A CONNECTED LOCALLY CONNECTED COUNTABLE SPACE
WHICH IS ALMOST REGULAR

BY

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1. Introduction. In [5] Urysohn proves that a countable connected space cannot be regular (cf. [3]). This poses the questions raised in [4] and [2] as to how closely a connected or locally connected countable space may resemble a regular space. In particular, can there be countable connected and locally connected almost regular spaces? We answer this question in the affirmative by constructing an example of such a space. The example we provide is obtained by combining the topologies of the Urysohn spaces constructed by Jones and Stone in [1] and Ritter in [3].

2. Definitions. A point $p$ in a topological space $X$ is called a regular point of $X$ if given any open set $U$ containing $p$ there is an open set $V$ containing $p$ such that $\text{Cl}(V) \subseteq U$. A topological space is called a Urysohn space if for each pair of distinct points $p$ and $q$ there are open sets $U$ and $V$, with disjoint closures, containing $p$ and $q$, respectively. An almost regular space is a Urysohn space which contains a dense subset of regular points.

3. A countable connected locally connected almost regular Urysohn space. Let $Q_n$ be a countable set of rational numbers such that, for each integer $n$, $Q_n$ is dense in the reals and $(Q_n)_{n=-\infty}^{\infty}$ is pairwise disjoint. Let $L_n \subseteq \mathbb{R}^2$ be defined by $L_n = \{(x, y) : x \in Q_n, y = n\}$. Similarly, let $(I_n)_{n=-\infty}^{\infty}$ be a countable collection of pairwise disjoint sets of irrational numbers each of which is dense in the reals. Set $K_n = \{(x, y) : x \in I_n, y = n\}$, $L'_n = L_n \cup K_n$ and let $(r_i)_{i=1}^{\infty}$ and $(w_i)_{i=1}^{\infty}$ denote the elements of $\bigcup_{n=-\infty}^{\infty} L_n$ and $\bigcup_{n=-\infty}^{\infty} K_n$, respectively. Henceforth, to simplify the notation we shall call the $r_i$'s rational points and the $w_i$'s irrational points. Let

$$X_0 = \bigcup_{n=-\infty}^{\infty} L'_n \quad \text{and} \quad X_1 = \bigcup_{n=-\infty}^{\infty} Y_n;$$

where each $Y_n$ denotes a copy of $X_0$ and the union ranges over all pairs
(m, n) of positive integers. Set Z₁ = X₀ ∪ X₁ and regard Z₁ and X₁ as disjoint unions. Thus, if p ∈ Z₁, then either p ∈ X₀ or p ∈ Yₙ for some unique pair (m, n) of positive integers. The basic open sets for Z₁ are defined as follows.

If \( p \in L'_{2k} \subset Y^m_n \) and \( \varepsilon > 0 \), let

\[
N^{*}_{i(1)}(p) = \{ q \in L'_{2k} \mid d(p, q) < \varepsilon \}.
\]

If \( p = (x, 2k+1) \in L'_{2k+1} \subset Y^m_n \) and \( \varepsilon > 0 \), let

\[
N^{*}_{i(1)}(p) = \{ q \in L'_{2k+j} \mid j = 0, 1, 2 \text{ and } d(p_j, q) < \varepsilon, \}
\]

where \( p_j = (x, 2k+j) \),

where \( L'_{2k} \) and \( L'_{2k+1} \) are the "lines" in \( Y^m_n \) one unit below and above \( L'_{2k+1} \), respectively. The sets \( N^{*}_{i(1)}(p) \) form a basis for a topology on \( X_1 \) which satisfies the Urysohn separation property.

For each \( w_i \in X_0 \) and \( \varepsilon > 0 \), let

\[
O^*_i(w_i) = \{ q \in L'_k \subset Y^i_n \mid n = 1, 2, \ldots \text{ and } k \geq 2[1/\varepsilon] \},
\]

where \( [1/\varepsilon] \) denotes the greatest integer less than or equal to \( 1/\varepsilon \). For each \( r_i \in X_0 \) and \( \varepsilon > 0 \), let

\[
O^*_i(r_i) = \{ q \in L'_k \subset Y^i_n \mid m = 1, 2, \ldots \text{ and } k \leq -2[1/\varepsilon] \}.
\]

If \( r_i \in L'_{2k} \subset X_0 \) and \( \varepsilon > 0 \), let

\[
U^*_i(r_i) = \{ q \in L'_{2k} \mid d(r_i, q) < \varepsilon \}.
\]

If \( r_i = (x, 2k+1) \in L'_{2k+1} \subset X_0 \), let

\[
U^*_i(r_i) = \{ q \in L'_{2k+j} \subset X_0 \mid j = 0, 1, 2 \text{ and } d(p_j, q) < \varepsilon, \}
\]

where \( p_j = (x, 2k+j) \).

The basic open sets for points in \( X_0 \) are defined as follows:

\[
N^*_0(w_4) = \{ w_i \} \cup O^*_i(w_4) \quad \text{and} \quad N^*_0(r_i) = U^*_i(r_i) \cup (\bigcup O^*_i(p)) \cup (\bigcup Y^m_n),
\]

where \( \bigcup O^*_i(p) \) ranges over all \( p \in U^*_o(r_i) \) and \( \bigcup Y^m_n \) ranges over all pairs \( (m, n) \) such that \( w_m \) and \( r_n \) are elements of \( U^*_o(r_i) \).

Since the following properties are easy to verify, we shall omit their formal proofs.

(i) The collection \( \{ N^*_i(p) \mid p \in Z_1, i = 0, 1 \text{ and } \varepsilon > 0 \} \) forms a basis for a topology on \( Z_1 \) which satisfies the Urysohn separation property.

(ii) \( Z_1 \) is connected.

(iii) The elements of \( X_0 \) have connected \( \varepsilon \)-neighborhoods.

(iv) \( E_0 = \{ w_i \in X_0 \mid i = 1, 2, \ldots \} \) is the set of all regular points of \( Z_1 \).
We note that (iii) follows from the observation that any set which is both open and closed and contains some \( w \), must contain \( Y_n^i \) for every positive integer \( n \), and hence must contain \( r_n \) for every positive integer \( n \).

To see (iii) note that an \( \varepsilon \)-neighborhood of an irrational point in \( X_0 \) is a slight modification of Roy's lattice space [6]. As for the rational points, any subset \( U \) of an \( \varepsilon \)-neighborhood of \( r_n \in X_0 \) which is both open and closed in \( U \) contains some \( w_k \). Thus \( Y_n^k \subset U \), which implies \( r_n \in U \).

In order to prove (iv), simply observe that for each \( w_i \in R_0 \) and for each sufficiently small \( \varepsilon > 0 \)

\[
N_{0(1)}^{e/2}(w_i) \subset \text{Cl}(N_{0(1)}^{e/2}(w_i)) \subset N_{0(1)}^{e}(w_i).
\]

On the other hand, if \( p \in Z_1 - R_0 \), then for every pair \( \varepsilon, \varepsilon' > 0 \)

\[
\text{Cl}(N_{i(1)}^{\varepsilon}(p)) \cap N_{i(1)}^{\varepsilon'}(p), \quad \text{where} \ i = 0, 1.
\]

We next extend our space so that the elements of \( X_1 \) have connected \( \varepsilon \)-neighborhoods and \( X_1 \) contains a dense subset of regular points.

With each \( Y_n^m \) we associate a countable collection \( \{Y_n^m,k\}_{k,l=1}^{\infty} \) of copies of \( X_0 \). Let \( Z_2 = \bigcup Y_n^m,k \), where the union ranges over all 4-tuples \( (k, l, m, n) \) of positive integers, and let

\[
Z_2 = \bigcup_{i=0}^{3} X_i.
\]

Again regard \( X_i, i = 1, 2 \), and \( Z_2 \) as disjoint unions. Basic open sets for \( Z_2 \) are defined as follows.

If \( p \in L'_{2k} \subset Y_n^m,k \) and \( \varepsilon > 0 \), let

\[
N_{2(3)}^{e}(p) = \{q \in L'_{2k} | d(p, q) < \varepsilon \}.
\]

If \( p = (x, 2k+1) \in L'_{2k+1} \subset Y_n^m,k \) and \( \varepsilon > 0 \), let

\[
N_{2(3)}^{e}(p) = \{q \in L'_{2k+j} \subset Y_n^m,k | j = 0, 1, 2 \text{ and } d(p_j, q) < \varepsilon, \text{ where } p_j = (x, 2k+j)\}.
\]

This defines basic open sets for points in \( X_2 \) which satisfy the Urysohn separation property.

For each \( w_j \in Y_n^m \) and \( \varepsilon > 0 \), let

\[
O_{2}^{e}(w_j) = \{q \in L'_k \subset Y_n^m,k | l = 1, 2, \ldots \text{ and } k \geq 2[1/\varepsilon] \}.
\]

For each \( r_j \in Y_n^m \) and \( \varepsilon > 0 \), let

\[
O_{2}^{e}(r_j) = \{q \in L'_k \subset Y_n^m,k | l = 1, 2, \ldots \text{ and } k \leq -2[1/\varepsilon] \}.
\]

We now redefine the neighborhoods of points in \( X_0 \cup X_1 \).

For each \( w_j \in Y_n^m \) and \( \varepsilon > 0 \), let

\[
N_{1(2)}^{e}(w_j) = N_{1(1)}^{e}(w_j) \cup (\bigcup O_{2}^{e}(p)) \cup (\bigcup Y_n^m,k),
\]
where $\bigcup \mathcal{O}^*_\varepsilon(p)$ ranges over all $p \in N^*_{1(1)}(w_j)$ and $\bigcup \mathcal{Y}^{m,k}_{n,i}$ ranges over all pairs $(k, l)$ such that $w_k$ and $r_l$ are elements of $N^*_{1(1)}(w_j)$.

For each $r_j \in \mathcal{Y}^*_n$ and $\varepsilon > 0$, let

$$N^*_{1(2)}(r_j) = \{r_j\} \cup \mathcal{O}^*_\varepsilon(r_j).$$

Points $p \in X_0$ have basic $\varepsilon$-neighborhoods of the form

$$N^*_{0(2)}(p) = N^*_{0(1)}(p) \cup (\bigcup \mathcal{O}^*_\varepsilon(q)) \cup (\bigcup \mathcal{Y}^{m,k}_{n,i}),$$

where $\bigcup \mathcal{O}^*_\varepsilon(q)$ ranges over all $q \in X_1 \cap N^*_{0(1)}(p)$, and $\bigcup \mathcal{Y}^{m,k}_{n,i}$ ranges over all $4$-tuples $(m, k, n, l)$ for which the pair $w_k, r_l$ is in $\mathcal{Y}^*_n \cap N^*_{0(1)}(p)$.

The proof of the following properties is analogous to the verification of these properties for the space $Z_1$.

(i) The collection $\{N^*_{i(2)}(p) \mid p \in Z_2, i = 0, 1, 2 \text{ and } \varepsilon > 0\}$ forms a basis for a topology on $Z_2$ which satisfies the Urysohn separation property.

(ii) $Z_2$ is connected.

(iii) The elements of $X_0 \cup X_1$ have connected $\varepsilon$-neighborhoods.

(iv) If $R_1$ is the set of all rational points of $X_1$, then $R_0 \cup R_1$ is the set of regular points of $Z_2$.

Our construction of $N^*_{i(2)}(p)$ also shows that

(v) If $N^*_{i(1)}(p)$ and $N^*_{j(1)}(q)$ have disjoint closures, then so do $N^*_{i(2)}(p)$ and $N^*_{j(2)}(q)$.

In order to obtain an almost regular space with the desired properties, we simply repeat the above process ad infinitum. The inductive argument should now be apparent. Suppose we have defined the connected Urysohn space

$$Z_k = \bigcup_{i=0}^{k} X_i$$

such that

(a) for each $i = 1, 2, \ldots, k$, $X_i = \bigcup_{j=1}^{i} \mathcal{Y}^{m(1), \ldots, m(i)}_{n(1), \ldots, n(i)}$, where $m(j), n(j)$, $1 \leq j \leq i$, range over all positive integers;

(b) the elements of $\bigcup X_i$ have connected $\varepsilon$-neighborhoods;

(c) $\bigcup R_i$ is the set of regular points of $Z_k$, where $R_i$ consists of all rational points of $X_i$ if $i$ is odd and $R_i$ consists of all irrational points of $X_i$ if $i$ is even;

(d) if $N^*_{i(n)}(p)$ and $N^*_{j(m)}(q)$ have disjoint closures for $n, m < k$, then so do $N^*_{i(k)}(p)$ and $N^*_{j(k)}(q)$.

Then

$$X_{k+1} = \bigcup \mathcal{Y}^{m(1), \ldots, m(k), m(k+1)}_{n(1), \ldots, n(k), n(k+1)},$$
where \( \{ Y_{m(n)}^{(1)}, \ldots, m(n), m(k+1) \mid m(k+1) \text{ and } n(k+1) \text{ range over all positive integers} \} \) is the countable collection of copies of \( X_0 \) associated with \( Y_{m(n)}^{(1)}, \ldots, m(n) \), and

\[
Z_{k+1} = \bigcup_{i=0}^{k+1} X_i.
\]

Again regard \( X_i, \ i > 0, \text{ and } Z_{k+1} \) as disjoint unions. Basic open sets for \( Z_{k+1} \) are defined as follows.

Let \( Y = Y_{m(n)}, \ldots, m(n), m(k+1) \). If \( p \in L_{2r}^r \subset Y \) and \( \varepsilon > 0 \), let

\[
N_{k+1}^*(p) = \{ q \in L_{2r}^r \mid d(p, q) < \varepsilon \}.
\]

If \( p = (x, 2r+1) \in L_{2r+1}^r \subset Y \) and \( \varepsilon > 0 \), let

\[
N_{k+1}^*(p) = \{ q \in L_{2r+j}^r \subset Y \mid j = 0, 1, 2 \text{ and } d(p_j, q) < \varepsilon, \text{ where } p_j = (x, 2r+j) \}.
\]

This defines the open sets for points in \( X_{k+1} \). For each \( w_j \in Y_{m(n)}, \ldots, m(n) \) and \( \varepsilon > 0 \), let

\[
O_{k+1}^*(w_j) = \{ q \in L_{r}^r \subset Y_{m(n)}, \ldots, m(n), l \mid l = 1, 2, \ldots \text{ and } r \geq 2[1/\varepsilon] \}.
\]

For each \( r_j \in Y_{m(n)}, \ldots, m(n) \) and \( \varepsilon > 0 \), let

\[
O_{k+1}^*(r_j) = \{ q \in L_{r}^r \subset Y_{m(n)}, \ldots, m(n), l \mid l = 1, 2, \ldots \text{ and } r \leq -2[1/\varepsilon] \}.
\]

Define \( \varepsilon \)-neighborhoods for \( w_j, r_j \in Y_{m(n)}, \ldots, m(n) \) as follows.

If \( k \) is even, let

\[
N_{k+1}^*(w_j) = \{ w_j \} \cup O_{k+1}^*(w_j)
\]

and

\[
N_{k+1}^*(r_j) = N_{k+1}^*(r_j) \cup (\bigcup O_{k+1}^*(p)) \cup (\bigcup Y_{m(n)}, \ldots, m(n), m(k+1), n(k+1)),
\]

where \( \bigcup O_{k+1}^*(p) \) ranges over all \( p \in N_{k+1}^*(r_j) \) and \( \bigcup Y_{m(n)}, \ldots, m(n), m(k+1), n(k+1) \) ranges over all pairs \( (m(k+1), n(k+1)) \) such that \( w_{m(k+1)} \) and \( r_{n(k+1)} \) are elements of \( N_{k+1}^*(r_j) \).

If \( k \) is odd, let

\[
N_{k+1}^*(w_j) = \{ w_j \} \cup O_{k+1}^*(w_j)
\]

and

\[
N_{k+1}^*(r_j) = \{ r_j \} \cup O_{k+1}^*(r_j)
\]

where \( O_{k+1}^*(p) \) ranges over all \( p \in N_{k+1}^*(w_j) \) and \( \bigcup Y_{m(n)}, \ldots, m(n), m(k+1), n(k+1) \) ranges over all pairs \( (m(k+1), n(k+1)) \) such that \( w_{m(k+1)} \) and \( r_{n(k+1)} \) are elements of \( N_{k+1}^*(r_j) \).

If \( p \in X_i \), \( 0 \leq i < k \), and \( \varepsilon > 0 \), let

\[
N_{i(k+1)}^*(p) = N_{i(k+1)}^*(p) \cup (\bigcup O_{k+1}^*(q)) \cup (\bigcup Y_{m(n)}, \ldots, m(k+1), n(k+1)),
\]
where \( \cup O_{i+1}(q) \) ranges over all \( q \in X_k \cap N_{i(k)}(p) \), and \( \cup Y_{n(1)}, \ldots, m(k+1) \) ranges over all \((2k+1)\)-tuples \((m(1), \ldots, m(k+1), n(1), \ldots, n(k+1))\) for which the pair \( w_{m(k+1)}, r_{n(k+1)} \) is in \( Y_{n(1)}, \ldots, m(k) \cap N_{i(k)}(p) \).

The proofs of the following properties are as before.

(i) The collection \( \{ N_{i(k)}^{s}(p) \mid p \in Z_{k+1}, i = 0, 1, \ldots, k+1 \text{ and } \varepsilon > 0 \} \) forms a basis for a topology on \( Z_{k+1} \) which satisfies the Urysohn separation property.

(ii) \( Z_{k+1} \) is connected.

(iii) The elements of \( \bigcup_{i=0}^{k} X_i \) have connected \( \varepsilon \)-neighborhoods.

(iv) \( \bigcup_{i=0}^{k} R_i \) is the set of regular points of \( Z_{k+1} \).

(v) If \( N_{i(n)}^{s}(p) \) and \( N_{j(m)}^{s}(q) \) have disjoint closures for \( n, m < k \), then so do \( N_{i(k+1)}^{s}(p) \) and \( N_{j(k+1)}^{s}(q) \).

This completes the inductive step.

Now let

\[
X = \bigcup_{i=0}^{\infty} X_i.
\]

Clearly, \( X \) is countable. If \( p \in X \), then \( p \in X_i \) for some unique integer \( i \). For each \( \varepsilon > 0 \), we define \( N_{i}^{s}(p) \) to be the smallest subset of \( X \) which contains \( N_{i(k)}^{s}(p) \) for each integer \( k > i \). It follows from our inductive construction that the sets \( N_{i}^{s}(p) \) form a basis for \( X \). Since each \( Z_k \) is connected, \( X \) is connected. Furthermore, the set

\[
\bigcup_{i=0}^{\infty} R_i = R
\]

is dense in \( X \).

If \( p \) and \( q \) are distinct points in \( X \), then \( p \in X_i \) and \( q \in X_j \) with \( i \) and \( j \) not necessarily distinct. Let \( k > \max \{i+1, j+1\} \). Since \( Z_k \) is a Urysohn space, there exists an \( \varepsilon > 0 \) such that \( N_{i(k)}^{s}(p) \) and \( N_{j(k)}^{s}(q) \) have disjoint closures. Thus, by property (iv), \( N_{i(n)}^{s}(p) \) and \( N_{j(m)}^{s}(q) \) have disjoint closures for every integer \( n \geq k \). It follows that

\[
\overline{\text{Cl}(N_{i}^{s}(p))} \cap \overline{\text{Cl}(N_{j}^{s}(q))} = \emptyset.
\]

Hence \( X \) is a Urysohn space.

It remains to show that every point in \( R \) is regular and that \( X \) is locally connected. If \( p \in R \), then \( p \in R_i \) for some unique integer \( i \). Thus for \( k > i+1 \) and \( \varepsilon > 0 \) there is a positive number \( \varepsilon' > 0 \) such that \( \overline{\text{Cl}(N_{i(k)}^{s}(p))} \subset [N_{i(k)}^{s}(p)] \). But then \( \overline{\text{Cl}(N_{i(n)}^{s}(p))} \subset N_{i(n)}^{s}(p) \) for every integer \( n \geq k \) and, therefore, \( \overline{\text{Cl}(N_{i}^{s}(p))} \subset N_{i}^{s}(p) \). Hence \( p \) is a regular
point of $X$. The local connectivity follows from the fact that if $p \in X$, then $p \in X_i$ for some unique integer $i$. Thus, by property (iii), $p$ has connected $s$-neighborhoods in $Z_k$ for every $k \geq i+1$.

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