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On univalent p-symmetric functions in the unit disc

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Abstract. Let $S_1^{(p)}$ be the class of functions $f(z) = \sum_{n=0}^{\infty} a_{np+1} z^{np+1}$, $|f(z)| < 1$, regular and univalent in the disc $K = \{z\colon |z| < 1\}$, where p is an arbitrary fixed natural number. We define for any $z \in K$, $z \neq 0$, and for any function from the class $S_1^{(p)}$ the quantities $a_1 = f'(0)$, $r = |z|$, $v = |f(z)|$, $v' = |f'(z)|$.

The present paper offers the definite solution to the problem of determining the sharp estimation from above of the functional $v' = |f'(z)|$ defined in the class $S_1^{(p)}$, depending on $a_1 = f'(0)$ and $v = |f(z)|$ for $z \in K$.

It contains also a discussion of relations among $r = |z|$, $V = |F(z)|$, $V' = |F'(z)|$, where F belongs to the class $S^{(p)}$ of functions which are regular, univalent and p-symmetric in K and are normalized by the conditions $F(0) = 0$, $F'(0) = 1$. The results concerning the class $S^{(p)}$ are obtained from those for the class $S_1^{(p)}$ by a suitable passage to the limit.

This paper belongs to the group of works by R. Robinson, Z. J. Jakubowski and L. Mikołajczyk and is their natural continuation.

I. Introduction. Let $S_1^{(p)}$ denote the class of functions $f(z) = \sum_{n=0}^{\infty} a_{np+1} z^{np+1}$ which are regular and univalent in the disc $K = \{z\colon |z| < 1\}$ and satisfy $|f(z)| < 1$ for $z \in K$. The grade of symmetry p is an arbitrary fixed natural number. Without any loss of generality we can assume in our considerations $f'(0) > 0$. For any $z \in K$, $z \neq 0$, and for any function $f \in S_1^{(p)}$ let us define the quantities

$$a_1 = f'(0), \quad r = |z|, \quad v = |f(z)|, \quad v' = |f'(z)|. \tag{1.1}$$

It is easy to see that

$$0 < a_1 \leqslant 1, \quad 0 < r < 1, \quad 0 < v \leqslant r < 1, \quad 0 < v'$$

and, clearly, if $a_1 = 1$ or $v = r$, then $f(z) = z$ and hence $v' = 1$.

R. Robinson (cf. [11]) determined non-trivial relations between quantities (1.1) for $p = 1$.

His results have been generalized by Z. J. Jakubowski. In [3], [4] he considered this problem in the class $S_1^{(p)}$ of p-symmetric functions for an arbitrary p and has obtained various relations between quantities (1.1).

The present paper belongs to the group of works mentioned above and is their natural continuation. It offers the definite solution to the problem of determining the sharp estimation from above of the functional $v' = |f'(z)|$ for $f \in S_1^{(p)}$, $z \in K$, depending on $a_1 = f'(0)$ and $v = |f(z)|$.

It contains also a discussion of relations between

$$r = |z|, \quad V = |F(z)|, \quad V' = |F'(z)|$$

for $F \in S^{(p)}$, where $S^{(p)}$ is the class of functions which are regular, univalent and p-symmetric in K and are normalized by the conditions

$$F(0) = 0, \quad F'(0) = 1.$$

The results concerning the class $S^{(p)}$ are obtained from those for the class $S_1^{(p)}$ by a suitable passage to the limit.

II. Preliminaries.

1. The procedure employed to determine the least upper bound of the functional v' depending on a_1 and v in the class $S_1^{(p)}$ is an example of an effective application of K. Löwner's [5] well-known method of parametric representation of some families of univalent functions. This method was generalized by I. Bazylewicz [1] to functions univalent and p-symmetric in K. From [1] it is known that if a function $f(z) \in S_1^{(p)}$ maps the disc K onto a region obtained from K by deleting a finite number of Jordan's arcs, then there exists a function $k(t)$, $|k(t)| = 1$, defined and piece-wise continuous for $t \in \langle 0; t_0\rangle$, $t_0 = -p\log f'(0)$, such that the solution $f = f(z, t)$ of Löwner's equality with the initial condition $f(z, 0) = z$ assumes the value $f(z, t_0) = f(z)$ at $t = t_0$.

Further considerations concerning the family $S_1^{(p)}$ may be restricted to functions $f(z)$ satisfying the assumptions of the above theorem.

Obviously, the function $f(z, t)$ which is the solution of Löwner's equality satisfies

$$p\frac{\partial}{\partial t}\log|f(z, t)| = -\frac{1-|f(z, t)|^{2p}}{|1-k(t)f^p(z, t)|^2}. \tag{2.1}$$

As seen from the form of (2.1), for a fixed z, $s = |f(z, t)|$ is a decreasing function of t. On the other hand, from (1.1) and from the boundary conditions $f(z, 0) = z$, $f(z, t_0) = f(z)$ it follows that when t increases from 0 to t_0, then s decreases from r to v. Thus the parameter t may be treated as a function of the independent variable $s \in \langle v; r\rangle$.

Let us now put

$$h(s) = \frac{|1-s^p g(s)|^2}{1-s^{2p}}, \tag{2.2}$$

where

$$g(s) = k(t)\frac{f^p(z,t)}{|f(z,t)|^p}.$$

From the properties of $k(t)$ and $f(z,t)$ it follows that $h(s)$ is defined and continuous everywhere in $\langle v; r\rangle$ but a finite number of discontinuity points.

Since $|g(s)| = 1$, we have

$$\frac{1-s^p}{1+s^p} \leqslant h(s) \leqslant \frac{1+s^p}{1-s^p} \tag{2.3}$$

for $s \in \langle v; r\rangle$. By (2.1) and (2.2)

$$t_0 = p\int_v^r h(s)\frac{ds}{s}. \tag{2.4}$$

Note that, conversely, for any piece-wise continuous function $h(s)$ satisfying (2.3) it is possible [4] to determine a function $g(s)$ connected with $h(s)$ by (2.2); to $g(s)$, in turn, there correspond functions $k(t)$ and $f(z,t)$, with which a certain function of class $S_1^{(p)}$ is associated.

Therefore, any function $h(s)$ of the form (2.2) determines some function $f(z)$ of the family $S_1^{(p)}$.

Making use of Löwner's equality for functions of class $S_1^{(p)}$ we obtain also [4]

$$v' = \frac{r^{p-1}(1-v^{2p})}{v^{p-1}(1-r^{2p})}e^{-R}, \tag{2.5}$$

where

$$R = p\int_v^r h^{-1}(s)\frac{ds}{s}. \tag{2.6}$$

The following inequalities are consequences of (2.3)–(2.6)

$$\frac{u_2(v)}{u_2(r)} \leqslant a_1 \leqslant \frac{u_1(v)}{u_1(r)}, \tag{2.7}$$

$$\frac{v}{r}\frac{u_3(r)}{u_3(v)} \leqslant v' \leqslant \frac{v}{r}\frac{u_3(v)}{u_3(r)}, \tag{2.8}$$

where

$$u_1(x) = \frac{x}{(1+x^p)^{2/p}}, \quad u_2(x) = \frac{x}{(1-x^p)^{2/p}}, \quad u_3(x) = \frac{1-x^p}{1+x^p}.$$

Inequalities (2.7) and (2.8), though obtained under the assumption $v < r$, remain still valid for $v = r$, because then $a_1 = v' = 1$.

2. We shall now present some important relations between integrals (2.4) and (2.6) and discuss their consequences.

Consider the functions

$$\begin{aligned} \varphi(x, r, v, p) &= p\int_v^x u_3(s)\frac{ds}{s} + p\int_x^r u_3(x)\frac{ds}{s} \\ &= \log\frac{u_1^p(x)}{u_1^p(v)} + u_3(x)\log\frac{r^p}{x^p}, \\ \psi(x, r, v, p) &= p\int_v^x u_3^{-1}(s)\frac{ds}{s} + p\int_x^r u_3^{-1}(x)\frac{ds}{s} \\ &= \log\frac{u_2^p(x)}{u_2^p(v)} + u_3^{-1}(x)\log\frac{r^p}{x^p}, \end{aligned} \tag{2.9}$$

defined for $x \in \langle v; r\rangle$. Setting $x = r$ or $x = v$ we obtain

$$\begin{aligned} \varphi(r, r, v, p) &= \log\frac{u_1^p(r)}{u_1^p(v)}, & \varphi(v, r, v, p) &= u_3(v)\log\frac{r^p}{v^p}, \\ \psi(r, r, v, p) &= \log\frac{u_2^p(r)}{u_2^p(v)}, & \psi(v, r, v, p) &= u_3^{-1}(v)\log\frac{r^p}{v^p}. \end{aligned} \tag{2.10}$$

It can be easily shown [4] that

$$\varphi(r, r, v, p) < \varphi(v, r, v, p) < \psi(v, r, v, p) < \psi(r, r, v, p), \tag{2.11}$$

and

$$\varphi(r, r, v, p) \leqslant t_0 \leqslant \psi(r, r, v, p). \tag{2.12}$$

Assuming t_0 to be an arbitrary fixed number satisfying (2.12) we find the sharp estimation [4]

$$R \geqslant \begin{cases} \psi(a, r, v, p) & \text{when } \varphi(r, r, v, p) \leqslant t_0 \leqslant \varphi(v, r, v, p), \\ t_0^{-1}\left(\log\dfrac{r^p}{v^p}\right)^2 & \text{when } \varphi(v, r, v, p) \leqslant t_0 \leqslant \psi(v, r, v, p), \\ \varphi(\beta, r, v, p) & \text{when } \psi(v, r, v, p) \leqslant t_0 \leqslant \psi(r, r, v, p), \end{cases} \tag{2.13}$$

where α and β are the roots of

$$\varphi(\alpha, r, v, p) = t_0, \tag{2.14}$$

$$\psi(\beta, r, v, p) = t_0, \tag{2.15}$$

respectively, and belong to $\langle v; r\rangle$.

The sharp estimation from above of the integral R [4] for an arbitrary fixed number t_0 from interval (2.12) is of the form

$$R \leqslant \varphi(r, r, v, p) + \psi(r, r, v, p) - t_0. \tag{2.16}$$

By (2.13) and (2.5) we have [4]:

THEOREM 1. *If $f(z)$ is an arbitrary function of class $S_1^{(p)}$, then a_1, r, v and v' satisfy*

$$\log v' \leqslant M(a_1, r, v, p), \tag{2.17}$$

where

$$M(a_1, r, v, p) = \log \frac{r^{p-1}(1-v^{2p})}{v^{p-1}(1-r^{2p})} - L(a_1, r, v, p), \tag{2.18}$$

and

$$L(a_1, r, v, p) \tag{2.19}$$

$$= \begin{cases} \psi(\alpha, r, v, p) & \text{when } \varphi(r, r, v, p) \leqslant t_0 \leqslant \varphi(v, r, v, p), \\ t_0^{-1}\left(\log \dfrac{r^p}{v^p}\right)^2 & \text{when } \varphi(v, r, v, p) \leqslant t_0 \leqslant \psi(v, r, v, p), \\ \varphi(\beta, r, v, p) & \text{when } \psi(v, r, v, p) \leqslant t_0 \leqslant \psi(r, r, v, p), \end{cases}$$

α and β being the roots of (2.14) *and* (2.15), *respectively.*

This theorem provides the essential relation between a_1, r, v, v' to be used in our further considerations.

As a consequence of (2.16) we obtain [4]:

THEOREM 2. *If f is an arbitrary function of class $S_1^{(p)}$, then a_1, r, v, v' satisfy*

$$v' \geqslant \frac{1}{a_1^p} \frac{v^{p+1}}{1-v^{2p}} \frac{1-r^{2p}}{r^{p+1}}. \tag{2.20}$$

Since estimations (2.13) and (2.16) are sharp, (2.17) and (2.20) are sharp, too.

III. Estimation of the functional v' depending on a_1 and v in the class $S_1^{(p)}$.

1. From (2.17) it readily follows that the admissible values of a_1, v, v' satisfy the inequality

$$\log v' \leqslant \sup_r M(a_1, r, v, p). \tag{3.1}$$

We shall find this upper bound.

(2.18) and (2.19) show that the function $M(a_1, r, v, p)$ is defined by three formulas depending on the position of the integral t_0 in the interval $\langle\varphi(r, r, v, p); \psi(r, r, v, p)\rangle$. To determine the upper bound in (3.1) we shall first express the three cases of the position of t_0 in terms involving of the position of r.

It is clear from (2.10) that all the four functions $\varphi(r, r, v, p)$, $\varphi(v, r, v, p)$, $\psi(v, r, v, p)$, $\psi(r, r, v, p)$ are increasing in r. They increase in $\langle v; 1)$ from 0 to

$$\begin{aligned} w_M &= \lim_{r\to 1}\varphi(r, r, v, p) &&= -\log 4u_1^p(v), \\ w_2 &= \lim_{r\to 1}\varphi(v, r, v, p) &&= u_3(v)\log v^{-p}, \\ w_1 &= \lim_{r\to 1}\psi(v, r, v, p) &&= u_3^{-1}(v)\log v^{-p}, \\ w_m &= \lim_{r\to 1}\psi(r, r, v, p) &&= +\infty, \end{aligned} \tag{3.2}$$

respectively. They satisfy also relations (2.11) (Fig. 1).

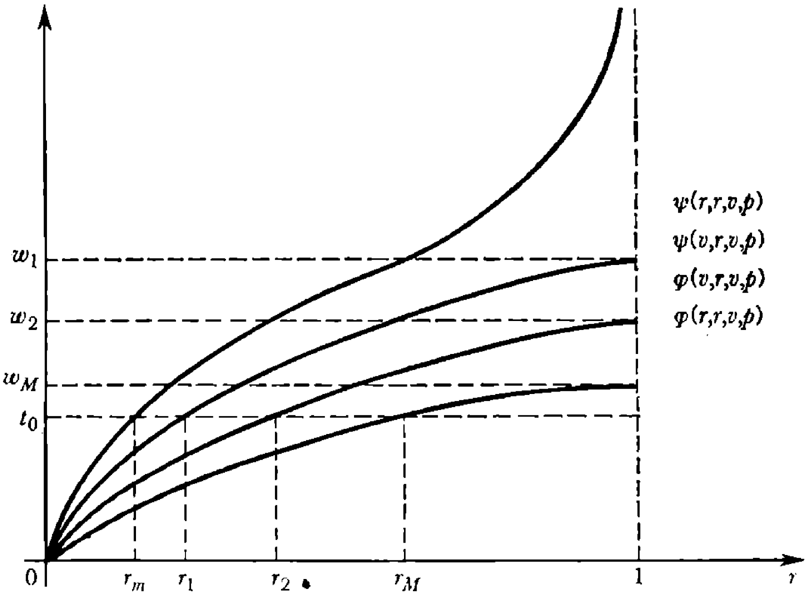

Fig. 1

Denote by r_m, r_1, r_2, r_M the roots belonging to $(v; 1)$ of the equations

$$\psi(r_m, r_m, v, p) = \psi(v, r_1, v, p) = \varphi(v, r_2, v, p) = \varphi(r_M, r_M, v, p) = t_0, \tag{3.3}$$

respectively. If those roots exist, then, by (2.11),

$$v < r_m < r_1 < r_2 < r_M < 1. \tag{3.4}$$

From the behaviour of $\psi(r, r, v, p)$ it is seen that for any value of t_0 there exists $r_m < 1$. It should be noted that all r_1, r_2, r_M are defined if and only if $t_0 < w_M$, i.e., if

$$a_1 > \frac{4^{1/p}v}{(1+v^p)^{2/p}} \tag{3.5}$$

holds.

From the behaviour of functions (2.10) of the variable r it follows that $r \in \langle r_m; r_M\rangle$ and the cases

$$\begin{gathered}\varphi(r, r, v, p) \leqslant t_0 \leqslant \varphi(v, r, v, p),\\ \varphi(v, r, v, p) \leqslant t_0 \leqslant \psi(v, r, v, p),\\ \psi(v, r, v, p) \leqslant t_0 \leqslant \psi(r, r, v, p),\end{gathered}$$

are equivalent to

$$r_2 \leqslant r \leqslant r_M, \qquad r_1 \leqslant r \leqslant r_2, \qquad r_m \leqslant r \leqslant r_1, \tag{3.6}$$

respectively.

If a prescribed pair (v, a_1) does not satisfy (3.5), then the variable r can take values arbitrarily close to 1.

2. We now show that for certain pairs (v, a_1) which do not satisfy (3.5) there exists no finite upper bound of the functional v'.

The following lemma is true:

LEMMA 1. *If the numbers a_1 and v satisfy*

$$a_1 < \frac{4^{1/p}v}{(1+v^p)^{2/p}}, \tag{3.7}$$

then there exists no finite upper bound of the functional $v' = |f'(z)|$ in the class of functions $f \in S_1^{(p)}$ for which a_1 and v are the values (1.1).

Proof. By (3.7) $r \in (r_m; 1)$ and, consequently, we may examine the behaviour of $L(a_1, r, v, p)$ in the left-hand side neighbourhood of the point $r = 1$. By (2.19) and (3.6) we have

$$\lim L(a_1, r, v, p) = \begin{cases}\psi(a, 1, v, p) & \text{if } r_M \text{ does not exist,}\\ p^2 t_0^{-1}\log v^{-1} & \text{if } r_2 \text{ and } r_M \text{ do not exist,}\\ \varphi(\beta, 1, v, p) & \text{if } r_1, r_2 \text{ and } r_M \text{ do not exist,}\end{cases}$$

as $r \to 1$.

It means that if (3.7) is true, then in all possible cases the above limit is finite and therefore

$$\lim M(a_1, r, v, p) = +\infty \quad \text{as } r \to 1.$$

Now notice that for any sequence $\{r_n\}$, $0 < r_n < 1$, $r_n \to 1$, there exists a sequence $\{h_n(s)\}$ of functions having the form (2.2) and realizing sharpness in estimation (2.13). The functions $h_n(s)$ determine a sequence of functions $\{f_n(z)\}$, $f_n \in S_1^{(p)}$, for which $v_n' = M(a_1, r_n, v, p)$, and this results in $\lim v_n' = +\infty$ as $n \to \infty$.

3. In the sequel, in view of Lemma 1, we assume that condition (3.5) is satisfied.

By (2.18), (2.14) and (2.15) we have

$$(3.8) \quad M_r'(a_1, r, v, p)$$

$$= \begin{cases} \dfrac{p-1+(p+1)r^{2p}}{r(1-r^{2p})} - \dfrac{2p}{r}\,\dfrac{1-\beta^p}{1+\beta^p} & \text{for } r_m \leqslant r \leqslant r_1, \\[2ex] \dfrac{p-1+(p+1)r^{2p}}{r(1-r^{2p})} - \dfrac{2p^2}{rt_0}\log\dfrac{r}{v} & \text{for } r_1 \leqslant r \leqslant r_2, \\[2ex] \dfrac{p-1+(p+1)r^{2p}}{r(1-r^{2p})} - \dfrac{2p}{r}\,\dfrac{1+a^p}{1-a^p} & \text{for } r_2 \leqslant r \leqslant r_M, \end{cases}$$

where a and β are defined by (2.14) and (2.15), respectively. Note that $M_r'(a_1, r, v, p)$ is a continuous function in the interval $\langle r_m; r_M\rangle$.

We shall now examine the sign of the derivative $M_r'(a_1, r, v, p)$ at the points r_m, r_1, r_2, r_M.

If $r = r_m$, then by (3.3) and (2.15) we have $\beta = r_m$ and thus

$$(3.9) \qquad M_r'(a_1, r_m, v, p) \begin{cases} < 0 & \text{when } r_m < v_m, \\ = 0 & \text{when } r_m = v_m, \\ > 0 & \text{when } r_m > v_m, \end{cases}$$

where

$$v_m = \begin{cases} \frac{1}{2} & \text{for } p = 1, \\ \left(\dfrac{2p-\sqrt{3p^2+1}}{p-1}\right)^{1/p} & \text{for } p > 1. \end{cases}$$

Let now $r = r_1$. This implies $\beta = v$ and by (3.8)

$$M_r'(a_1, r_1, v, p) = \frac{1}{r_1}\,\frac{[3p+1+(1-p)v^p]r_1^{2p} - [p+1+(1-3p)v^p]}{(1-r_1^{2p})(1+v^p)}.$$

To examine the sign of the derivative we shall consider two cases:

$$v^p \geqslant \frac{p+1}{3p-1}, \tag{3.10}$$

and

$$v^p < \frac{p+1}{3p-1}. \tag{3.11}$$

If (3.10) holds, then $M'_r(a_1, r_1, v, p) > 0$ for any admissible values of a_1 and v. In the case of (3.11) being true we readily obtain

$$M'_r(a_1, r_1, v, p) \begin{cases} < 0 & \text{for } r_1 < v_1, \\ = 0 & \text{for } r_1 = v_1, \\ > 0 & \text{for } r_1 > v_1, \end{cases} \tag{3.12}$$

where

$$v_1 = v_1(v) = \left(\frac{p+1+(1-3p)v^p}{3p+1+(1-p)v^p}\right)^{1/2p}$$

If $r = r_2$, then by (3.3), (2.14) we have $a = v$. Thus by (3.8) we get

$$M'_r(a_1, r_2, v, p) \begin{cases} < 0 & \text{for } r_2 < v_2, \\ = 0 & \text{for } r_2 = r_2, \\ > 0 & \text{for } r_2 > v_2, \end{cases} \tag{3.13}$$

where

$$v_2 = v_2(v) = \left(\frac{p+1+(3p-1)v^p}{3p+1+(p-1)v^p}\right)^{1/2p}$$

In the case of $r = r_M$ we have $a = r_M$ and thus by (3.8)

$$M'_r(a_1, r_M, v, p) < 0. \tag{3.14}$$

It should be noted that the inequalities

$$v < r_m, \qquad v_m < \left(\frac{p+1}{3p-1}\right)^{1/p}$$

are true; thus if $r_m < v_m$, inequality (3.11) holds.

Since the functions in (2.10) are increasing in r, we can replace (3.9), (3.12), (3.13) and (3.14) by the equivalent relations

$$M'_r(a_1, r_m, v, p) \begin{cases} < 0 & \text{for } a_1 > k_m(v), \\ = 0 & \text{for } a_1 = k_m(v), \\ > 0 & \text{for } a_1 < k_m(v), \end{cases} \tag{3.15}$$

$$M'_r(a_1, r_1, v, p) \begin{cases} < 0 & \text{for } a_1 > k_1(v), \\ = 0 & \text{for } a_1 = k_1(v), \\ > 0 & \text{for } a_1 < k_1(v), \end{cases} \tag{3.16}$$

$$M'_r(a_1, r_2, v, p) \begin{cases} < 0 & \text{for } a_1 > k_2(v), \\ = 0 & \text{for } a_1 = k_2(v), \\ > 0 & \text{for } a_1 < k_2(v), \end{cases} \tag{3.17}$$

$$M'_r(a_1, r_M, v, p) < 0 \quad \text{for } a_1 > k_M(v), \tag{3.18}$$

respectively, where

$$\begin{aligned} k_m(v) &= u_2(v)\cdot u_2^{-1}(v_m) && \text{for } v \in \langle 0; v_m\rangle, \\ k_1(v) &= \big(v\cdot v_1^{-1}(v)\big)^{u_3^{-1}(v)} && \text{for } v \in \langle 0; v_m\rangle, \\ k_2(v) &= \big(v\cdot v_2^{-1}(v)\big)^{u_3(v)} && \text{for } v \in \langle 0; 1\rangle, \\ k_M(v) &= \sqrt[p]{4u_1(v)} && \text{for } v \in \langle 0; 1\rangle. \end{aligned} \tag{3.19}$$

4. We now turn to the problem of the behaviour of the functions (3.19) of the variable v and examine their graphs.

It can be easily seen that k_M is concave and increasing in $\langle 0; 1\rangle$ and $k_M(0) = 0$, $k_M(1) = 1$. On the other hand, k_m is convex and increasing in $\langle 0; v_m\rangle$. Since k_m and k_M are continuous, $k_m(v_m) = 1$ and $v_m < 1$, we deduce that there exists a point at which the two curves meet. The abscissa of that point is

$$v_3 = \left(\frac{1-2v_m^{p/2}-v_m^p}{-1-2v_m^{p/2}+v_m^p}\right)^{1/p}. \tag{3.20}$$

The function k_1 increases in $\langle 0; v_m\rangle$, $k_1(0) = 0$, $k_1(v_m) = 1$. Similarly, k_2 increases in $\langle 0; 1\rangle$, $k_2(0) = 0$, $k_2(1) = 1$.

Notice that for $v \in (0; 1)$ we have $k_M(v) > k_2(v)$. Indeed, the function $\varkappa(v)$ defined by

$$\varkappa(v) = \log k_M(v) - \log k_2(v)$$

is positive for $v \in (0; 1)$.

Let us look at the relationship between $k_m(v)$ and $k_1(v)$ for $v \in \langle 0; v_m\rangle$. As it is known, $k_m(v_m) = k_1(v_m) = 1$ and $k_m(0) = k_1(0) = 0$. Note that if

$$\mu(v) = \log k_m(v) - \log k_1(v), \qquad v \in \langle 0; v_m\rangle,$$

then $\mu(v_m) = 0$ and $\lim \mu(v) = \log A_p$ as $v \to 0$, where

$$A_p = \left(\frac{(1-v_m^p)^2}{v_m^p}\right)^{1/p}\left(\frac{p+1}{3p+1}\right)^{1/2p}. \tag{3.21}$$

We shall examine in detail the behaviour of $\mu(v)$. It is easy to see that

$$\mu'(v) = -\frac{v^{p-1}}{(1-v^p)^2}\cdot G(v),$$

where

$$G(v) = \log\frac{v^{2p}[3p+1+(1-p)v^p]}{p+1+(1-3p)v^p} + \frac{4p^2(1-v^{2p})}{[3p+1+(1-p)v^p][p+1+(1-3p)v^p]}.$$

Since $G'(v) > 0$ for $v \in (0; v_m)$, $\lim G(v) = -\infty$ as $v \to 0$ and $G(v_m) > 0$, $G(v)$ increases from $-\infty$ to $G(v_m) > 0$. Hence there exists $v = v_4$ such that $G(v_4) = \mu'(v_4) = 0$ and $\mu(v)$ increases from $\mu(0) = \log A_p$ to $\mu(v_4)$ and then decreases from $\mu(v_4)$ to $\mu(v_m) = 0$. On the other hand, if $\mu(0) \geqslant 0$, then $\mu(v) > 0$ for $v \in (0; v_m)$ and $k_m(v) > k_1(v)$ in that interval. If, however, $\mu(0) < 0$, then there exists $v = v_5$ such that $\mu(v_5) = 0$ and

$$\mu(v)\begin{cases} < 0 & \text{when } v \in (0; v_5), \\ > 0 & \text{when } v \in (v_5; v_m). \end{cases}$$

Hence we obtain

$$k_m(v)\begin{cases} < k_1(v) & \text{if } v \in (0; v_5), \\ = k_1(v) & \text{if } v = v_5, \\ > k_1(v) & \text{if } v \in (v_5; v_m). \end{cases} \tag{3.22}$$

Considering the behaviour of the sequence $\{A_p\}$ we can see that $A_p < 1$ for $p < 10$ and $A_p > 1$ for $p \geqslant 10$. Moreover, for $p < 10$ we have $k_1(v_3) < k_m(v_3)$, where v_3 is defined by (3.20), and thus, by (3.22), $v_5 < v_3$ (Fig. 2).

The curves k_1 and k_M also meet at exactly one point $v = v_6$, where $v_3 < v_6 < v_m$.

5. Consider the following sets:

$$D_1^* = D_1 \cap D_4, \qquad D_2^* = D_1 \cap D_2 \cap D_3, \qquad D_3^* = D_1 \cap (D_5 \cup D_6),$$

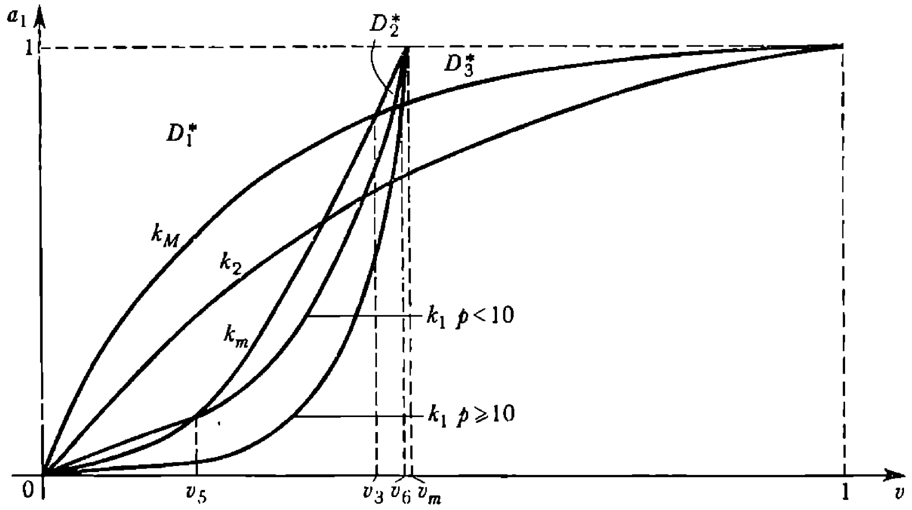

Fig. 2

where

$$\begin{aligned}
D_1 &= \{(v, a_1)\colon\ 0 < v < 1 \wedge k_M(v) < a_1 \leqslant 1\},\\
D_2 &= \{(v, a_1)\colon\ 0 < v < v_m \wedge 0 < a_1 < k_m(v)\},\\
D_3 &= \{(v, a_1)\colon\ 0 < v < v_m \wedge k_1(v) < a_1 < 1\},\\
D_4 &= \{(v, a_1)\colon\ 0 < v < v_m \wedge k_m(v) \leqslant a_1 \leqslant 1\},\\
D_5 &= \{(v, a_1)\colon\ 0 < v < v_m \wedge 0 < a_1 \leqslant k_1(v)\},\\
D_6 &= \{(v, a_1)\colon\ v_m \leqslant v < 1 \wedge 0 < a_1 \leqslant 1\}.
\end{aligned}$$

LEMMA 2. *If* $(v, a_1) \in \bigcup\limits_k D_k^*$, *then* $M(a_1, r, v, p)$ *is a function decreasing in exactly one subinterval of* $\langle r_m; r_M\rangle$, *namely*

a) *if* $(v, a_1) \in D_1^*$, *then* $M(a_1, r, v, p)$ *decreases in the interval* $\langle r_m; r_M\rangle$;

b) *if* $(v, a_1) \in D_2^*$, *then* $M(a_1, r, v, p)$ *increases for* $r \in \langle r_m; \tilde{r}\rangle$ *and decreases for* $r \in \langle \tilde{r}; r_M\rangle$, $r_m < \tilde{r} < r_1$, *where* $(\beta_2, \tilde{r}(\beta_2))$ *is the unique solution of the system*

$$M_r'(a_1, \tilde{r}, v, p) = 0, \qquad \psi(\beta_2, r, v, p) = t_0;$$

c) *if* $(v, a_1) \in D_3^*$, *then* $M(a_1, r, v, p)$ *increases for* $r \in \langle r_m; \hat{r}\rangle$ *and decreases for* $r \in \langle \hat{r}; r_M\rangle$, $r_1 \leqslant \hat{r} < r_2$, *where* $r = \hat{r}$ *is the unique root of the equation*

$$M_r'(a_1, \hat{r}, v, p) = 0.$$

Proof. 1° To examine the behaviour of $M(a_1, r, v, p)$ in the sets D_k^*, $k = 1, 2, 3$, for $r \in \langle r_m; r_1\rangle$ we shall consider the system of equations

$$M_r'(a_1, r, v, p) = 0, \qquad \psi(\beta, r, v, p) = t_0, \tag{3.23}$$

where the parameter β takes values in the interval $\langle v; r_m\rangle$. By (3.8) and

(2.9) the system can be rewritten as follows:

$$r^{2p} = \frac{p+1+(1-3p)\beta^p}{3p+1+(1-p)\beta^p},$$

$$\log\frac{\beta^p}{(1-\beta^p)^2}+\frac{1+\beta^p}{1-\beta^p}\log\frac{r^p}{\beta^p} = \log\frac{v^p}{a_1^p(1-v^p)^2},$$

and hence we get the equation

$$P(\beta) = 0, \tag{3.24}$$

where

$$P(\beta) = \log\frac{\beta^p}{(1-\beta^p)^2}+\frac{1}{2}\,\frac{1+\beta^p}{1-\beta^p}\log\frac{p+1+(1-3p)\beta^p}{\beta^{2p}[3p+1+(1-p)\beta^p]} - \log\frac{v^p}{a_1^p(1-v^p)^2}.$$

For $\beta = v$ we can obtain

$$P(v)\begin{cases} > 0 & \text{for } a_1 > k_1(v), \\ = 0 & \text{for } a_1 = k_1(v), \\ < 0 & \text{for } a_1 < k_1(v). \end{cases} \tag{3.25}$$

Since

$$\log\frac{v^p}{a_1^p(1-v^p)^2} = \log\frac{r_m^p}{(1-r_m^p)^2},$$

therefore

$$P(r_m)\begin{cases} > 0 & \text{for } a_1 > k_m(v), \\ = 0 & \text{for } a_1 = k_m(v), \\ < 0 & \text{for } a_1 < k_m(v). \end{cases} \tag{3.26}$$

It is easy to see that

$$P'(\beta) = -\frac{p\beta^{p-1}}{(1-\beta^p)^2}H(\beta),$$

where

$$H(\beta) = \log\frac{\beta^{2p}[3p+1+(1-p)\beta^p]}{p+1+(1-3p)\beta^p}+\frac{4p^2(1-\beta^{2p})}{[p+1+(1-3p)\beta^p][3p+1+(1-p)\beta^p]}.$$

To show that $M(a_1, r, v, p)$ decreases for $r \in \langle r_m; r_1\rangle$ in D_1^* we shall investigate the behaviour of $P(\beta)$ in the interval $\langle v; r_m\rangle$, where, by (3.15) and (3.9), $r_m \leqslant v_m$. For $(v, a_1) \in D_1^*$, by (3.25) and (3.26), we obtain $P(v) > 0$ and $P(r_m) \geqslant 0$. Moreover, for $\beta \in \langle v; r_m\rangle$, $r_m \leqslant v_m$, for sufficiently small values of β, we have $H(\beta) < 0$, while for β close to v_m we have $H(\beta) > 0$ ($H(v_m) > 0$).

The fact that $H(\beta)$ is increasing and the above considerations imply the existence of β_1 such that the function $P(\beta)$ increases for $\beta \in \langle v;\ \beta_1\rangle$ and decreases for $\beta \in \langle \beta_1; v_m\rangle$.

Thus from the behaviour of $P(\beta)$ it follows that for $(v, a_1) \in D_1^*$, $\beta = r_m$ is the unique root of equation (3.24) and hence $M'_r\ (a_1, r, v, p) \neq 0$ if $r \in (r_m\ ; r_1)$ and $M'_1(a_1, r, v, p) = 0$ if $a_1 = k_m(v)$. However, in the set D_1^* we have $M'_1(a_1, r_m, v, p) \leqslant 0$ and $M'_r(a_1, r_1, v, p) < 0$. Thus $M(a_1, r, v, p)$ decreases in D_1^* for $r \in \langle r_m; r_1\rangle$.

We now prove that if $(v, a_1) \in D_2^*$ and $r \in \langle r_m; r_1\rangle$, then there exists a unique $r = \tilde{r}, r_m < \tilde{r} < r_1$, such that $M(a_1, r, v, p)$ increases in the interval $\langle r_m; \tilde{r}\rangle$ and decreases in $\langle \tilde{r}; r_1\rangle$.

Indeed, from (3.25) and (3.26) it follows that in the set D_2^* $P(v) > 0$ and $P(r_m) < 0$.

Notice also that in D_2^*, $v \in (v_3; v_m)$, where, by (3.15) and (3.9), $v_m < r_m$ and hence the parameter β can take values arbitrarily close to v_3. As it has already been stated, $H(v_m) > 0$ and $H(\beta)$ is an increasing function. Thus, depending on the sign of this function in the sufficiently small neighbourhood of v_3, $P(\beta)$ either decreases of first increases and then decreases in the interval $\langle v;\ r_m\rangle$. Hence in both cases (3.24) has a unique root $\beta = \beta_2$, $v < \beta_2 < r_m$, and consequently the existence of a unique solution $(\beta_2, \tilde{r})$, $r_m < \tilde{r} < r_1$, of the system (3.23) follows. On the other hand, in the set D_2^* we have $M'_1(a_1, r_m, v, p) > 0$ and $M'_r(a_1, r_1, v, p) < 0$, which results in $M(a_1, r, v, p)$ being increasing for $r \in \langle r_m; \tilde{r}\rangle$ and decreasing for $r \in \langle \tilde{r}; r_1\rangle$.

We shall show that the function $M(a_1, r, v, p)$ increases in $\langle r_m; r_1\rangle$ for $(v, a_1) \in D_3^*$.

By (3.25) and (3.26) $P(v) \leqslant 0$ and $P(r_m) < 0$ in the set D_3^*. It should also be noted that in D_3^*, $v \in (v_6; 1)$ and since $\beta \in \langle v; r_m\rangle$, the parameter β takes values arbitrarily close to v_6. On the other hand, in view of (3.15) and (3.9), $v_m < r_m$, and from the form of $P(\beta)$ it follows that $r_m < \left(\dfrac{p+1}{3p-1}\right)^{1/p}$.

Since for any p we have $H(v_6) > 0$, $H(v_m) > 0$ and $H(\beta)$ is an increasing function, therefore $H(\beta) > 0$ for $\beta \in \langle v; v_m\rangle$. It is easy to verify that $H(\beta) > 0$ also for $\beta \in \left\langle v_m; \left(\dfrac{p+1}{3p-1}\right)^{1/p}\right\rangle$. Hence $P(\beta)$ decreases in $\langle v; r_m.\rangle$

The behaviour of $P(\beta)$ implies that (3.24) possesses the unique solution $\beta = v$ and thus $M'_r(a_1, r, v, p) \neq 0$ for $r \in (r_m; r_1)$ and $M'_r(a_1, r, v, p) = 0$ for $a_1 = k_1(v)$. In D_3^*, however, $M'_r(a_1, r_m, v, p) > 0$ and $M'_r(a_1, r_1, v, p) \geqslant 0$, which completes the proof of the fact that $M(a_1, r, v, p)$ increases in the interval $\langle r_m; r_1\rangle$.

2° Let us now consider the interval $\langle r_1; r_2\rangle$. By (3.8)

$$M_r'(a_1, r, v, p) = \frac{1}{r} W(r),$$

where

$$W(r) = \frac{p-1+(p+1)r^{2p}}{1-r^{2p}} - \frac{2p^2}{t_0}\log\frac{r}{v}.$$

It can be shown that $W(r)$ decreases for $r \in \langle r_1; r_0\rangle$ and increases for $r \in \langle r_0; r_2\rangle$, where $r_0 = \sqrt[2p]{1+t_0-\sqrt{2t_0+t_0^2}}$. If $r_0 < r_1$ or $r_0 > r_2$, then $W(r)$ is strictly monotone in $\langle r_1; r_2\rangle$.

Let $(v, a_1) \in D_1^* \cup D_2^*$ and $r \in \langle r_1; r_2\rangle$.

Then $M_r'(a_1, r_1, v, p) < 0$ and $M_r'(a_1, r_2, v, p) < 0$ and the behaviour of $W(r)$ shows that $M(a_1, r, v, p)$ decreases.

If, however, $(v, a_1) \in D_3^*$, then $M_r'(a_1, r_1, v, p) \geqslant 0$ and $M_r'(a_1, r_2, v, p) < 0$. Thus the behaviour of $W(r)$ in the interval $\langle r_1; r_2\rangle$ implies the existence of a unique $r = \hat{r}$, $r_1 \leqslant \hat{r} < r_2$, which is the solution of

$$W(\hat{r}) = 0 \tag{3.27}$$

and such that $M(a_1, r, v, p)$ increases for $r \in \langle r_1; \hat{r}\rangle$ and then decreases for $r \in \langle \hat{r}; r_2\rangle$.

3° Finally we examine the behaviour of $M(a_1, r, v, p)$ in the sets D_k^* for $r \in \langle r_2; r_M\rangle$. Consider the system of equations

$$M_r'(a_1, r, v, p) = 0, \qquad \varphi(a, r, v, p) = t_0,$$

where $a \in \langle v; r_M\rangle$. By (3.8) and (2.9) the above system can be rewritten as

$$P(a) = 0, \tag{3.28}$$

where

$$P(a) = \log\frac{a^p}{(1+a^p)^2} + \frac{1}{2}\,\frac{1-a^p}{1+a^p}\log\frac{p+1+(3p-1)a^p}{a^{2p}[3p+1+(p-1)a^p]} - \log\frac{v^p}{a_1^p(1+v^p)^2}.$$

It is easy to show that

$$P(v)\begin{cases} > 0 & \text{for } a_1 > k_2(v), \\ = 0 & \text{for } a_1 = k_2(v), \\ < 0 & \text{for } a_1 < k_2(v), \end{cases}$$

and

$$P(r_M) > 0 \quad \text{for } a_1 > k_M(v).$$

Moreover,

$$P'(a) = \frac{pa^{p-1}}{(1+a^p)^2}\,G(a),$$

where

$$G(a) = \log\frac{a^{2p}[3p+1+(p-1)a^p]}{p+1+(3p-1)a^p} + \\ + \frac{4p^2(1-a^{2p})}{[p+1+(3p-1)a^p][3p+1+(p-1)a^p]}.$$

We shall prove that $M(a_1, r, v, p)$ decreases for $(v, a_1) \in \bigcup_k D_k^*$ and $r \in \langle r_2; r_M\rangle$.

To show this we should first notice that $G(a)$ is an increasing function, i.e. $G(a) < G(1) = 0$. Hence $P(a)$ decreases for $a \in \langle v; r_M\rangle$. In the set $\bigcup_k D_k^*$ we have $P(v) > 0$ and $P(r_M) > 0$, therefore (3.28) has no solution. Thus we have shown that $M_r'(a_1, r, v, p) \neq 0$ for $r \in \langle r_2; r_M\rangle$ and since in $\bigcup_k D_k^*$, $M_r'(a_1, r_2, v, p) < 0$ and $M_r'(a_1, r_M, v, p) < 0$, we get the desired result, namely that $M(a_1, r, v, p)$ is a decreasing function in $\langle r_2; r_M\rangle$.

The results obtained for the three cases prove the lemma.

Consider the following sets:

$$E_1 = D_1^* \cup \{(v, a_1):\ 0 < v \leqslant v_3 \wedge a_1 = k_M(v)\},$$
$$E_2 = D_2^* \cup \{(v, a_1):\ v_3 < v < v_6 \wedge a_1 = k_M(v)\},$$
$$E_3 = D_3^* \cup \{(v, a_1):\ v_6 \leqslant v < 1 \wedge a_1 = k_M(v)\}.$$

Making use of Lemma 2 we shall prove

THEOREM 3. *For any function f of the family $S_1^{(p)}$ satisfying $f'(0) = a_1$ and $|f(z)| = v$, where $a_1 \in (0; 1\rangle$ and $v \in (0; 1)$, the following sharp estimation holds:*

$$\log v' \tag{3.29}$$
$$\leqslant \begin{cases} \log\dfrac{v(1-v^p)(1+r_m^p)}{r_m(1+v^p)(1-r_m^p)} & \text{if } (v, a_1) \in E_1, \\ \log\dfrac{\tilde{r}^{p-1}(1-v^{2p})}{v^{p-1}(1-\tilde{r}^{2p})} - \varphi(\beta_2, \tilde{r}, v, p) & \text{if } (v, a_1) \in E_2, \\ \log\dfrac{\hat{r}^{p-1}(1-v^{2p})}{v^{p-1}(1-\hat{r}^{2p})} - \dfrac{p-1+(p+1)\hat{r}^{2p}}{2p(1-\hat{r}^{2p})}\log\dfrac{\hat{r}}{v} & \text{if } (v, a_1) \in E_3, \end{cases}$$

where

1° $r = r_m$ *is the root of* $\psi(r_m, r_m, v, p) = t_0$,

2° $(\beta_2, \tilde{r})$, $r_m < \tilde{r} < r_1$, $v < \beta_2 < r_m$ *is the unique solution of the system*

$$\frac{p-1+(p+1)\tilde{r}^{2p}}{1-\tilde{r}^{2p}} = 2p\frac{1-\beta_2^p}{1+\beta_2^p}, \quad \psi(\beta_2, \tilde{r}, v, p) = t_0,$$

3° $r = \hat{r}$, $r_1 \leqslant \hat{r} \leqslant r_2$, *is the unique solution of*

$$\frac{p-1+(p+1)\hat{r}^{2p}}{1-\hat{r}^{2p}} = \frac{2p^2}{t_0}\log\frac{\hat{r}}{v}.$$

Proof. If $(v, a_1) \in D_1^*$, then, by Lemma 2, $M(a_1, r, v, p)$ will attain its upper bound at the point $r = r_m$. The form of the estimation follows from (2.17), (3.3) and (2.9).

If $(v, a_1) \in D_2^*$, then by Lemma 2 there exists $r = \tilde{r}$, $\tilde{r} \in (r_m; r_1)$, such that $(\beta_2, \tilde{r})$ is the solution of system (3.23) for which $M(a_1, \tilde{r}, v, p) = \sup M(a_1, r, v, p)$ for $(v, a_1) \in D_2^*$.

If $(v, a_1) \in D_3^*$, then (cf. Lemma 2) $M(a_1, r, v, p)$ increases in $\langle r_m; \hat{r}\rangle$ and then decreases in $\langle \hat{r}; r_M\rangle$. Hence the upper bound for $(v, a_1) \in D_3^*$ will be attained at the point $r = \hat{r}$ defined by (3.27).

The case

$$a_1 = k_M(v) \tag{3.30}$$

remains to be examined. Notice that if (3.30) holds, then there exist only three of the four roots of (3.3), namely r_m, r_1 and r_2. Hence the function $M(a_1, r, v, p)$ has the form

$$M(a_1, r, v, p) = \log\frac{r^{p-1}(1-v^{2p})}{v^{p-1}(1-r^{2p})} - L(a_1, r, v, p), \tag{3.31}$$

where

$$L(a_1, r, v, p) = \begin{cases} \varphi(\beta, r, v, p) & \text{for } r_m \leqslant r \leqslant r_1, \\ t_0^{-1}\left(\log\dfrac{r^p}{v^p}\right)^2 & \text{for } r_1 \leqslant r \leqslant r_2, \\ \psi(a, r, v, p) & \text{for } r_2 \leqslant r < 1, \end{cases} \tag{3.32}$$

a and β being the roots of (2.14) and (2.15), respectively.

Thus if (3.30) holds, then $M(a_1, r, v, p)$ does not change its behaviour in $\langle r_m; r_1\rangle$ and $\langle r_1; r_2\rangle$. The variable r, however, can now take values arbitrarily close to 1, so we shall examine the behaviour of $M(a_1, r, v, p)$ in $\langle r_2; 1)$. By (3.31) and (3.32)

$$M(a_1, r, v, p) = \log\frac{r^{p-1}(1-v^{2p})}{v^{p-1}(1-r^{2p})} - \psi(a, r, v, p),$$

where a satisfies (2.14). From (3.30)

$$t_0 = \log\frac{(1+v^p)^2}{4v^p},$$

and hence (2.14) can be rewritten as

$$\log\frac{4a^p}{(1+a^p)^2}+\frac{1-a^p}{1+a^p}\log\frac{r^p}{a^p} = 0. \tag{3.33}$$

Note that $a\to 1$ as $r\to 1$. Indeed, assuming that a does not tend to 1 we obtain that a is the solution of

$$\log\frac{4a^p}{(1+a^p)^2}+\frac{1-a^p}{1+a^p}\log a^{-p} = 0. \tag{3.34}$$

It is not difficult to verify that $a = 1$ is the only root of equation (3.34).

Now consider the expression

(3.35)

$$\exp M(a_1, r, v, p) = \frac{r^{p-1}}{1-r^{2p}}\,\frac{v(1+v^p)}{2(1-v^p)}\,\frac{(1-a^p)^2}{a^p}\exp\left(-p\frac{1+a^p}{1-a^p}\log\frac{r}{a}\right).$$

Using (3.33) we get

$$p\frac{1+a^p}{1-a^p}\log\frac{r}{a} = \left(\frac{1+a^p}{1-a^p}\right)^2\log\frac{(1+a^p)^2}{4a^p}.$$

Thus

$$p\lim\frac{1+a^p}{1-a^p}\log\frac{r}{a} = 1 \quad \text{as } r\to 1. \tag{3.36}$$

Also by (3.33)

$$r^p = a^p\exp\left(\frac{1+a^p}{1-a^p}\log\frac{(1+a^p)^2}{4a^p}\right),$$

and hence

$$\lim\frac{1-a^p}{1-r^p} = 2 \quad \text{as } r\to 1. \tag{3.37}$$

By (3.35), (3.36) and (3.37) we see that as $r\to 1$, $\lim\exp M(a_1, r, v, p) = 0$ and thus

$$\lim M(a_1, r, v, p) = -\infty.$$

Summing up we can say that if (3.30) holds, then $M(a_1, r, v, p)$ decreases to $-\infty$ in $\langle r_2; 1)$. Thus, under the condition $a_1 = k_M(v)$, the upper bound $\sup\limits_{r} M(a_1, r, v, p)$ in the sets E_1, E_2 and E_3 will be attained for r_m, $\tilde{r}$ and $\hat{r}$, respectively. This completes the proof of Theorem 3.

Setting in (3.29) $p = 1$ we get the results obtained by R. Robinson [11].

The first of estimations (3.29) was obtained earlier [4] in the part of the region E_1 consisting of (v, a_1) with $v \in (0; r_0)$, where $r_0 = \sqrt[p]{\sqrt{p^2+1}-p}$.

Löwner–Robinson's method has been also employed in the works of L. Mikołajczyk [8], [9], [10] for examining relations between quantities (1.1) with respect to the class of regular univalent p-symmetric in $|z| > 1$ functions of the form

$$W(z) = z + \sum_{n=1}^{\infty} \frac{a_{np-1}}{z^{np-1}},$$

and satisfying $|W(z)| > m$, where $m \in (0; 1)$.

IV. Functions univalent and p-symmetric in the unit disc.

1. Denote by $S^{(p)}$ the class of functions F of the form

$$F(z) = z + \sum_{n=1}^{\infty} A_{np+1} z^{np+1}$$

for $z \in K$, $K = \{z\colon |z| < 1\}$, which are regular and univalent in K.

Denote also

$$r = |z|, \quad v = |F(z)|, \quad v' = |F'(z)|, \tag{4.1}$$

where $z \neq 0$ is an arbitrary fixed point of K and F is an arbitrary function of $S^{(p)}$.

For any $F \in S^{(p)}$ let us consider the function

$$\varphi_\varrho(z) = \varrho^{-1} F(\varrho z), \tag{4.2}$$

where $\varrho \in (0; 1)$. Note that function (4.2) is regular and univalent for $z \in K$, satisfies $\varphi_\varrho(0) = 0$, $\varphi_\varrho'(0) = 1$ and [2]

$$|\varphi_\varrho(z)| \leqslant \frac{1}{\varrho} \frac{\varrho |z|}{(1-\varrho|z|)^2},$$

thus

$$|\varphi_\varrho(z)| < M_\varrho$$

for $z \in K$, where $M = 1/(1-\varrho)^2$. As $\varrho \to 1$, we have $\varphi_\varrho(z) \to F(z)$ and thus any function of the class $S^{(p)}$ can be approximated by bounded functions of the form (4.2).

Now consider for any $\varphi_\varrho(z)$ defined by (4.2), the function

$$f(z) = (1-\varrho)^2 \varphi_\varrho(z). \tag{4.3}$$

It can be easily verified that $f(z)$ belongs to the class $S_1^{(p)}$ considered earlier, provided $a_1 = f'(0) = (1-\varrho)^2$.

If for any arbitrarily fixed $z \neq 0$, $z \in K$, we set

$$r = |z|, \quad V_\varrho = |\varphi_\varrho(z)|, \quad V'_\varrho = |\varphi'_\varrho(z)|, \tag{4.4}$$

then from (4.3) we get

$$v = a_1 V_\varrho, \quad v' = a_1 V'_\varrho, \tag{4.5}$$

where $v = |f(z)|$, $v' = |f'(z)|$, according to the notation introduced in Section I. Thus using (4.5) and the relations obtained in Section I, holding between the quantities a_1, r, v, v', and assuming $a_1 \to 0$ or, which results in the same, $\varrho \to 1$, we obtain the relation between quantities (4.1). Those relations are the mere consequences of Löwner–Robinson's method employed in the previous part of the paper.

2. We shall show that certain estimations of functionals on $S^{(p)}$ can be obtained by a passage to the limit from inequalities valid for functions of the family $S_1^{(p)}$.

The following lemma holds (cf. [2]):

Lemma 3. *If $F \in S^{(p)}$ and $z \in K$, then the sharp estimation*

$$\frac{r}{(1+r^p)^{2/p}} \leqslant V \leqslant \frac{r}{(1-r^p)^{2/p}} \tag{4.6}$$

is true.

To show this we must put $v = a_1 V_\varrho$ in (2.7) and then pass to the limit as $a_1 \to 0$. Note also the equality signs in (4.6) are realized for the p-symmetric Koebe function

$$w = \frac{z}{(1-e^{i\alpha}z^p)^{2/p}} \tag{4.7}$$

at the points $z_1 = re^{i(\pi-\alpha)/p}$ and $z_2 = re^{-i\alpha/p}$, respectively, $\alpha \in R$ (an arbitrary real number).

We next prove

Lemma 4. *If $F \in S^{(p)}$ and $z \in K$, then*

$$V' \geqslant \frac{1-r^{2p}}{r^{p+1}} V^{p+1}. \tag{4.8}$$

To prove the lemma we put $v = a_1 V_\varrho$ and $v' = a_1 V'_\varrho$ in (2.20) and pass to the limit as $a_1 \to 0$. The equality sign in (4.8) is realized by function (4.7) at the point $z_0 = re^{-i\alpha/2p}$, $\alpha \in R$.

Inequality (4.8) defines the lower bound of the functional $V' = |F'(z)|$ depending on r and V.

Let us now discuss the more complicated problem of determining the upper bound of V', provided that r and V are given. Consider the functions

$$(4.9)\qquad \begin{aligned} \Phi(x, r, V_\varrho, p) &= \log\frac{x^p}{V_\varrho^p(1+x^p)^2}+\frac{1-x^p}{1+x^p}\log\frac{r^p}{x^p},\\ \Psi(x, r, V_\varrho, p) &= \log\frac{x^p}{V_\varrho^p(1-x^p)^2}+\frac{1+x^p}{1-x^p}\log\frac{r^p}{x^p}, \end{aligned}$$

defined for $x \in (0; r\rangle$. At $x = 0$ functions (4.9) are defined additionally by

$$(4.10)\qquad \Phi(0, r, V_\varrho, p) = \Psi(0, r, V_\varrho, p) = \log\frac{r^p}{V_\varrho^p}.$$

Thus the values of functions (4.9) at $x = 0$ are the limits of those functions as $x \to 0$. Note also that the equalities

$$(4.11)\qquad \begin{aligned} \varphi(x, r, v, p) &= \varphi(x, r, a_1 V_\varrho, p) = \Phi(x, r, V_\varrho, p)-\log\frac{a_1^p}{(1+a_1^p V_\varrho^p)^2},\\ \psi(x, r, v, p) &= \psi(x, r, a_1 V_\varrho, p) = \Psi(x, r, V_\varrho, p)-\log\frac{a_1^p}{(1-a_1^p V_\varrho^p)^2} \end{aligned}$$

hold.

Using (4.11) we pass to the limit as $a_1 \to 0$ in the inequalities

$$\begin{aligned} \varphi(r, r, v, p) &\leqslant t_0 \leqslant \varphi(v, r, v, p),\\ \varphi(v, r, v, p) &\leqslant t_0 \leqslant \psi(v, r, v, p),\\ \psi(v, r, v, p) &\leqslant t_0 \leqslant \psi(r, r, v, p), \end{aligned}$$

defined in Theorem 1 and we obtain

$$(4.12)\qquad \frac{r}{(1+r^p)^{2/p}} \leqslant V \leqslant r,$$

$$(4.13)\qquad V = r,$$

$$(4.14)\qquad r \leqslant V \leqslant \frac{r}{(1-r^p)^{2/p}};$$

respectively.

Employing (4.11) we can rewrite (2.14) and (2.15) as

$$\begin{aligned} \Phi(a, r, V_\varrho, p)+2\log(1+a_1^p V_\varrho^p) &= 0,\\ \Psi(\beta, r, V_\varrho, p)+2\log(1-a_1^p V_\varrho^p) &= 0, \end{aligned}$$

respectively. Hence as $a_1 \to 0$ we obtain

$$(4.15)\qquad \Phi(a, r, V, p) = 0,$$

$$(4.16)\qquad \Psi(\beta, r, V, p) = 0.$$

Putting $v = a_1 V_\varrho$ and $v' = a_1 V'_\varrho$ in (2.17) and passing to the limit as $a_1 \to 0$ we get from (4.12)–(4.14) and (4.15), (4.16)

LEMMA 5. *If* $F \in S^{(p)}$ *and* $z \in K$, *then the sharp estimation*

$$\log V' \leqslant M(r, V, p) \tag{4.17}$$

holds, where

$$M(r, V, p) = \log \frac{r^{p-1}}{V^{p-1}(1-r^{2p})} - L(r, V, p) \tag{4.18}$$

and

$$L(r, V, p) = \begin{cases} \Psi(a, r, V, p) & \text{for } V \leqslant r, \\ 0 & \text{for } V = r, \\ \Phi(\beta, r, V, p) & \text{for } V \geqslant r. \end{cases} \tag{4.19}$$

Parameters a and β from the interval $\langle 0; r\rangle$ are the unique solutions of (4.15) and (4.16), respectively.

If V takes the minimal values $V_{\min} = r/(1+r^p)^{2/p}$, then

$$V' = \frac{1-r^p}{(1+r^p)^{1+2/p}}, \tag{4.20}$$

and if V takes the maximal value $V_{\max} = r/(1-r^p)^{2/p}$, then

$$V' = \frac{1+r^p}{(1-r^p)^{1+2/p}}. \tag{4.21}$$

Indeed, if $V = r/(1+r^p)^{2/p}$, then by (4.8)

$$V' \geqslant \frac{1-r^p}{(1+r^p)^{1+2/p}},$$

but on the other hand, by (4.15), $a = r$ and hence

$$L(r, V_{\min}, p) = \Psi(r, r, V_{\min}, p) = \log\left(\frac{1+r^p}{1-r^p}\right)^2.$$

Thus from (4.17)

$$V' \leqslant \frac{1-r^p}{(1+r^p)^{1+2/p}}.$$

Combining the results we obtain equality (4.20). In a similar manner (4.21) can be proved.

From (4.20) and (4.21) it follows that for the extremal values of V the equality in (4.17) is realized by function (4.7) at the points $z_1 = re^{i(\pi-a)/p}$ and $z_2 = re^{-ia/p}$, $a \in R$.

3. Employing the results obtained we shall prove

THEOREM 4. *If* $F \in S^{(p)}$ *and* $z \in K$, $z \neq 0$, *then the sharp estimation*

$$\frac{1-r^p}{(1+r^p)^{1+2/p}} \leqslant V' \leqslant \frac{1+r^p}{(1-r^p)^{1+2/p}} \tag{4.22}$$

holds.

Proof. By (4.8) V' attains its minimal value when V is minimal. From this and (4.6) we obtain the form of the lower bound of the functional V'.

Notice also that in view of (4.18), (4.15) and (4.16) we have

$$M'_V(r, V, p) = \begin{cases} \dfrac{1}{V}+\dfrac{p}{V}\left(\dfrac{1+a^p}{1-a^p}\right)^2 & \text{for } V \leqslant r, \\ \dfrac{1}{V}+\dfrac{p}{V}\left(\dfrac{1-\beta^p}{1+\beta^p}\right)^2 & \text{for } V \geqslant r, \end{cases} \tag{4.23}$$

which means that $M(r, V, p)$ is an increasing function of the variable V. From this and (4.21) we obtain the estimation from above of the functional V'. The equality signs in (4.22) are realized by function (4.7) at the points $z_1 = re^{i(\pi-a)/p}$ and $z_2 = re^{-ia/p}$, respectively, $a \in R$.

As an easy consequence of Lemmas 3, 4 and 5 we get

THEOREM 5. *If* $F \in S^{(p)}$ *and* $z \in K$, $z \neq 0$, *then the sharp estimation*

$$\frac{1-r^p}{1+r^p} \leqslant \frac{rV'}{V} \leqslant \frac{1+r^p}{1-r^p} \tag{4.24}$$

holds.

Indeed, the lower bound of the functional in question can be easily obtained from (4.8) and (4.6).

From (4.23) it follows also that $M(r, V, p)-\log V$ is an increasing function and hence the considered functional attains its upper bound only for the maximal value of V.

The extremal function is the Koebe function of the form (4.7).

THEOREM 6. *If* $F \in S^{(p)}$ *and* $z \in K$, $z \neq 0$, *then the sharp estimation*

$$\frac{1-r^{2p}}{r^p} \leqslant \frac{rV'}{V^{p+1}} \leqslant \frac{1}{r^p(1-r^{2p})} \tag{4.25}$$

holds.

Indeed, the estimation from below has already been found (cf. Lemma 4).

Using (4.23) we can show that the function $M(r, V, p)-(p+1)\log V$ attains its maximum for $V = r$. Then, by (4.17),

$$V' \leqslant \frac{1}{1-r^{2p}},$$

and hence we obtain the estimation from above in (4.25). The sharpness of the estimation is the result of (4.8) and (4.17) being sharp.

Finally we shall examine relations between V and V' for the functions of the class $S^{(p)}$.

THEOREM 7. *If* $F \in S^{(p)}$ *and* $z \in K$, $z \neq 0$, *then for* $V < 4^{-1/p}$ *we have*

(4.26)

$$4^{-1/p}\sqrt{1-4V^p}\,(1+\sqrt{1-4V^p})^{2/p} \leqslant V' \leqslant 4^{-1/p}\sqrt{1+4V^p}\,(1+\sqrt{1+4V^p})^{2/p}.$$

Equalities in (4.26) *are realized for the Koebe function* (4.7) *at the points* $z_1 = re^{i(\pi-a)}$ *and* $z_2 = re^{-ia/p}$, *respectively, where* a *is an arbitrary real number.*

Proof. From (4.8) it follows that to determine the lower bound of the functional V' in (4.26) it suffices to calculate

$$\min_r \frac{1-r^{2p}}{r^{p+1}}.$$

The above minimum is taken for the maximal admissible value of r. Note that, given any value of $V < 4^{-1/p}$, the maximal admissible value is attained by r satisfying

$$V = \frac{r}{(1+r^p)^{2/p}}.$$

Form this and (4.8) we obtain the form of estimation (4.26) from below.

We shall now determine the estimation from above of the functional V' for a given value of V.

By (4.18), (4.19), (4.15) and (4.16) we have

$$(4.27)\qquad M_r'(r, V, p) = \begin{cases} \dfrac{(p+1)r^{2p}+p-1}{r(1-r^{2p})} - 2\dfrac{p(1-\beta^p)}{r(1+\beta^p)} & \text{for } r \leqslant V, \\[2ex] \dfrac{(p+1)r^{2p}+p-1}{r(1-r^{2p})} - 2\dfrac{p(1+a^p)}{r(1-a^p)} & \text{for } r \geqslant V, \end{cases}$$

where a and β are the solutions of (4.15) and (4.16), respectively. Note that $M_r'(r, V, p)$ is a continuous function at $r = V$, since then $a = \beta = 0$.

Since $\Phi(a, r, V, p)$ is a decreasing function of the variables a and V, the derivative $a_V'(V) = -\Phi_V'/\Phi_a'$ is negative. Thus if V increases from its minimal value to r, then the parameter a decreases from r to 0.

Moreover, since $\Psi(\beta, r, V, p)$ is increasing in the variable β and decreasing in V, the derivative $\beta_V'(V) = -\Psi_V'/\Psi_\beta'$ is positive. Thus if V increases from r to the maximal admissible value, then the parameter β increases from 0 to r.

Consequently, for an arbitrary fixed value of V satisfying $V \leqslant r$, there exists a unique value of $a \in \langle 0; r\rangle$ which is the solution of (4.15).

Similarly, to an arbitrary value of V satisfying $V \geqslant r$, the unique $\beta \in \langle 0; r\rangle$ being solution of (4.16) is assigned.

It can be easily verified that the only roots of the equation $M'_r(r, V, p) = 0$ are

$$\beta_0 = \left(\frac{(3p+1)r^{2p}-(p+1)}{(p-1)r^{2p}-(3p-1)}\right)^{1/p} \quad \text{for } r \leqslant V,$$

and

$$\alpha_0 = \left(\frac{(3p+1)r^{2p}-(p+1)}{(1-p)r^{2p}+(3p-1)}\right)^{1/p} \quad \text{for } r \geqslant V.$$

Since $0 \leqslant \alpha_0 \leqslant r$, $0 \leqslant \beta_0 \leqslant r$ and α_0, β_0 are the solutions of (4.15) and (4.16), respectively, $M'_r(r, V, p) = 0$ if and only if

$$\Psi(\beta_0, r, V, p) = 0 \quad \text{for } r^*(p) \leqslant r \leqslant \left(\frac{p+1}{3p+1}\right)^{1/2p},$$

or

$$\Phi(\alpha_0, r, V, p) = 0 \quad \text{for } \left(\frac{p+1}{3p+1}\right)^{1/2p} \leqslant r < 1,$$

where

$$r^*(p) = \begin{cases} 1/2 & \text{for } p = 1, \\ \left(\dfrac{2p-\sqrt{3p^2+1}}{p-1}\right)^{1/p} & \text{for } p > 1. \end{cases}$$

Hence the equation $M'_r(r, V, p) = 0$ is satisfied only on the curve

$$V = 4^{-1/p}\big(N(r)\big)^{1/p}, \tag{4.28}$$

where

$$N(r) = \exp\left[\frac{(3p+1)r^{2p}-(p+1)}{(p+1)r^{2p}+p-1}\log\left|\frac{(3p+1)r^{2p}-(p+1)}{(p-1)r^{2p}-(3p-1)}\right| + \right.$$
$$\left. + \frac{2p^2(1-r^{2p})}{(p+1)r^{2p}+p-1}\log r + \log\left(\frac{(p-1)r^{2p}-(3p-1)}{(p+1)r^{2p}+p-1}\right)^2\right]$$

for $r^*(p) \leqslant r < 1$ and where for $r = \left(\dfrac{p+1}{3p+1}\right)^{1/2p}$ we define the value of $N(r)$ by its limit at this point.

To examine the curve defined by (4.28) we shall consider the behaviour of the function

$$g(h) = \log(4V^p)^2, \tag{4.29}$$

where $h = r^{-2p}$ ranges over the interval $1 < h \leqslant \big(r^*(p)\big)^{-2p}$.

By (4.29) and (4.28)

$$g'(h) = \frac{4p^2}{[(p+1)+(p-1)h]^2} B(h),$$

where

$$B(h) = \log\frac{[(p-1)-(3p-1)h]^2}{h[(3p+1)-(p+1)h]^2} + \frac{(p+1)-2h-(p-1)h^2}{2ph}.$$

Since $B(h)$ increases for $h \in \left(1; \frac{3p+1}{p+1}\right)$ and $B(1) = 0$, therefore $B(h) > 0$ in that interval, while in the interval $\frac{3p+1}{p+1} < h \leqslant \left(r^*(p)\right)^{-2p}$, $B(h)$ decreases, and

$$\lim B(h) = +\infty \quad \text{as } h \to \frac{3p+1}{p+1}.$$

However, for sufficiently large values of h, e.g. $h = 6$, $B(h) < 0$ for every p and thus the existence of a unique $h = h_0$ such that

$$B(h) \begin{cases} > 0 & \text{for } \frac{3p+1}{p+1} < h < h_0, \\ = 0 & \text{for } h = h_0, \\ < 0 & \text{for } h_0 < h < \left(r^*(p)\right)^{-2p} \end{cases}$$

follows.

Hence $g(h)$ attains its maximum for $h = h_0$.

Summing up we can state that in the interval $(1; h_0)$ the function $g(h)$ increases and attains its maximal value at $h = h_0$. In the interval $h_0 < h < \left(r^*(p)\right)^{-2p}$, $g(h)$ decreases.

Thus for $r^*(p) < r < r_0$, by (4.29), the values of V increase from $V\left(r^*(p)\right)$ to $V(r_0)$, where $r_0 = h_0^{-2p}$, while for $r \in (r_0; 1)$ they decrease and $\lim V(r) = 4^{-1/p}$ as $r \to 1$.

As a result we obtain that the curve defined by (4.28) lies above the line $V = 4^{-1/p}$ for $r < 1$ and $\lim V(r) = 4^{-1/p}$ as $r \to 1$. On the other hand, by Lemma 3, the curve defined by (4.28) divides the set

$$D_1 = \left\{(r, V):\ 0 < r < 1, \frac{r}{(1+r^p)^{2/p}} \leqslant V \leqslant \frac{r}{(1-r^p)^{2/p}}\right\}$$

into two subsets (Fig. 3).

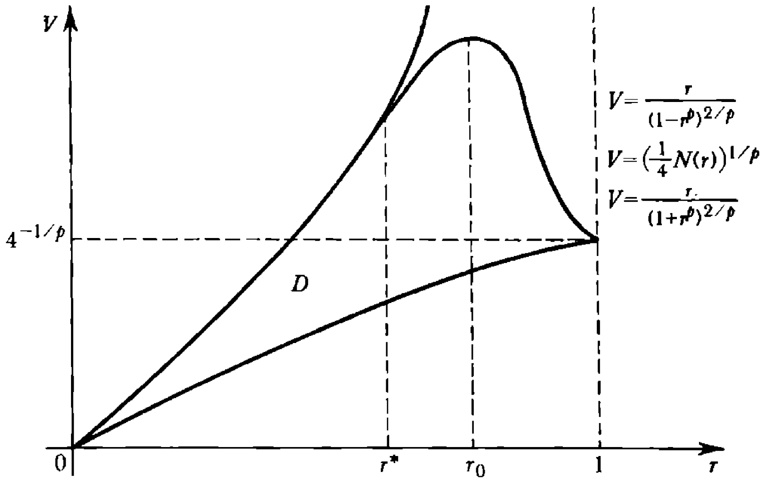

Fig. 3

It can be easily verified that below the curve $V = 4^{-1/p}(N(r))^{1/p}$, $M'_r(r, V, p) < 0$ and above the curve, $M'_r(r, V, p) > 0$. Let $D = D_1 \cap D_2$ where

$$D_2 = \{(r, V)\colon\ 0 < r < 1,\ 0 < V < 4^{-1/p}\}.$$

$M(r, V, p)$ decreases with respect to r in D and thus at the prescribed value of V it attains its maximal value for the minimal admissible value of r which is the solution of

$$V = \frac{r}{(1-r^p)^{2/p}}. \tag{4.30}$$

If, however, (4.30) holds, then, as we have already shown, $M(r, V, p) = \dfrac{1+r^p}{(1-r^p)^{1+2/p}}$. Thus from (4.17) and (4.30) we obtain the upper bound of the functional V' in (4.26); this completes the proof.

As a supplement to Theorem 7, we remark that for $V \geqslant 4^{-1/p}$ the variable r takes values arbitrarily close to 1 and thus, by (4.8), there exists no positive lower bound of the functional V'.

Furthermore, we shall examine whether for $V \geqslant 4^{-1/p}$ there exists an upper bound of V. As already remarked, the variable r takes then values arbitrarily close to 1. We shall first examine the behaviour of the function $L(r, V, p)$ as $r \to 1$.

If $V \geqslant 1$, then $V > r$ and, consequently, $L(r, V, p) = \Phi(\beta, r, V, p)$, where the parameter β is the solution of (4.16). Since $\Psi(\beta, r, V, p)$ is an increasing function of the variables r and β, the derivative $\beta'_r(r) = -\Psi'_r/\Psi'_\beta$ is negative. Hence if $r \to 1$, then $\beta \to 0$ and $\lim L(r, V, p)$ as $r \to 1$ is finite.

If $4^{-1/p} < V < 1$, then by the Schwarz lemma, $V \leqslant r$ and thus $L(r, V, p) = \Psi(a, r, V, p)$, where the parameter a is the solution of (4.15). It is easy to verify that $\Phi(a, r, V, p)$ is increasing in the variable

r and decreasing in a. Thus if $r \to 1$, then the value of the parameter a increases, which implies that $L(r, V, p)$ also increases and $\lim L(r, V, p)$ is finite.

Thus we have shown that for $V > 4^{-1/p}$ the limit $\lim L(r, V, p)$ as $r \to 1$ is finite.

The case of $V = 4^{-1/p}$ has remained to be considered. In this case (4.15) takes the form

$$\log \frac{4a^p}{(1+a^p)^2} + \frac{1-a^p}{1+a^p} \log \frac{r^p}{a^p} = 0. \tag{4.31}$$

Notice that if $r \to 1$, then $a \to 1$. Indeed, otherwise a would have to satisfy

$$g(a) = \log \frac{4a^p}{(1+a^p)^2} + \frac{1-a^p}{1+a^p} \log a^{-p} = 0. \tag{4.32}$$

But the only root of (4.23) is $a = 1$.

Using now (4.31) we get

$$\lim \frac{1-a^p}{1-r^p} = 2 \quad \text{as } r \to 1 \tag{4.33}$$

and

$$\lim p \frac{1+a^p}{1-a^p} \log \frac{r}{a} = 1 \quad \text{as } r \to 1. \tag{4.34}$$

For $V = 4^{-1/p}$ we have $L(r, V, p) = \Psi(a, r, V, p)$ and so, by (4.32) and (4.33),

$$\lim [L(r, V, p) + 2 \log(1-r^p)] = 1 \quad \text{as } r \to 1.$$

The above considerations results in

$$\lim M(r, V, p) = +\infty \quad \text{as } r \to 1 \quad \text{for } V > 4^{-1/p},$$

and

$$\lim M(r, V, p) = -\infty \quad \text{as } r \to 1 \quad \text{for } V = 4^{-1/p}.$$

Hence for $V > 4^{-1/p}$, in view of the fact that estimation (4.17) is sharp, the upper bound of V' does not exist for any function of the family $S^{(p)}$. If, however, $V = 4^{-1/p}$, then $M(r, V, p)$ is a decreasing function and thus it attains its maximum for the minimal admissible value of r defined by the equation $V = r/(1-r^p)^{2/p}$.

Thus the final conclusion is that the estimation from above in (4.26) holds for $V \leqslant 4^{-1/p}$.

The results of this section for $p = 1$ have been obtained with the aid of the consistent method due to R. Robinson [11]. Some of his results had been, of course, known before. For instance, in the family $S_b = S^{(1)}$ L. Bieberbach [2], R. Nevanlinna [7], and K. Löwner [6] had otained the sharp estimations of $|f'(z)|$, $|f'(z)|/|f(z)|$ and $|f'(z)/|f^2(z)|$, respectively.

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