Error estimates for the finite element solutions
of variational inequalities

by M. A. Noor and K. I. Noor (Ahwaz, Iran)

Abstract. For piecewise linear approximation of variational inequalities associated with the
mildly non-linear elliptic boundary value problems having auxiliary constraint conditions, we
prove that the error estimate for \( u - u_h \) in the \( W^{1,2} \)-norm is of order \( h \), i.e., \( \|u - u_h\| = O(h) \).

In this paper, we derive the finite element error estimates for the
approximate solution of mildly non-linear elliptic boundary value problems
having auxiliary constraint conditions. A much used approach with any
elliptic is to reformulate it in a weak or variational forms. It has been shown
by Noor and Whiteman [7] that in the presence of a constraint, such
approach leads to a variational inequality which is the weak formulation. An
approximate formulation of the variational inequality is then defined, and the
error estimates involving the difference between the solution of the exact and
approximate formulation in the \( W^1_2 \)-norm is obtained, which is in fact of
order \( h \). This result is an extension of that obtained by Falk [2] and Mosco

For simplicity, we consider the problem of the following type:

\[
-\Delta u(x) = f(x, u), \quad x \in \Omega,
\]
\[
u(x) = 0, \quad x \in \partial \Omega,
\]

where \( \Omega \) is a simply connected open domain in \( \mathbb{R}^n \) with boundary \( \partial \Omega \) and its
closure \( \bar{\Omega} = \Omega \cup \partial \Omega \). It is assumed that the boundary \( \partial \Omega \) and \( f \) are smooth
enough to ensure the existence and uniqueness of the solution \( u \) of (1). We
study this problem in the usual Sobolev space \( W^1_2(\Omega) = H^1 \), the space of
functions which together with their generalized derivatives of order one are
in \( L_2(\Omega) \). The subspace of functions from \( H^1 \), which in a generalized sense
satisfy the homogenous boundary conditions on \( \partial \Omega \) is \( \widetilde{W}^1_2(\Omega) = H^1_0 \).

It has been shown by Tonti [9] that in its direct variational formulation,
(1) is equivalent to finding \( u \in H^1_0 \) such that

\[
I[u] \leq I[v] \quad \text{for all} \; v \in H^1_0,
\]
where
\[
I[v] = \int_\Omega \left( \frac{\partial v}{\partial n} \right)^2 d\Omega - 2 \int_\Omega \int_0^v f(x, \eta) d\eta d\Omega \equiv a(v, v) - 2F(v),
\]

is the energy functional associated with (1).

We here consider the case, when the solution \( u \) of (1) is required to satisfy the condition \( u \geq \psi \), where \( \psi \) is a given function on \( \Omega \). In this situation, our problem is to find
\[
u \in K \overset{\text{def}}{=} \{ v; v \in H^1_0, v \geq \psi \text{ on } \Omega \},
\]
a closed convex subset of \( H^1_0 \), see Mosco [3], such that \( u \) minimizes \( I[v] \) on \( K \). It has been shown by Noor and Whiteman [7] that the minimum of \( I[v] \) on \( K \) can be characterized by a class of variational inequalities
\[
a(u, v-u) \geq \langle F'(u), v-u \rangle \quad \text{for all } v \in K,
\]
where \( F'(u) \) is the Fréchet differential of \( F(u) \) and is, in fact,
\[
\langle F'(u), v \rangle = \int_\Omega f(u)v d\Omega.
\]

The finite dimensional form of (3) is to find \( u_h \in K_h \) such that
\[
a(u_h, v_h-u_h) \geq \langle F'(u_h), v_h-u_h \rangle \quad \text{for all } v_h \in K_h.
\]

Here \( K_h \) is a finite dimensional convex subset of \( H^1_0 \) for the construction of \( K_h \), see Mosco [3]. Let \( \Omega \) be the convex polygon. We partition it into triangles of side less than \( h \). We consider \( S_h \subset H^1_0 \), the subspace of continuous piecewise linear functions on the triangulation of \( \Omega \), vanishing on its boundary \( \partial\Omega \). Let \( \psi_h \) be the interpolant of \( \psi \) such that \( \psi_h \) agrees with \( \psi \) at all the vertices of the triangulation. For our purposes, it is enough to choose the finite dimensional convex subset \( K_h = S_h \{ v_h > \psi_h \text{ on } \Omega \} \). For other choices of convex subsets \( K_h \), see Natterer [5] and Nitsche [6], where they have chosen \( K_h = K \cap S_h \).

We also want to know the regularity of the solution \( u \in K \) satisfying (3). In this case Brezis and Stampacchia [1] have shown that if \( \psi \) lies in both \( H^1_0 \) and \( H^2 \), then the solution \( u \in K \) satisfying (3) also lies in \( H^2 \). Its norm can be estimated from the data:
\[
\|u\|_2 < e\|\psi\|_2.
\]

Moreover, if \( \bar{u} \) is the interpolant of \( u \), which agrees with \( u \) at every vertex of \( \bar{\gamma} \), then \( \bar{u} \) lies in \( K_h \). It is well known from the approximation theory, see Strang and Fix [8] that
\[
\|u - \bar{u}\| \leq Ch\|u\|_2.
\]
We also note that in certain cases, the equality holds instead of inequality in (3). This happens, when \( v \), together with \( 2u - v \) also lies in \( K \). In this case, we get

\[
(7) \quad a(u, v-u) = \langle F'(u), v-u \rangle.
\]

Finally let \( C \) and \( C_h \) be the cones composed of non-negative functions on \( H^1_0 \) and its subspace \( S_h \). Thus it is clear that

\[
U = u - \psi \quad \text{is in } C,
\]

\[
U_h = u_h - \psi_h \quad \text{is in } C_h.
\]

From these relations it follows that

\[
(8) \quad u - u_h = U - U_h + \psi - \psi_h.
\]

**Definition.** An operator \( T \) on \( H^1_0 \) is said to be quasi-monotone, if for all \( u, v, w, z \in H^1_0 \),

\[
(9) \quad \langle Tu - Tz, w - z \rangle \geq 0.
\]

We also need the following result of Mosco and Strang [4].

**Theorem 1.** Suppose that \( U > 0 \) in the plane polygon \( \Omega \) and that \( U \) lies in both \( H^1_0 \) and \( H^2 \). Then there exists a \( V_h \) in \( S_h \) such that

\[
0 \leq V_h \leq U \quad \text{in } \Omega
\]

and

\[
(10) \quad \|U - V_h\| \leq Ch\|U\|_2.
\]

Now we state and prove the main result.

**Theorem 2.** Let \( a(u, v) \) be a continuous coercive bilinear form and \( F'(u) \)
be a quasi-monotone operator on \( H^1_0 \). If \( V_h \in C_h \) and \( 2U - V_h \in C \), then

\[
\|u - u_h\| = O(h),
\]

where \( u \) and \( u_h \) are the solutions of (3) and (5) respectively.

**Proof.** Since both \( v = \psi + V_h \) and \( 2u - v = \psi + (2U - V_h) \) are in \( K \), we have from (3) and (7) that

\[
(11) \quad a(u, V_h - U) = \langle F'(u), V_h - U \rangle.
\]

Letting \( v_h = \psi_h + V_h \) and \( u_h = \psi_h + U_h \) in (5), we have

\[
(12) \quad a(u_h, V_h - U_h) \geq \langle F'(u_h), V_h - U_h \rangle.
\]

and taking \( v = \psi + U_h \) in (3), we get

\[
(13) \quad a(u, U_h - U) \geq \langle F'(u), U_h - U \rangle.
\]
From (11) and (13), we obtain
\begin{equation}
 a(u, U_h - V_h) \geq \langle F'(u), U_h - V_h \rangle.
\end{equation}
and from (12) and (14), we get
\[ a(u - u_h, U_h - V_h) \geq \langle F'(u) - F'(u_h), U_h - V_h \rangle. \]
Thus using the quasi-monotonicity of $F'(u)$, we have
\[ a(u - u_h, U_h - V_h) \geq 0, \]
which can be written as
\begin{equation}
 a(u - u_h, U - U_h) \leq a(u - u_h, U - V_h).
\end{equation}
Now by the coercivity of $a(u, v)$, it follows that there exists a constant $\varrho > 0$ such that
\[ \varrho \|u - u_h\|^2 \leq a(u - u_h, u - u_h) \]
\[ = a(u - u_h, \psi - \psi_h) + a(u - u_h, U - U_h) \quad \text{from (8)} \]
\[ \leq a(u - u_h, \psi - \psi_h) + a(u - u_h, U - V_h) \]
\[ \leq \mu \|u - u_h\| \{ \|\psi - \psi_h\| + \|U - V_h\| \}, \]
where $\mu$ is a continuity constant of the bilinear form $a(u, v)$.

Hence it follows that
\[ \|u - u_h\| \leq \frac{\mu}{\varrho} \{ \|\psi - \psi_h\| + \|U - V_h\| \} \]
\[ \leq \frac{\mu}{\varrho} \varepsilon h \{ \|\psi\|_2 + \|U\|_2 \}, \quad \text{by (6) and (10)} \]
from which the required estimate follows.

Remark. The problem of deriving the $L^p$-norm for the mildly non-linear problems having constraint conditions is still open.

Acknowledgement. The authors would like to thank Prof. Dr. J. R. Whiteman for his helpful criticism and encouragement.

References


MATHEMATICS DEPARTMENT, JUNDI SHAPUR UNIVERSITY, AHWAZ, IRAN

*Reçu par la Rédaction le 16.6.1978*