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# On Mikusiński's functional equation

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**Abstract.** The conditional equation (4) is considered for functions $f\colon X \to Y$, where $(X, +)$ and $(Y, +)$ are (not necessarily commutative) groups. The general solution of (4) is described and, in particular, the problem of the equivalence of equations (4) and (2) is investigated.

**1.** Certain geometrical considerations have led J. Mikusiński to the functional equation

$$f(x+y)[f(x+y)-f(y)-f(x)] = 0, \tag{1}$$

where $f$ denotes a continuous real-valued function of a real variable. It is not very difficult to see that in such a case $f$ must satisfy Cauchy's functional equation

$$f(x+y) = f(x)+f(y), \tag{2}$$

and hence it must be linear $f(x) = cx$. But it is less obvious that in fact equation (1) implies (2) (the converse implication is trivial) without any regularity hypotheses. (Then $f$ need not be linear, equation (2) admits also non-measurable solutions; cf. [1].)

It is still more interesting that this conclusion fails to hold if we restrict the domain of $f$ to the set of integers. (A suitable example is given in Section 3.) This fact suggests the investigation of equation (1) for functions $f$ having the domain and range in more general algebraic structures than the additive group, resp. the field, of real numbers.

In the present paper we investigate equation (1), or rather slightly more general conditional equation (4), in the case where $+$ denote not necessarily commutative group operations. We shall find also those soutions of (1) that do not satisfy (2), in the case where they exist.

**2.** Let $X = (X, +)$ and $Y = (Y, +)$ be two (not necessarily commutative) groups. Without a fear of ambiguity we shall use the same symbol $+$ to denote the operation in both groups. Similarly, the neutral element in both groups will be denoted by 0, the inverse element to $x$ will be denoted by $-x$, and $2x$ will denote $x+x$.

Suppose that in the set $Y$ we have also another inner operation, the multiplication, fulfilling the following condition

(3) $\quad ab = 0$ *if and only if* $a = 0$ *or* $b = 0$.

(Condition (3) is fulfilled e.g. if $Y$ is a ring without divisors of zero). Then equation (1) for functions $f\colon X \to Y$ is equivalent to the condition

(4) $\quad$ *if* $f(x+y) \neq 0$, *then* $f(x+y) = f(x)+f(y)$.

But in (4) the multiplication does not occur, and we may solve the conditional equation (4) instead of (1), without assuming that there is a second inner operation defined in $Y$.

Now suppose that $f\colon X \to Y$ is a solution of (4) and write

$$K = \{x \in X\colon f(x) = 0\}, \qquad K' = X \setminus K.$$

LEMMA 1. $\boldsymbol{K} = (K, +)$ *is a subgroup of* $\boldsymbol{X}$.

Proof. It is readily seen that $0 \in K$ and that $x+y \in K$ whenever $x, y \in K$. Now suppose that for an $x \in K$ we have $-x \in K'$, i.e. $f(-x) \neq 0$. Then by (4)

(5) $\quad f(-x) = f[(-2x)+x] = f(-2x)+f(x) = f(-2x)$,

and again by (4) (since $f(-2x) = f(-x) \neq 0$),

(6) $\quad f(-2x) = f(-x)+f(-x)$.

Relations (5) and (6) yield $f(-x) = f(-x)+f(-x)$, i.e. $f(-x) = 0$ contrary to the assumption.

LEMMA 2. *If* $x \in K'$, *then also* $-x \in K'$.

This is an immediate consequence of Lemma 1.

LEMMA 3. *If* $x \in K$ *and* $y \in K'$, *then* $x+y \in K'$ *and* $y+x \in K'$.

Proof. By Lemma 1 $-x \in K$. If we had $x+y \in K$, then $y = (-x)+ +(x+y)$ would belong to $K$, and similarly $y+x \in K$ would imply $y = (y+x)+(-x) \in K$.

Now we shall distinguish two cases.

(*) *For every* $x \in K'$ *such that* $2x \in K$ *we have* $2f(x) = 0$.

(**) *There exists a* $u \in K'$ *such that* $2u \in K$ *and* $2f(u) \neq 0$.

In particular, case (*) occurs if for every $x \in K'$ we have $2x \in K'$.

LEMMA 4. *In case* (*) *we have*

(7) $\quad f(2x) = 2f(x) \quad$ *and* $\quad f(-x) = -f(x)$

*for every* $x \in X$.

Proof. If $x \in K$, then $2x \in K$ (Lemma 1) and $f(2x) = 0 = 2f(x)$. If $x \in K'$ and $2x \in K'$, then by (4) $f(2x) = 2f(x)$. If $x \in K'$ but $2x \in K$, then by (*) we have $f(2x) = 0 = 2f(x)$.

If $x \in K$, then $-x \in K$ (Lemma 1) and $f(-x) = 0 = -f(x)$. If $x \in K'$, then by (4) and the first relation of (7)

$$f(x) = f(2x-x) = f(2x)+f(-x) = 2f(x)+f(-x),$$

whence $f(-x) = -f(x)$.

LEMMA 5. *In case* (*) *relation* (4) *implies* (2).

Proof. Take arbitrary $x, y \in X$. If $f(y) \neq 0$, then by (4) and (7)

$$f(y) = f[(-x)+(x+y)] = f(-x)+f(x+y) = -f(x)+f(x+y),$$

which implies (2). If $f(x) \neq 0$, then by (4) and (7)

$$f(x) = f[(x+y)+(-y)] = f(x+y)+f(-y) = f(x+y)-f(y),$$

which again implies (2). Finally, if $f(x) = f(y) = 0$, then by Lemma 1 $f(x+y) = 0$ and (2) holds, too.

LEMMA 6. *In case* (**) *there exists a* $c \in Y$, $c \neq 0$, *such that*

$$f(x) = \begin{cases} 0 & \text{for } x \in K, \\ c & \text{for } x \in K'. \end{cases} \tag{8}$$

Proof. Write $f(u) = c$. We have $c \neq 0$ by the definition of $K'$.

Take an arbitrary $x \in K'$. Then by (4)

$$f(x) = f[(x-2u)+2u] = f(x-2u)+f(2u) = f(x-2u), \tag{9}$$

since $2u \in K$. Similarly,

$$f(x) = f[(x-u)+u] = f(x-u)+f(u),$$

whence

$$f(x-u) = f(x)-f(u). \tag{10}$$

If we had

$$f(x) \neq f(u), \tag{11}$$

then (4) would imply in view of (10) and (9)

$$f(x-u) = f[(x-2u)+u] = f(x-2u)+f(u) = f(x)+f(u),$$

whence, again by (10),

$$f(x)-f(u) = f(x)+f(u). \tag{12}$$

Relation (12) yields $2f(u) = 0$ contrary to the supposition. Consequently (11) is impossible, i.e. $f(x) = f(u) = c$ for every $x \in K'$. Hence (8) results in view of the definition of $K$.

Let us note two further facts concerning case (**).

LEMMA 7. *In case* (**) *we have* $x+y \in K$ *for* $x, y \in K'$.

Proof. If we had $x+y \in K'$ for some $x, y \in K'$, then we would get by (5) and Lemma 6

$$c = f(x+y) = f(x)+f(y) = 2c,$$

which contradicts the inequality $c \neq 0$.

LEMMA 8. *In case* (**) $\boldsymbol{K}$ *is a normal subgroup of* $\boldsymbol{X}$ *of index* 2.

Proof[1]. Take arbitrary $x \in K$, $y \in X$. If $y \in K$, then $y+x-y \in K$ by Lemma 1. If $y \in K'$, then also $-y \in K'$ (Lemma 2), and $y+x-y = (y+x)+(-y) \in K$ by Lemmas 3 and 7. Consequently $\boldsymbol{K}$ is a normal subgroup of $\boldsymbol{X}$.

Since for arbitrary $x, y \in K'$ we have $x-y = x+(-y) \in K$ (Lemmas 7 and 2), the index of $\boldsymbol{K}$ is equal to 2.

Lemmas 5, 6 and 8 give the complete description of the solutions of (4). We summarize them in the following

THEOREM 1. *If the group* $\boldsymbol{X}$ *has no normal subgroup of index* 2, *then the conditional equation* (4) *for functions* $f\colon X \to Y$ *is equivalent to the Cauchy equation* (2). *If* $\boldsymbol{X}$ *has normal subgroups of index* 2, *then the family of all solutions* $f\colon X \to Y$ *of* (4) *consists of all solutions of equation* (2) *and all functions* $f$ *of form* (8), *where* $c \neq 0$ *is an arbitrary element of* $Y$, *and* $K \subset Y$ *is an arbitrary set such that* $\boldsymbol{K} = (K, +)$ *is a normal subgroup of* $X$ *of index* 2.

Proof. It follows from Lemmas 5, 6 and 8 that all solutions of (4) must be of the form described above. Conversely, it is clear that all functions satisfying Cauchy's equation (2) fulfil also the conditional equation (4). It remains to show that functions (8) also fulfil relation (4).

Since all groups with two elements are isomorphic, the operation $+$ in the quotient group $\boldsymbol{X}/\boldsymbol{K} = (\{K, K'\}, +)$ must be described by the formulae

$$K+K = K'+K' = K, \quad K+K' = K'+K = K'. \tag{13}$$

In other words, we have $x+y \in K$ for $x, y \in K$ or $x, y \in K'$, and $x+y \in K'$ for $x \in K$, $y \in K'$ or $x \in K'$, $y \in K$. Now suppose that $f$ is given by (8) with $c \neq 0$. Then $f(x+y) \neq 0$ implies $x+y \in K'$, whence either $x \in K$, $y \in K'$ and

$$f(x+y) = c = f(y) = f(x)+f(y),$$

or $x \in K'$, $y \in K$ and

$$f(x+y) = c = f(x) = f(x)+f(y).$$

Consequently relation (4) holds.

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[1] Groups having subgroups of index 2 have been investigated by Z. Moszner and J. Tabor in [6]. In particular, properties (I) and (II) in [6], p. 324, might also beused in proving our Lemma 8 and Theorem 3.

Note that if $2c \neq 0$, then functions (8) do not satisfy (2), since for $x, y \in K'$ we have $x+y \in K$ and

$$f(x+y) = 0 \neq 2c = f(x)+f(y).$$

However, it may happen that $2c = 0$ for every $c \in Y$.

THEOREM 2. *If $2c = 0$ for every $c \in Y$, then the conditional equation* (4) *for functions $f\colon X \to Y$ is equivalent to Cauchy's equation* (2).

We obtain the same conclusion if we assume that the division by 2 is performable in the group $\boldsymbol{X}$.

THEOREM 3. *If every $x \in X$ may be written in the form $x = 2x'$, $x' \in X$, then the conditional equation* (4) *for functions $f\colon X \to Y$ is equivalent to Cauchy's equation* (2).

Proof. It is enough to show that $\boldsymbol{X}$ cannot have a normal subgroup of index 2. Supposing the contrary, let $\boldsymbol{K} = (K, +)$ be such a subgroup. By (13) $2X = \{x\colon x = 2x',\ x' \in X\} \subset K$, and since the hypotheses of the present theorem imply that $X \subset 2X$, we obtain finally $X = K$, which is impossible.

**3.** In this section we shall discuss a few examples.

1. $\boldsymbol{X} = \boldsymbol{Y} = (R, +)$ is the additive group of real numbers. According to Theorem 3 the only solutions of the conditional equation (4) [or, equivalently, of Mikusiński's equation (1)] are additive functions, i.e. the solutions of (2): continuous ones $f(x) = cx$, and non-measurable Hamel solutions (cf. [1]).

The same conclusion holds if the additive group of real numbers is replaced by the additive group of complex numbers.

2. $\boldsymbol{X} = \boldsymbol{Y} = (Q, +)$ is the additive group of rational numbers. The conclusion is as above with the exception that now we have no irregular solutions (cf. [1]). The family of solutions of (4) [or, equivalently, of (1)] consists of the linear functions $f(x) = cx$, $c$ rational.

3. $\boldsymbol{X} = \boldsymbol{Y} = (Z, +)$ is the additive group of integers. Then the family of solutions of (4) [or, equivalently, of (1)] consists of the linear functions $f(x) = cx$ ($c$ — an integer) and of the functions

$$f(x) = \begin{cases} 0 & \text{for even } x, \\ c & \text{for odd } x \end{cases}$$

($c \neq 0$ an integer).

In examples 2 and 3 $\boldsymbol{Y}$ may also be e.g. $(R, +)$; then the constant $c$ need not be rational resp. integral.

4. $\boldsymbol{X} = (R^*, \cdot)$ is the multiplicative group of non-zero real numbers, $\boldsymbol{Y} = (R, +)$ is the additive group of real numbers. Then the family of

solutions of equation (4), or, equivalently, of equation (1), which in this case takes the form

$$f(xy)[f(xy)-f(y)-f(x)] = 0, \quad xy \neq 0, \tag{14}$$

consists of the solutions of the Cauchy equation

$$f(xy) = f(x)+f(y), \tag{15}$$

and of the functions of the form

$$f(x) = \begin{cases} 0 & \text{for } x > 0, \\ c & \text{for } x < 0, \end{cases} \tag{16}$$

$c \neq 0$ being a real constant.

5. $\boldsymbol{X} = (\mathscr{F}_n, \circ)$ is the group of non-singular linear transformations of a real $n$-dimensional vector space onto itself (with the composition as the group operation), $\boldsymbol{Y} = (Y, \cdot) = \mathrm{GL}(n, R)$ is the multiplicative group of non-singular $n \times n$ real matrices. In this case the conditional equation (4) has the form

$$\textit{if } f(T_1 \circ T_2) \neq e, \textit{ then } f(T_1 \circ T_2) = f(T_1)f(T_2), \tag{17}$$

where $e$ is the $n \times n$ unit matrix. Since there is no other inner operation defined in a natural way in $Y$, we do not consider equation (1).

Denoting by $a_T$ the matrix of the transformation $T$, we obtain as the general solution of (17) the two families of functions:

$$f(T) = \varphi(a_T) \tag{18}$$

and

$$f(T) = \begin{cases} e & \text{if } \det a_T > 0, \\ c & \text{if } \det a_T < 0. \end{cases} \tag{19}$$

In formulae (18) and (19) $c \neq e$ is an arbitrary non-singular real $n \times n$ matrix, and $\varphi\colon Y \to Y$ is an arbitrary solution of the matrix functional equation

$$\varphi(ab) = \varphi(a)\varphi(b). \tag{20}$$

The general solution of equation (20) has been found by M. Kucharzewski and A. Zajtz [3].

6. Let $\boldsymbol{X} = \boldsymbol{E} = (E, \cdot)$ be the multiplicative group of the matrices

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{21}$$

and let $\boldsymbol{Y} = (R, +)$ be the additive group of real numbers. Equation (1) takes again form (14), where now $x$, $y$ range over the set $E$ of matrices (21).

The only solution of equation (15) is the trivial one $f(x) \equiv 0$. But besides this, equation (14) has also three families of other solutions corresponding to the three normal subgroups of the group $\boldsymbol{E}$:

$$f(x) = \begin{cases} 0 & \text{for } x = e_0, e_1, \\ c & \text{for } x = e_2, e_3, \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{for } x = e_0, e_2, \\ c & \text{for } x = e_1, e_3, \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{for } x = e_0, e_3, \\ c & \text{for } x = e_1, e_2, \end{cases}$$

where $c \neq 0$ is a real constant.

7. $\boldsymbol{X} = (R^*, \cdot)$ is the multiplicative group of non-zero real numbers, $\boldsymbol{Y} = \boldsymbol{E} = (E, \cdot)$ is the multiplicative group of matrices (21). Then according to Theorem 2, all solutions of the conditional equation (4), which in this case takes the form

$$\textit{if } f(xy) \neq e_0, \textit{ then } f(xy) = f(x)f(y),$$

satisfy the corresponding Cauchy equation.

**4.** It might be interesting to observe that in some cases very weak regularity suppositions allow us to eliminate those solutions of (4) which do not satisfy (2). The proof of this fact is based on a topological version of a theorem of Steinhaus. In the case where $X$ is a Banach space or a linear topological space, such a version of the theorem of Steinhaus was proved by W. Orlicz and Z. Ciesielski [7] and by Z. Kominek [2]. The proof given below is patterned on that found in [7] (cf. also [4]).

LEMMA 9. *Let $\boldsymbol{X} = (X, +, \mathscr{I})$ be a (not necessarily commutative) topological group (with the algebraic operation $+$ and the family $\mathscr{I}$ of open sets). If $A, B \subset X$ are second category Baire sets, then the set $A+B = \{x \in X: x = a+b,\ a \in A,\ b \in B\}$ has a non-void interior.*

Proof. Since $A, B$ are second category Baire sets, there exist non-empty open subsets $G, H$ of $X$ such that the sets $G \setminus A$ and $H \setminus B$ are first category. Let $g \in G$ and $h \in H$ be fixed and write

$$G_0 = -g+G, \quad A_0 = -g+A, \quad H_0 = H-h, \quad B_0 = B-h.$$

Then the sets $G_0 \setminus A_0$ and $H_0 \setminus B_0$ are first category. Put $S = G_0 \cap H_0$. $S$ is a neighbourhood of zero. For every $t \in S$ write

$$U_t = S \cap (t-S).$$

$U_t$ is a non-empty open set (in particular, $t \in U_t$), and hence it is second category ([5], Lemma 1).

We have

$$U_t \setminus A_0 \subset S \setminus A_0 \subset G_0 \setminus A_0$$

and

$$U_t \setminus (t-B_0) \subset (t-S) \setminus (t-B_0) = t-(S \setminus B_0) \subset t-(H_0 \setminus B_0).$$

Consequently the sets $U_t \setminus A_0$ and $U_t \setminus (t-B_0)$ are first category. Therefore $U_t \cap A_0 \cap (t-B_0)$ must be second category, whence in particular

$$A_0 \cap (t-B_0) \neq \emptyset$$

for every $t \in S$. This means that $t = a_0+b_0$ with suitable $a_0 \in A_0$, $b_0 \in B_0$, i.e.

(22) $$t = -g+a+b-h, \quad a \in A, \ b \in B.$$

Since every $t \in S$ may be written in form (22), we obtain

$$g+S+h \subset A+B,$$

which was to be proved.

THEOREM 4. *Let $\mathbf{X} = (X, +, \mathscr{I})$ be a second category topological group (not necessarily commutative), let $\mathbf{Y} = (Y, +)$ be a group (not necessarily commutative, either), and suppose that $\mathbf{X}$ fulfils the following condition:*

$(\substack{**\\ *})$ *for every neighbourhood $V$ of 0 we have* $\bigcup_{n=1}^{\infty} nV = X$.

*If $f$ satisfies the conditional equation* (4) *and the set $K = \{x \in X: f(x) = 0\}$ is a Baire set, then $f$ satisfies Cauchy's equation* (2).

Proof. We consider two cases.

I. $K$ is second category. By Lemma 1, $\mathbf{K} = (K, +)$ is a group, whence $K+K \subset K$. By Lemma 9, $K$ has a non-void interior and therefore it must contain a neighbourhood $V$ of zero. Being a group, $K$ must contain $nV$ for each positive integer $n$, whence $K = X$. Thus $f$ is identically zero and in particular satisfies equation (2).

II. $K$ is first category. Suppose that $f$ does not satisfy (2). By Lemma 5 such a solution of (4) may exist only in case (**). But, in view of Lemma 8, in case (**) we have $K' = x+K$ for arbitrary $x \in K'$. Therefore $K'$ is first category and so is $X = K \cup K'$, contrary to the supposition.

This completes the proof.

In examples 4 and 5 disciussed in Section 3 $\mathbf{X}$ may be regarded as a topological group with the obvious natural topology. Also in examples 3 and 6 we may regard $\mathbf{X}$ as endowed with the discrete topology. In all these cases condition $(\substack{**\\ *})$ is not fulfilled. In order to show that the Baire condition in Theorem 4 is essential, we shall exhibit an example in which condition $(\substack{**\\ *})$ is fulfilled, but there exist solutions of (4) which do not satisfy (2).

To this purpose we modify example 4. Instead of the topological group $(R^*, \cdot, \mathscr{I}_0)$, where $\mathscr{I}_0$ denotes the family of open sets in $R^*$ in the usual sense, we consider the topological group $(R^*, \cdot, \mathscr{I})$, where $\mathscr{I}$ denotes the family of those subsets of the set $R^* = (-\infty, 0) \cup (0, \infty)$ which belong to $\mathscr{I}_0$ and are symmetric with respect to zero. The topology $\mathscr{I}$ is rather pathological; in particular, the topological space $(R^*, \mathscr{I})$ is not Hausdorff. But $(R^*, \cdot, \mathscr{I})$ is a topological group, since the functions $xy$ and $x^{-1}$ are continuous in the topology $\mathscr{I}$.

It is not difficult to see that the closure $A^c$ of a set $A$ in the topological space $(R^*, \mathscr{I})$ is equal to $A^c = \bar{A} \cup (-\bar{A})$, where $\bar{A}$ is the closure of $A$ in the topological space $(R^*, \mathscr{I}_0)$. Hence it follows that a set $A \subset R^*$ is first category in the space $(R^*, \mathscr{I})$ if and only if it is first category in the space $(R^*, \mathscr{I}_0)$. Consequently $R^*$ is second category.

Condition $(^{**}_{*})$ is now fulfilled. But, besides the solutions satisfying equation (15), equation (14) has also solutions (16). However, in the case of those solutions the set $K = (0, \infty)$ is a second category set without the Baire property.

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