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ON A THEOREM OF H. P. LOTZ ON QUASI-COMPACTNESS OF MARKOV OPERATORS

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We denote by $C(X)$ the continuous real-valued functions on a compact Hausdorff space X. A linear operator T on $C(X)$ which is positive ($f \geqslant 0 \Rightarrow Tf \geqslant 0$) and takes 1 into 1 is said to be *Markov*. It is well known (see, e.g., [5]) that for every Markov operator T there exists a unique family of probability (Radon) measures $P(x, \cdot)$, $x \in X$, such that

(a) for every Borel set A in X the mapping $x \to P(x, A)$ is Borel measurable;

(b) for every $f \in C(X)$, $Tf = \int f(y) P(\cdot, dy)$.

In fact, we have $P(x, \cdot) = T^* \delta_x$. By $B(X)$ we denote the bounded Borel functions on X. Using (b) we see that T can be canonically extended to an operator $T\colon B(X) \to B(X)$. Recall that a linear operator $T\colon C(X) \to C(X)$ is compact if and only if the mapping $x \to T^* \delta_x$ is norm continuous (see [4], Proposition 5.9). In particular, every compact Markov operator takes $B(X)$ into $C(X)$. We say that T is *weak* mean ergodic* (w*.m.e.) if for every $f \in C(X)$ the Cesàro means

$$A_n f = n^{-1}(f + Tf + \ldots + T^{n-1} f)$$

converge pointwise (see [1]). If $A_n f$ converge uniformly for every $f \in C(X)$, then T is said to be *uniformly mean stable* (u.m.s.). The operator T^* is said to be *strongly ergodic* (s.e.) if the Cesàro means A_n^* converge in the strong operator topology. T is said to be *uniformly ergodic* (u.e.) if the Cesàro means A_n converge in the uniform topology, and T is said to be *quasi-compact* if the A_n converge uniformly to a finite-dimensional projection (see [2]). Let $P_T(X)$ denote the set of all T^*-invariant probability (Radon) measures. Lotz [3] has proved the following:

Let T be a Markov operator such that

(i) T^* is s.e.;

(ii) $P_T(X)$ has a weak order unit;

(iii) every T^*-invariant probability measure has a non-meager support. Then T is quasi-compact.

He asked whether (ii) was necessary. We obtain quasi-compactness of T without the assumption (ii).

THEOREM. *Let T be a Markov operator on $C(X)$. If* (i) *and* (iii) *hold, then T is quasi-compact.*

Proof. Similarly as in [3] we show that T satisfies Doeblin's condition (see [3], Proposition 3). For $\mu \in P_T(X)$, as in [3] we put

$$g_{n,\mu}(t) = \|T^{*n}\delta_t \wedge \mu\|,$$

$$H_{n,\mu} = \{f \in C(X)\colon 0 \leqslant f \leqslant g_{n,\mu}\}, \qquad H_\mu = \bigcup_{n=1}^{\infty} H_{n,\mu},$$

$$A_\mu = \{t \in X\colon h(t) = 0 \text{ for all } h \in H_\mu\}.$$

Let $A = \bigcap A_\mu$, $\mu \in P_T(X)$. If $A = \emptyset$, then by compactness we have

$$A_{\mu_1} \cap A_{\mu_2} \cap \ldots \cap A_{\mu_n} = \emptyset \quad \text{for some } \mu_1, \mu_2, \ldots, \mu_n \in P_T(X).$$

Observe that $A_{\bar{\mu}} \subset A_{\mu_1} \cap \ldots \cap A_{\mu_n}$, where $\bar{\mu} = n^{-1}(\mu_1 + \ldots + \mu_n)$. Indeed, if $t \notin A_{\mu_k}$, then $h(t) > 0$ for some continuous function h with $0 \leqslant h \leqslant g_{m,\mu_k}$. Thus

$$0 \leqslant \frac{h(s)}{n} \leqslant \|T^{*m}\delta_s \wedge n^{-1}(\mu_1 + \ldots + \mu_n)\| \quad \text{for all } s \in X$$

and, consequently, $n^{-1}h \in H_{m,\bar{\mu}}$ and $t \notin A_{\bar{\mu}}$. Now, as in [3], it can be proved that T satisfies Doeblin's condition with the invariant measure $\bar{\mu}$. It remains to show that A is empty. Suppose $A \neq \emptyset$. By the inclusion $TH_\mu \subset H_\mu$, the closed set A is T-invariant, so there exists a minimal invariant set $B \subset A$. By (i) and [1], Theorem 5, the set B is the support of a unique T^*-invariant probability ν (the separability assumption in Theorem 5 of [1] is unnecessary since, as the proof of Lemma 2 in [1] shows, every non-empty open set is visited infinitely many times even if X is non-metrizable). Let $B_{n,k} = \{t\colon g_{n,\nu}(t) \geqslant 1/k\}$. By the same argument as in [3], Theorem 1, the sets $B \cap B_{n,k}$ are meager. Since $\operatorname{int} B \neq \emptyset$, by assumption (ii) the set $B - \bigcup_{n,k} B_{n,k}$ is non-empty. For $t \in B - \bigcup_{n,k} B_{n,k}$ we have $\|T^{*n}\delta_t \wedge \nu\| = 0$ for every natural n. This is a contradiction since $\|A_n^*\delta_t - \nu\| \to 0$ by (i).

Remark. By the last part of the proof it is clear that (i) in the Theorem can be replaced by

(i′) $A_n f(t)$ converge uniformly on the unit ball of $C(X)$ for every t in a co-meager subset of X.

The following corollary strengthens one part of Corollary 1 in [3].

COROLLARY 1. *Let T be an irreducible Markov operator on $C(X)$. If T^* is s.e., then T is quasi-compact.*

As in [6], let A be the family of non-negative lower semicontinuous functions such that $Tf \leqslant f \leqslant 1$ and $T^n f \to 0$ pointwise. By the *Foguel boundary* F of T we mean the intersection of all zero-sets $\{x: \hat{f}(x) = 0\}$ for the functions f in A. If $F = X$, we say that T is *conservative*. By the *center* M of T we mean the closure of the supports of all measures in $P_T(X)$. It is well known that $M \subset F$ (see [7]). If T^* is s.e., then $M = F$. Indeed, we always have $T1_{M^c} \leqslant 1_{M^c}$ (see [6]). By the strong ergodicity of T we have $\lim A_n^* \delta_x \in P_T(X)$, so $T^n 1_{M^c}(x) \leqslant A_n 1_{M^c}(x) \to 0$ for all $x \in X$. Thus $1_{M^c} \in A$, so $M \supset F$. Now we have the following

COROLLARY 2. *Let T be a conservative Markov operator on $C(X)$. Then T is quasi-compact if and only if T^* is s.e., T is u.m.s., and* $\operatorname{ex} P_T(X)$ *is finite.*

Proof. If T^* is s.e., then by conservativeness the center of T is X, so $\bigcup \operatorname{supp} \mu = X$, $\mu \in \operatorname{ex} P_T(X)$. Thus $\operatorname{supp} \mu$ is non-meager for $\mu \in P_T(X)$ and, by the Theorem, T is quasi-compact.

Example 2 in [3] shows that the conservativeness cannot be dropped in Corollary 2.

COROLLARY 3. *Let T be a Markov operator on $C(X)$. Then T is quasi-compact if and only if T is u.e. and the support of every invariant probability is non-meager in the relative topology on M.*

Proof. If T is quasi-compact, then $T|C(M)$ is also quasi-compact, so we can assume that $M = X$. For $\mu \in P_T(X)$ we have $T1_{\operatorname{supp}\mu} \geqslant 1_{\operatorname{supp}\mu}$ (see [6]). By the quasi-compactness of T, the Cesàro means $A_n 1_{\operatorname{supp}\mu}$ norm converge to $P1_{\operatorname{supp}\mu}$. Since P is a compact projection, $P1_{\operatorname{supp}\mu} \in C(X)$. Thus

$$1_{\operatorname{supp}\mu} = T1_{\operatorname{supp}\mu} = \ldots = P1_{\operatorname{supp}\mu},$$

and $\operatorname{supp}\mu$ is clopen, so non-meager. Conversely, let T be a u.e. Markov operator and $\operatorname{int}\operatorname{supp}\mu \neq \emptyset$ on M for every $\mu \in P_T(X)$. From the Theorem we infer that $T|C(M)$ is quasi-compact. By the uniform ergodicity of T, for every finite measure μ, $A_n^* \mu(M^c) \to 0$ uniformly on the unit ball of $C(X)^*$. Thus T is quasi-compact on $C(X)$.

REFERENCES

[1] A. Iwanik, *On pointwise convergence of Cesàro means and separation properties for Markov operators on $C(X)$*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 29 (1981), p. 515–520.

[2] M. Lin, *Quasi-compactness and uniform ergodicity of positive operators*, Israel Journal of Mathematics 29 (1978), p. 309–311.

[3] H. P. Lotz, *Uniform ergodic theorems for Markov operators on $C(X)$*, Mathematische Zeitschrift 178 (1981), p. 145–156.

[4] D. Revuz, *Markov chains*, North-Holland Mathematical Library 1975.

[5] M. Rosenblatt, *Equicontinuous Markov operators*, Teorija Verojatnosteĭ i ee Primenenija 9 (1964), p. 205–222.

[6] R. Sine, *Geometric theory of a single Markov operator*, Pacific Journal of Mathematics 27 (1968), p. 155–166.

[7] — *Sample path convergence of stable Markov processes. II*, Indiana University Mathematics Journal 25 (1976), p. 23–43.

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