l(t, x) \geqslant 0$ for all $(t, x)\in [0, T]\times R_n$ and $l \neq k$ $(l, k = 1, \ldots, N)$.

It is clear that condition II' is equivalent to parabolicity in the sense of Petrovskiĭ.

It follows [10] from III' that $\Gamma_{ij}(t, x;\ \tau, y) \geqslant 0$ $(i, j = 1, \ldots, N)$ for (t, x), $(\tau, y)\in [0, T]\times R_n$ $(\tau < t)$. In paper [10] it is shown that all elements $\Gamma_{ij}(t, x;\ \tau, y)$ of the fundamental matrix of the system (1) (with $b = 1$) are non-negative if and only if

$$A_k^{ij}(t, x) = 0 \text{ in } [0, T]\times R_n \quad \text{for } |k| = 1, 2 \text{ and } i \neq j,$$

$$A_0^{ij}(t, x) \geqslant 0 \text{ in } [0, T]\times R_n \quad \text{for } i \neq j.$$

In this section we have always defined the norms $\|\cdot\|_{L_\beta^p}$ by

$$\|f\|_{L_\beta^p} = \Big[\int_{R_n} \exp(-p\beta|x|^2)\,|f(x)|^p\,dx\Big]^{1/p} \quad \text{if } 1 \leqslant p < \infty$$

and

$$\|f\|_{L_p^\infty} = \operatorname*{ess\,sup}_{R_n}[\exp(-\beta|x|^2)\,|f(x)|].$$

We shall extend Theorem 1 to the solutions of (14) satisfying the growth conditions

$$\|u_j^-(t, \cdot)\|_{L_\beta^p} \leqslant M \quad \text{for } 0 < t \leqslant T \ (j = 1, \ldots, N),$$

where $u_j^-(t, x) = \max[0, -u(t, x)]$. We begin with the following lemma:

LEMMA 2. *Let* $\{u_j(t, x)\}$ $(j = 1, \ldots, N)$ *be a solution of* (14) *in* $(0, T]\times R_n$ *such that*

$$\|u_j^-(t, \cdot)\|_{L_\beta^p} \leqslant M \quad (j = 1, \ldots, N)$$

for $0 < t \leqslant T$. *Then there exists a positive constant* δ_1 *such that*

$$u_i(t, x) + \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ \tau, y)\,u_j^-(\tau, y)\,dy \geqslant 0 \quad (i = 1, \ldots, N)$$

for $(t, x)\in \big(\tau, \min(T, \tau+\delta_1)\big]\times R_n$, *where* $0 < \tau < T$.

Proof. Introduce the function

$$w_i(t, x) = u_i^+(t, x) - u_i^-(t, x) + \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ \tau, y) u_j^-(\tau, y) dy \qquad (i = 1, \ldots, N).$$

It is clear that there exists a positive constant δ_1 such that $\{w_i\}$ $(i = 1, \ldots, N)$ satisfies the system (14) in $(\tau, \min(T, \tau+\delta_1)] \times R_n$. Note that

$$w_i^-(t, x) \leqslant u_i^-(t, x) \qquad (i = 1, \ldots, N),$$

which implies that the inequality $\|w_i^-(t, \cdot)\|_{L^p_\beta} \leqslant M$ $(i = 1, \ldots, N)$ holds for $t \in (\tau, \min(\tau+\delta_1, T)]$.

Since

$$\lim_{t\to\tau} w_j(t, x) = u_j^+(\tau, x) \geqslant 0 \qquad (j = 1, \ldots, N)$$

in R_n, it follows from the maximum principle (Theorem 5 of [12]) that $w_j(t, x) \geqslant 0$ $(j = 1, \ldots, N)$ in $(\tau, \min(\tau+\delta_1, T)] \times R_n$.

Now with the aid of Lemma 3 we can generalize Theorem 1:

THEOREM 3. *Let* $\{u_j(t, x)\}$ $(j = 1, \ldots, N)$ *be a solution of* (14) *in* $(0, T] \times R_n$ *such that*

$$\|u_j^-(t, \cdot)\|_{L^p_\beta} \leqslant M \qquad (j = 1, \ldots, N)$$

for $0 < t \leqslant T$, *where* $1 \leqslant p \leqslant \infty$. *Then there are* $f_j \in L^p_\beta$ $(j = 1, \ldots, N)$ *such that*

$$u_i(t, x) + \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x;\ 0, y) f_j(y) dy \geqslant 0 \qquad (j = 1, \ldots, N)$$

in $(0, \delta_1] \times R_n$ *with the usual modification when* $p = 1$.

Proof. We know that the sets

$$\left\{u_i^-\left(\frac{1}{k}, \cdot\right),\ k = 1, \ldots\right\} \qquad (i = 1, \ldots, N)$$

are bounded in L^p_β. There are $f_j \in L^p_\beta$ $(j = 1, \ldots, N)$ and a subsequence $1/k_s$ such that, for any $g_i \in L^{p'}_{-\beta}$,

$$\lim_{s\to\infty} \int_{R_n} u_j^-\left(\frac{1}{k_s}, y\right) g_j(y) dy = \int_{R_n} f_j(y) g_j(y) dy \qquad (j = 1, \ldots, N).$$

It is clear that $\Gamma_{ij}(t, x; 0, y)$ is in $L^{p'}_{-\beta}$. Hence

$$(15)\qquad \lim_{s\to\infty} \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x; 0, y) u_j^-\left(\frac{1}{k_s}, y\right) dy = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x; 0, y) f_j(y)\, dy.$$

From Lemma 2 it follows that

$$u_i(t, x) + \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}\left(t, x; \frac{1}{k_s}, y\right) u_j^-\left(\frac{1}{k_s}, y\right) dy \geqslant 0 \qquad (i = 1, \ldots, N),$$

so that it suffices to prove that

$$\lim_{s\to\infty} \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}\left(t, x; \frac{1}{k_s}, y\right) u_j^-\left(\frac{1}{k_s}, y\right) dy = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x; 0, y) f_j(y)\, dy \qquad (i = 1, \ldots, N).$$

Write

$$\int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}\left(t, x; \frac{1}{k_s}, y\right) u_j^-\left(\frac{1}{k_s}, y\right) dy - \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x; 0, y) f_j(y)\, dy$$
$$= \left[\int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}\left(t, x; \frac{1}{k_s}, y\right) u_j^-\left(\frac{1}{k_s}, y\right) dy - \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x; 0, y) u_j^-\left(\frac{1}{k_s}, y\right) dy\right] +$$
$$+ \left[\int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x; 0, y) u_j^-\left(\frac{1}{k_s}, y\right) dy - \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t, x; 0, y) f_j(y)\, dy\right] = J_1 + J_2.$$

By (15) we get $\lim_{s\to\infty} J_2 = 0$. Using the method of the proof of Theorem 1 one can show that $\lim_{s\to\infty} J_1 = 0$.

In paper [11] a representation theorem for non-negative solutions is proved in the form of

$$(16)\qquad u_i(t,x) = \int_{R_n} \sum_{j=0}^{N} \Gamma_{ij}(t,x;\ 0,y)\mu_j(dy) \qquad (j = 1,\ldots,N),$$

where μ_j are non-negative measures satisfying condition

$$(17)\qquad \int_{R_n} \exp(-\beta|x|^2)\mu_j(dx) < \infty \qquad (j = 1,\ldots,N).$$

As an immediate consequence of Theorem 3 and (16) we obtain

COROLLARY 1. *Let* $\{u_j(t,x)\}$ $(j = 1,\ldots,N)$ *be a solution of* (14) *in* $(0,T]\times R_n$ *such that*

$$\|u_i^-(t,\cdot)\|_{L^p} \leqslant M \qquad (j = 1,\ldots,N)$$

for $0 < t \leqslant T$. *Then there are measure* μ_j *satisfying* (17) *and* $f_j \in L^p_\beta$ $(j = 1,\ldots,N)$ *if* $1 < p \leqslant \infty$ *with the usual modification when* $p = 1$, *such that*

$$(18)\quad u_i(t,x) = \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t,x;\ 0,y)\mu_j(dy) - \int_{R_n} \sum_{j=1}^{N} \Gamma_{ij}(t,x;\ 0,y)f_j(y)\,dy \qquad (i = 1,\ldots,N)$$

for $(t,x)\in(0,\delta_1]\times R_n$.

In view of Lemma 1 and the representation (16) we can state the following corollary:

COROLLARY 2. *Let* $\{u_j(t,x)\}$ $(j = 1,\ldots,N)$ *be a non-negative solution of* (14) *in* $(0,T]\times R_n$ *such*

$$\lim_{t\to 0} u_i(t,\cdot) = 0 \qquad (i = 1,\ldots,N)$$

in the topology of weak convergence of measures. Then $u_j(t,x) = 0$ *in* $(0,T]\times R_n$ $(j = 1,\ldots,N)$.

Using the representation (16) we deduce the following

COROLLARY 3. *Let* $\{u_j(t,x)\}$ $(j = 1,\ldots,N)$ *be a solution of* (14) *in* $(0,T]\times R_n$ *such that*

$$\|u_i^-(t,\cdot)\|_{L^p_\beta} \leqslant M \qquad (i = 1,\ldots,N)$$

for $0 < t \leqslant T$ *and*

$$\lim_{t\to 0} u_i(t,\cdot) = 0 \qquad (i = 1,\ldots,N)$$

in the topology of weak convergence of measures. Then $u_j(t,x) = 0$ $(j = 1,\ldots,N)$ *in* $(0,T]\times R_n$.

COROLLARY 4. *Let* $\{u_j(t, x)\}$ $(j = 1, \ldots, N)$ *be a solution of* (14) *in* $(0, T] \times R_n$ *such that*

$$\|u_j^-(t, \cdot)\|_{L^p_\beta} \leqslant M \quad (j = 1, \ldots, N)$$

for $0 < t \leqslant T$, *where* $1 \leqslant p \leqslant \infty$. *Then* u_j $(j = 1, \ldots, N)$ *have limits almost everywhere in* R_n *if* $t \to 0$.

REFERENCES

[1] D. G. Aronson, *On initial value problem for parabolic system of differential equations*, Bulletin of the American Mathematical Society 65 (1959), p. 310-318.

[2] — *Uniqueness of positive solutions of second order parabolic equations*, Annales Polonici Mathematici 16 (1965), p. 285-303.

[3] — *Non-negative solutions of linear parabolic equations*, Annali della Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche, 22 (1968), p. 607-694.

[4] — *Non-negative solutions of linear parabolic equations*; An addendum, ibidem 25 (1971), p. 221-228.

[5] — and P. Besala, *Parabolic equations with unbounded coefficients*, Journal of Differential Equations 3 (1967), p. 1-14.

[6] — *Uniqueness of positive solutions of parabolic equations with unbounded coefficients*, Colloquium Mathematicum 18 (1967), p. 125-135.

[7] P. Besala, *Limitations of solutions of non-linear parabolic equations in unbounded domains*, Annales Polonici Mathematici 17 (1965), p. 25-47.

[8] W. Bodanko, *Les propriétés des solutions non négatives de l'équation linéaire normale parabolique*, ibidem 20 (1968), p. 107-117.

[9] J. Chabrowski, *Bemerkungen über Zeichen der Elemente der Matrix der Grundlösungen für parabolische Systeme von partiellen Differentialgleichungen zweiter Ordnung*, ibidem 19 (1967), p. 287-300.

[10] — *Les solutions non négatives d'un système parabolique d'équations*, ibidem 19 (1967), p. 193-197.

[11] — *Les propriétés des solutions non négatives d'un système parabolique d'équations*, ibidem 22 (1970), p. 223-231.

[12] — *Sur l'unicité du problème de Cauchy dans une classe de fonctions non bornées*, ibidem 24 (1971), p. 127-135.

[13] — *Certaines propriétés des solutions non négatives d'un système parabolique d'équations*, ibidem 24 (1971), p. 137-143.

[14] J. Chabrowski, *Propriétés des solutions faibles non négatives de l'équation parabolique*, Nagoya Mathematical Journal 39 (1970), p. 119-125.

[15] S. D. Eidelman, *Parabolic systems*, Amsterdam 1969.

[16] A. Friedman, *Partial differential equations of parabolic type*, Englewood Cliffs 1964.

[17] R. Johnson, *Representation theorems and Fatou theorems for second-order linear parabolic partial differential equations*, Proceedings of the London Mathematical Society 23 (1971), p. 325-347.

[18] — *Representation theorems for the heat equations*, University of Maryland, TR 70-95, 1970.

[19] M. Kato, *On positive solutions of the heat equation*, Nagoya Mathematical Journal 30 (1968), p. 203-207.

[20] M. Krzyżański, *Sur les solutions non négatives de l'équation linéaire normale parabolique*, Revue Roumaine de Mathématiques Pures et Appliquées 9 (5) (1964), p. 393-408.

[21] W. Pogorzelski, *Étude de la matrice des solutions fondamentales du système parabolique d'équations aux dérivées partielles*, Ricerche di Matematica 7 (1958), p. 153-185.

[22] — *Propriétés des intégrales généralisées de Poisson-Weierstrass et problème de Cauchy pour un système parabolique*, Annales Scientifiques de l'École Normale Supérieure 76 (1959), p. 125-149.

[23] D. Widder, *Positive temperatures on an infinite rod*, Transactions of the American Mathematical Society 55 (1944), p. 85-95.

Reçu par la Rédaction le 23. 3. 1972