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THE FILLING SCHEME AND THE ERGODIC THEOREMS OF KESTEN AND TANNY

BY

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1. Introduction. Let $T: X \to X$ be a measure-preserving transformation of a probability space $(X, \mathscr{A}, \mu)$ and let h be a finite measurable function on X. In [12] Tanny proved that the sequence $(h \circ T^n)/n$ converges a.e. as $n \to \infty$ if it is bounded a.e. The proof uses some previous results of Tanny in the theory of branching processes in a random environment [13]. The above theorem is used in Kesten's paper [6] to obtain the following general result: if the sequence of ergodic averages $n^{-1} \sum_{k=0}^{n-1} h \circ T^k$ is bounded a.e., then it is convergent a.e. (Another proof was recently given by J. Aaronson [2].) Moreover, in [6] Kesten proved that the sequence $\sum_{k=0}^{n} h \circ T^k$ cannot tend to infinity slower than linearly, i.e., we have $\liminf_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} h \circ T^k > 0$ a.e. on the set $\{\sum_{k=0}^{n} h \circ T^k \to \infty\}$.

The aim of the present paper is to give simple proofs of the above theorems (see the Proposition and Corollaries 1 and 2) as well as some improvements and generalizations. We shall mostly consider conservative transformations T which preserve a σ-finite measure μ. The ergodic averages will be replaced by the ergodic ratios

$$D_n(h, g) = \sum_{k=0}^{n} h \circ T^k / \sum_{k=0}^{n} g \circ T^k,$$

where g is a strictly positive integrable function.

The idea of proofs is to relate the properties of the sums of iterates $\sum_{k=0}^{n} h \circ T^k$ to some properties of the filling scheme for h, especially when h is not integrable. It should be noted that the method of the filling scheme reduces in our case (where T commutes with the lattice operations) to the analysis of the partial supremum

$$H_n = \sup_{0 \leq l \leq n} \sum_{k=0}^{l} h \circ T^k.$$

However, we shall use the language of the filling scheme because of its intuitive value.

The negative part H^- of the supremum

$$H = \sup_{n \geq 0} \sum_{k=0}^{n} h \circ T^k$$

can be interpreted as the amount of the antimatter h^- which remains after the application of the filling scheme procedure to h (see Section 3). For simplicity, suppose that T is an ergodic measure-preserving transformation of a probability space and let h be a finite function. Then

$$\lim_{n \to \infty} \sum_{k=0}^{n} h \circ T^k = -\infty \text{ a.e.} \quad \text{iff} \quad \int H^- > 0$$

(cf. Theorem 1), while

$$\lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} h \circ T^k = -\infty \text{ a.e.} \quad \text{iff} \quad \int H^- = \infty$$

(cf. Theorem 3). Moreover, if $\int H^- > 0$, then $H^+ < \infty$ a.e. and $h = -H^- + H^+ - H^+ \circ T$ a.e. (cf. Theorem 1). The proof of the theorem of Tanny is then a simple application of these results (see the Proposition).

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2. Preliminaries. Throughout the paper we fix a σ-finite measure space $(X, \mathscr{A}, \mu)$ and a measurable non-singular transformation T: $X \to X$ (this means that $T^{-1}A \in \mathscr{A}$ for $A \in \mathscr{A}$, and $\mu(T^{-1}A) = 0$ if $\mu(A) = 0$).

By a *function* we mean a measurable mapping f from X to the extended real line. All inequalities and limit operations appearing in this paper are understood to be μ-almost everywhere, unless otherwise stated. For example, we write $\lim_{n \to \infty} f_n = f$ on A if $\lim_{n \to \infty} f_n(x) = f(x)$ for μ-almost all $x \in A \subseteq X$.

Throughout the paper we fix a strictly positive μ-integrable function g.

We say that T is *conservative* if for every function f with $f \circ T \geq f$ on X we have $f \circ T = f$ on X. In this definition we can restrict ourselves to the indicators $f = 1_A$ of measurable sets (see [7], Lemma). Clearly, T is conservative iff for every non-negative function f the series $\sum_{k=0}^{\infty} f \circ T^k$ can take on X only two values, 0 or ∞.

By $\mathscr{C}$ we denote the σ-field of invariant sets ($A \in \mathscr{C}$ iff $1_A \circ T = 1_A$ on X).

If $\mu(A) = 0$ or $\mu(A^c) = 0$ for every $A \in \mathscr{C}$, then T is said to be *ergodic*. If $A \in \mathscr{A}$, we set

$$A^* = \bigcup_{k=0}^{\infty} T^{-k} A.$$

If the restriction of μ to $\mathscr{C}$ is σ-finite and if h is an arbitrary non-negative function, we denote by $\mathrm{E}_\mu[h \mid \mathscr{C}]$ the conditional expectation of h with respect to $\mathscr{C}$ and to the measure μ.

We say that T is *measure-preserving* if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathscr{A}$. If T is conservative and measure-preserving, then the operator $h \to h \circ T$ is a positive conservative contraction on $L_1(\mu)$ (in the terminology of [10]). The following theorem is well known (see [10], Chapter 4, Theorem 3.2, for h integrable, and Chapter 4, Exercise 3.8, for arbitrary h).

HOPF ERGODIC THEOREM. *Suppose that T is conservative and measure-preserving and let h be an arbitrary non-negative function. Then the ergodic ratios*

$$D_n(h, g) = \sum_{k=0}^{n} h \circ T^k / \sum_{k=0}^{n} g \circ T^k$$

converge on X (as $n \to \infty$) to

$$\mathrm{E}^1[h \mid \mathscr{C}] = \mathrm{E}_\mu[h \mid \mathscr{C}]/\mathrm{E}_\mu[g \mid \mathscr{C}]. \tag{2.1}$$

When the restriction of μ to $\mathscr{C}$ is not σ-finite, the right-hand side of (2.1) is understood as

$$\mathrm{E}_{f\mu}[h/f \mid \mathscr{C}]/\mathrm{E}_{f\mu}[g/f \mid \mathscr{C}],$$

where f is an arbitrary strictly positive μ-integrable function on X.

Clearly, the above ratio is well defined without the additional assumptions on T. If T is ergodic, then simply $\mathrm{E}^1[h \mid \mathscr{C}] = \int h / \int g$.

We shall need the following supplement to the above theorem:

LEMMA 1. *Assume that T is conservative. Let h be an arbitrary non-negative function and let $A \in \mathscr{A}$. Then $\mathrm{E}^1[h \mid \mathscr{C}] = 0$ on A iff*

$$\sum_{k=0}^{\infty} h \circ T^k = 0 \quad \textit{on } A.$$

Proof. The function $\mathrm{E}^1[h \mid \mathscr{C}]$ is always invariant, while $\sum_{k=0}^{\infty} h \circ T^k$ is invariant since T is conservative. Consequently, it suffices to consider the case where $A = X$. However, $\mathrm{E}^1[h \mid \mathscr{C}] = 0$ on X iff $h = 0$ on X iff

$$\sum_{k=0}^{\infty} h \circ T^k = 0 \quad \text{on } X,$$

which completes the proof.

We shall frequently use also the following

LEMMA 2. *If T is conservative and if h is an arbitrary function, then*

$$\liminf_{n\to\infty} h\circ T^n \leqslant h \quad \text{on } X.$$

If T is additionally ergodic, then

$$\liminf_{n\to\infty} h\circ T^n = \operatorname{ess\,inf} h.$$

Proof. Let

$$f_n = \inf_{k\geqslant n} h\circ T^k \quad \text{for } n = 0, 1, \ldots$$

Then $f_n \leqslant f_{n+1} = f_n\circ T$ and, since T is conservative, $f_n = f_0$ on X for $n = 1, 2, \ldots$ Consequently,

$$\liminf_n h\circ T^n = \sup_n f_n = f_0 \leqslant h \quad \text{on } X.$$

If T is ergodic, then $f_0 = f_0\circ T$ is a constant lying between h and $\operatorname{ess\,inf} h$. Hence $f_0 = \operatorname{ess\,inf} h$.

3. The filling scheme. The filling scheme associates to a finite function h the sequence $(h_n)_{n\geqslant 0}$ which is defined inductively by

$$h_0 = h, \quad h_{n+1} = h_n^+\circ T - h_n^- \ (n = 0, 1, \ldots).$$

This sequence has the following intuitive interpretation. Let h^+ be a density (with respect to the measure μ) of a matter and let h^- be a density of an antimatter. In the first step, the matter h^+ is transported by T and is absorbed by the antimatter h^- so that we are left with the matter $h_1^+ = (h_0^+\circ T - h_0^-)^+$ and the antimatter $h_1^- = (h_0^- - h_0^+\circ T)^+$. In the next step we transport the remaining matter h_1^+ by T and after an absorption we obtain the matter h_2^+ and the antimatter h_2^-. This procedure is continued indefinitely.

The filling scheme may also be defined alternatively in the following way. Let h be a finite function and let the sequence $(H_n)_{n\geqslant 0}$ be defined inductively by

$$H_0 = h, \quad H_{n+1} = h + H_n^+\circ T \ (n = 0, 1, \ldots). \tag{3.1}$$

This sequence is increasing and related to the sequence h_n by the following formulas: $h_n = H_n - H_{n-1}^+$, $h_n^+ = H_n^+ - H_{n-1}^+$, and $h_n^- = H_n^-$ $(n = 0, 1, \ldots)$, where $H_{-1}^+ = 0$ (see [8], Lemma 2). Let us put

$$H = \lim_n\uparrow H_n.$$

Then

$$H^-(x) = \lim_n\downarrow H_n^-(x) = \lim_n\downarrow h_n^-(x)$$

represents the amount of antimatter which remains at the point x, and

$$H^+(x) = \lim_n \uparrow H_n^+(x) = \sum_{n=0}^{\infty} h_n^+(x)$$

represents the total amount of matter which passes over x during the filling scheme procedure. (Such an interpretation is correct if at each step the matter h_n^+ is thrown into the air and then falls according to the distribution of $h_n^+ \circ T$.)

Using (3.1) we can always write $H = h + H^+ \circ T$ and, if $H^+(x)$ is finite, the function h has the following representation:

$$h(x) = -H^-(x) + H^+(x) - H^+(Tx). \tag{3.2}$$

Since h and H^- are finite ($0 \leqslant H^- \leqslant h^-$), it follows from (3.2) that the set $A = \{x \in X \colon H^+(x) < \infty\}$ has the property $A \subseteq T^{-1}A$. Therefore we obtain

LEMMA 3. *If T is conservative, then the set $\{H^+ < \infty\}$ is invariant.*

Note also that if $h \leqslant v - v \circ T$ for a finite non-negative function v, then $H^+ \leqslant v$ (cf. [11], Lemma 5).

Finally, since the operator $h \to h \circ T$ commutes with the lattice operations, it follows from (3.1) that

$$H_n = \max_{0 \leqslant l \leqslant n} \sum_{k=0}^{l} h \circ T^k$$

and

$$H = \sup_{n \geqslant 0} \sum_{k=0}^{n} h \circ T^k. \tag{3.3}$$

4. Pointwise behaviour of sums of iterates. In this section we shall investigate various conditions equivalent to the fact that the sequence $\sum_{k=0}^{n} h \circ T^k$ converges to infinity on A (Theorem 1) or is bounded on A (Theorem 2).

THEOREM 1. *Let h be a finite function and let $A \in \mathscr{A}$. Consider the following conditions:*

(i) $\lim_{n \to \infty} \sum_{k=0}^{n} h \circ T^k = -\infty$ *on* A;

(ii) $\limsup_{n \to \infty} \sum_{k=0}^{n} h \circ T^k < 0$ *on* A;

(iii) $\mathrm{E}^1[H^- \mid \mathscr{C}] > 0$ *on* A;

(iv) *there exist a finite non-negative function u with* $\mathrm{E}^1[u \mid \mathscr{C}] > 0$ *on A and a finite non-negative function v such that $h = -u + v - v \circ T$ on A^**;

(v) $\limsup_{n \to \infty} D_n(h, g) < 0$ *on* A.

If T is conservative, then conditions (i)–(iv) *are equivalent and are implied by* (v). *If T is conservative and measure-preserving, then all the above conditions are equivalent.*

Moreover, if T is conservative and if one of the above conditions holds, then

(vi) $H^+ < \infty$ *on* A^* *and* $h = -H^- + H^+ - H^+ \circ T$ *on* A^*.

Proof. Assume that T is conservative. The implications (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (i) are obvious.

(ii) $\Rightarrow$ (iii). If (ii) holds, then H^+ is finite on A in view of (3.3). Thus it is finite on A^* in virtue of Lemma 3. Consequently, $h = -H^- + H^+ - H^+ \circ T$ on A^* by (3.2). Applying Lemma 2 we obtain

$$\limsup_n \sum_{k=0}^{n} h \circ T^k = -\sum_{k=0}^{\infty} H^- \circ T^k + H^+ - \liminf_n H^+ \circ T^{n+1}$$
$$\geqslant -\sum_{k=0}^{\infty} H^- \circ T^k \quad \text{on } A^*.$$

Using once more the assumption (ii) we infer that

$$\sum_{k=0}^{\infty} H^- \circ T^k > 0 \quad \text{on } A, \tag{4.1}$$

which is equivalent to (iii) by Lemma 1.

(iii) $\Rightarrow$ (iv) and (vi). If (iii) holds, then (4.1) is satisfied by Lemma 1. Hence $A \subseteq B^*$, where $B = \{H^- > 0\}$. Next, $H^+ = 0$ on B and, by Lemma 3, $H^+ < \infty$ on B^*. Consequently, $H^+ < \infty$ on A^* and, by (3.2), $h = -H^- + H^+ - H^+ \circ T$ on A^*.

(iv) $\Rightarrow$ (i). Assume that (iv) is valid. Then

$$\sum_{k=0}^{\infty} u \circ T^k = \infty \quad \text{on } A$$

by Lemma 1, and

$$\sum_{k=1}^{n} h \circ T^k \leqslant -\sum_{k=0}^{n} u \circ T^k + v \quad \text{on } A^*,$$

which yields (i).

Finally, if T is conservative measure-preserving and if (iv) holds, then

$$D_n(h, g) \leqslant -D_n(u, g) + \Big(v / \sum_{k=0}^{n} g \circ T^k\Big) \tag{4.2}$$

on A^*. By the Hopf ergodic theorem, $D_n(u, g)$ converges on X to $\mathrm{E}^1[u \mid \mathscr{C}]$, which is assumed to be positive on A, and $(v / \sum_{k=0}^{n} g \circ T^k)$ converges to zero on X. Therefore we have (v), and the proof of the theorem is completed.

The following corollary implies the result of Kesten ([6], Theorem) that for a measure-preserving transformation T of a probability space and for a finite function h

$$\liminf_{n\to\infty} n^{-1}\sum_{k=0}^{n-1} h\circ T^k > 0 \quad \text{on } \{\lim_{n\to\infty}\sum_{k=0}^{n} h\circ T^k = \infty\}.$$

COROLLARY 1. *If T is a measure-preserving transformation of a probability space and if h is a finite function, then*

$$\liminf_{n\to\infty} n^{-1}\sum_{k=0}^{n-1} h\circ T^k > 0 \quad \textit{on } \{\liminf_{n\to\infty}\sum_{k=0}^{n} h\circ T^k > 0\}.$$

Remark 1. If T is conservative, ergodic and measure-preserving and if $h\in L_1$, the implication (iii) $\Rightarrow$ (vi) in Theorem 1 follows also from a result of Neveu ([8], Proposition 3).

The next theorem is similar to a result in topological dynamics (see [4], Theorem 14.11, (3) and (4)).

THEOREM 2. *Assume that T is conservative and ergodic. Let h be a finite function and let $A\in\mathscr{A}$, $\mu(A)>0$. Then the following conditions are equivalent*:

(i) $\sup_{n\geq 0}|\sum_{k=0}^{n} h\circ T^k| < \infty$ *on* A;

(ii) $(\sup_{n\geq 0}|\sum_{k=0}^{n} h\circ T^k|)\in L_\infty(\mu)$;

(iii) *there exists a non-negative function $v\in L_\infty(\mu)$ such that $h = v - v\circ T$ on X.*

Proof. Since the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious, we prove only (i) $\Rightarrow$ (iii). Clearly, we may assume that $A = X$. It follows from (i) that

$$\limsup_{n}\sum_{k=0}^{n} h\circ T^k > -\infty \quad \text{on } X.$$

Consequently, $H^- = 0$ on X by Theorem 1. On the other hand, it follows from (i) and (3.3) that H^+ is finite on X. Hence $h = H^+ - H^+\circ T$ on X by (3.2). Now, by Lemma 2,

$$\liminf_{n}\sum_{k=0}^{n} h\circ T^k = H^+ - \limsup_{n} H^+\circ T^{n+1} = H^+ - \operatorname{ess\,sup} H^+.$$

Using once more the assumption (i) we infer that $H^+\in L_\infty$, which gives (iii) and completes the proof.

5. Pointwise behaviour of ergodic ratios. In this section we shall exhibit some conditions equivalent to the fact that the sequence $D_n(h, g)$ converges to infinity on A (Theorem 3) or is bounded on A (Theorem 4).

Let h be a finite function and let

$$f = \limsup_{n\to\infty} D_n(h, g).$$

If T is conservative, then

$$f\circ T = \limsup_{n\to\infty}\Big(\big(\sum_{k=0}^{n+1} h\circ T^k - h\big)/\sum_{k=0}^{n+1} g\circ T^k\Big)\Big(\sum_{k=0}^{n+1} g\circ T^k/\sum_{k=1}^{n+1} g\circ T^k\Big)$$
$$= \limsup_{n\to\infty} D_{n+1}(h, g) = f \quad \text{on } X,$$

i.e., f is invariant on X.

THEOREM 3. *Assume that T is conservative and measure-preserving. Let h be a finite function and let $A\in\mathscr{A}$. Then the following conditions are equivalent*:

(i) $\lim_{n\to\infty} D_n(h, g) = -\infty$ *on* A;

(ii) $\mathrm{E}^1[H^-\,|\,\mathscr{C}] = \infty$ *on* A;

(iii) *there exist a finite non-negative function u with $\mathrm{E}^1[u\,|\,\mathscr{C}] = \infty$ on A and a finite non-negative function v such that $h = -u+v-v\circ T$ on A^*.*

Proof. Clearly, we may assume that $A = A^* = X$. If (i) holds, then $H^+ < \infty$ on X and, by Theorem 1, $h = -H^- + H^+ - H^+\circ T$ on X. Hence

$$D_n(h, g) = -D_n(H^-, g) + \Big(H^+/\sum_{k=0}^{n} g\circ T^k\Big) - \Big(H^+\circ T^{n+1}/\sum_{k=0}^{n} g\circ T^k\Big)$$

on X. By the Hopf ergodic theorem,

$$\lim_n D_n(H^-, g) = \mathrm{E}^1[H^-\,|\,\mathscr{C}] \quad \text{on } X.$$

Next,

$$\lim_n \Big(H^+/\sum_{k=0}^{n} g\circ T^k\Big) = 0 \quad \text{on } X$$

and

$$\liminf_n \Big(H^+\circ T^{n+1}/\sum_{k=0}^{n} g\circ T^k\Big) = 0 \quad \text{on } X$$

by Lemma 2. Thus, using once more the assumption (i) we obtain (ii).

Clearly, (ii) $\Rightarrow$ (iii) by Theorem 1. Finally, if (iii) is valid, then, by (4.2) and the Hopf ergodic theorem,

$$\limsup_n D_n(h, g) \leqslant -\mathrm{E}^1[u\,|\,\mathscr{C}] = -\infty,$$

which yields (i) and completes the proof.

The following result was essentially proved by Tanny [12] for a measure-preserving transformation T of a probability space.

PROPOSITION. *Assume that T is conservative and measure-preserving. Let v be a finite non-negative function and let $A \in \mathscr{A}$. Then the following conditions are equivalent*:

(i) $\sup_{n\geqslant 0}(v\circ T^{n+1}/\sum_{k=0}^{n} g\circ T^k) < \infty$ *on* A;

(ii) $\lim_{n\to\infty}(v\circ T^{n+1}/\sum_{k=0}^{n} g\circ T^k) = 0$ *on* A;

(iii) *there exists a finite function w such that $w \geqslant v$ on X and we have* $\mathrm{E}^1[|w\circ T - w| \,|\mathscr{C}] < \infty$ *on* A.

Proof. Since the function

$$f = \limsup_n D_n(v\circ T - v, g) = \limsup_n (v\circ T^{n+1}/\sum_{k=0}^{n} g\circ T^k)$$

is invariant, we may assume that $A = A^* = X$.

(i)$\Rightarrow$(iii). Let $A_n = \{N-1 \leqslant f < N\}$ for $N = 1, 2, \ldots$ The sets A_N are invariant and pairwise disjoint. Since, by (i), f is finite and non-negative on X, we have

$$\bigcup_{N=1}^{\infty} A_N = X \quad \text{on } X.$$

Put

$${}_Nh = v\circ T - v - Ng \quad \text{and} \quad {}_NH = \sup_{n\geqslant 0}\sum_{k=0}^{n} {}_Nh\circ T^k.$$

Then

$$D_n({}_Nh, g) = (v\circ T^{n+1}/\sum_{k=0}^{n} g\circ T^k) - (v/\sum_{k=0}^{n} g\circ T^k) - N.$$

It follows from Lemma 2 that

$$-\infty < -N = \liminf_n D_n({}_Nh, g) \quad \text{on } X, \tag{5.1}$$

and the definition of A_N implies

$$\limsup_n D_N({}_Nh, g) < 0 \quad \text{on } A_N. \tag{5.2}$$

Now, by (5.1) and Theorem 3 we have

$$\mathrm{E}^1[{}_NH^- \,|\mathscr{C}] < \infty \quad \text{on } X, \tag{5.3}$$

while (5.2) and Theorem 1 imply that ${}_NH^+ < \infty$ on A_N and ${}_Nh = -{}_NH^- + {}_NH^+ - {}_NH^+\circ T$ on A_N. Thus

$$(v + {}_NH^+)\circ T - (v + {}_NH^+) = Ng - {}_NH^- \quad \text{on } A_N$$

by the definition of ${}_Nh$. Since $g \in L_1$, we infer from (5.3) that

$$\mathrm{E}^1\,[|(v+{}_NH^+)\circ T-(v+{}_NH^+)|\;|\mathscr{C}] < \infty \quad \text{on } A_N.$$

Now, it is clear that the function

$$w = \sum_{N=1}^{\infty} (v+{}_NH^+)\,1_{A_N}$$

has the desired properties.

(iii) $\Rightarrow$ (ii). If (iii) holds, then the limit

$$\lim_n \Big(w\circ T^{n+1}/\sum_{k=0}^{n} g\circ T^k\Big) = \lim_n D_n(w\circ T-w,\, g)$$

exists on X by the Hopf ergodic theorem. By Lemma 2 this limit is equal to zero on X. Since $0 \leqslant v \leqslant w$, the assertion (ii) is obvious.

Finally, the implication (ii) $\Rightarrow$ (i) is trivial.

If $A \in \mathscr{A}$, we denote by $\mathscr{T}_A$ the vector lattice consisting of all finite functions v such that

$$\sup_{n\geqslant 0}\Big(|v|\circ T^{n+1}/\sum_{k=0}^{n} g\circ T^k\Big) < \infty \quad \text{on } A.$$

Remark 2. The implication (i) $\Rightarrow$ (ii) of the Proposition was obtained by Tanny ([12], Corollary 5) for a measure-preserving transformation T of a probability space. The proof given by Tanny goes also through the condition (iii). Namely, in the notation of [12] and [13], the function $w \geqslant v$ satisfying $w\circ T-w \in L_1$ (T is assumed to be ergodic) is of the form $w = -\log\big(1-q(\xi)\big)$, while v has a representation $v = -\log\big(1-p_0(\xi_0)\big)$ (see [13], the proof of Proposition 5.7, and [12], the proof of Theorem 1).

We give an example, based on an example in [1], p. 172 (which in turn is adapted from [12]), showing that in general the function w in the Proposition cannot be taken to be equal to v.

Example. Let τ be an ergodic measure-preserving transformation of a probability space $(\Omega, \mathfrak{S}, P)$ and let $\varphi\colon \Omega \to N$ be such that $\int \varphi\, dP < \infty$ and $\int \varphi^2\, dP = \infty$. Let $(X, \mathscr{A}, \mu, T)$ be a tower on $(\Omega, \mathfrak{S}, P, \tau)$ with φ as the height function, i.e.,

$$X = \{(\omega, n)\colon\, n \geqslant 1,\ \varphi(\omega) \geqslant n\}, \qquad \mathscr{A} = \bigvee_{n=1}^{\infty} (\mathfrak{S} \cap \{\varphi \geqslant n\},\, n),$$

$$\mu = (1/C) \sum_{n=1}^{\infty} P|(\mathfrak{S} \cap \{\varphi \geqslant n\},\, n), \qquad \text{where } C = \int \varphi\, dP,$$

and

$$T(\omega, n) = \begin{cases} (\omega, n+1) & \text{if } \varphi(\omega) \geqslant n+1, \\ (\tau\omega, 1) & \text{if } \varphi(\omega) = n. \end{cases}$$

Then T is an ergodic measure-preserving transformation of the probability space $(X, \mathscr{A}, \mu)$ (see [5]). Let $w(\omega, n) = n$ and let

$$v(\omega, n) = \begin{cases} n & \text{if } l(\varphi(\omega)) = l(n), \\ 0 & \text{otherwise,} \end{cases}$$

where $l(k) = 1$ if k is odd, and $l(k) = 0$ if k is even. Clearly, $0 \leqslant v \leqslant w$. An easy calculation shows that $v \circ T - v \notin L_1$, $w \notin L_1$, but $w \circ T - w \in L_1$. Thus, both v and w belong to $\mathscr{T}_X$.

Note also that if T is an ergodic measure-preserving transformation of a probability space and if f is a non-negative integrable function, then the maximal function

$$f^* = \sup_{n \geqslant 1} n^{-1} \sum_{k=0}^{n-1} f \circ T^k$$

always belongs to $\mathscr{T}_X$. Indeed, it suffices to check this for $g = 1$, in which case

$$\sup_{n \geqslant 0} \Big(f^* \circ T^{n+1} / \sum_{k=0}^{n} g \circ T^k\Big) = \sup_{n \geqslant 0} \big(f^* \circ T^{n+1}/(n+1)\big) \leqslant 2f^*.$$

However, by Ornstein's theorem (see [9]), $f^* \in L_1$ iff $f \in L \log L$.

The implication (i) $\Rightarrow$ (ii) in the Proposition is a particular case of the following

THEOREM 4. *Assume that T is conservative and measure-preserving. Let h be a finite function and let $A \in \mathscr{A}$. Then the following conditions are equivalent*:

(i) $\sup_{n \geqslant 0} |D_n(h, g)| < \infty$ *on* A;

(ii) $\lim_{n \to \infty} D_n(h, g)$ *exists and is finite on* A;

(iii) *there exist a function u with* $\mathrm{E}^1 [|u| \mid \mathscr{C}] < \infty$ *on A and a non-negative function $v \in \mathscr{T}_A$ such that $h = u + v - v \circ T$ on A^*.*

Proof. We may assume that $A = A^* = X$. Since the implication (ii) $\Rightarrow$ (i) is trivial, and (iii) $\Rightarrow$ (ii) by the Hopf ergodic theorem and by the Proposition, we prove only (i) $\Rightarrow$ (iii). Let $A_1 = \{f < 1\}$ and let $A_N = \{N-1 \leqslant f < N\}$ for $N = 2, 3, \ldots$, where $f = \limsup_n D_n(h, g)$. The sets A_N are invariant, pairwise disjoint, and

$$\bigcup_{N=1}^{\infty} A_N = X \quad \text{on } X$$

by the assumption (i). Put

$${}_N h = h - Ng \quad \text{and} \quad {}_N H = \sup_{n \geqslant 0} \sum_{k=0}^{n} {}_N h \circ T^k.$$

It follows from (i) that

$$-\infty < \liminf_n D_n({}_N h, g) \quad \text{on } X, \tag{5.4}$$

and the definition of A_N implies

$$\limsup_n D_n({}_N h, g) < 0 \quad \text{on } A_N. \tag{5.5}$$

Now, by (5.4) and Theorem 3 we have

$$\mathrm{E}^1[{}_N H^- \mid \mathscr{C}] < \infty \quad \text{on } X, \tag{5.6}$$

while (5.5) and Theorem 1 imply ${}_N H^+ < \infty$ on A_N and ${}_N h = -{}_N H^- + {}_N H^+ - {}_N H^+ \circ T$ on A_N. Thus

$$h = Ng - {}_N H^- + {}_N H^+ - {}_N H^+ \circ T \quad \text{on } A_N \tag{5.7}$$

and, since $g \in L_1$,

$$\mathrm{E}^1[|Ng - {}_N H^-| \mid \mathscr{C}] < \infty \quad \text{on } X \tag{5.8}$$

by (5.6). It follows from (5.7) that

$${}_N H^+ \circ T^{n+1} / \sum_{k=0}^{n} g \circ T^k \leqslant |D_n(h, g)| + D_n({}_N H^-, g) + \left({}_N H^+ / \sum_{k=0}^{n} g \circ T^k\right) + N$$

on A_N. Thus, using (i), (5.6), and the Hopf ergodic theorem we infer that

$${}_N H^+ \in \mathscr{T}_{A_N}. \tag{5.9}$$

Now, it follows from (5.7)–(5.9) that the functions

$$u = \sum_{N=1}^{\infty} (Ng - {}_N H^-) 1_{A_N} \quad \text{and} \quad v = \sum_{N=1}^{\infty} {}_N H^+ 1_{A_N}$$

have the desired properties.

The following corollary was proved in [6], p. 211.

COROLLARY 2. *If T is a measure-preserving transformation of a probability space and if h is a finite function, then the limit*

$$\lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} h \circ T^k$$

exists on the set

$$\left\{\sup_{n \geqslant 1} \left|n^{-1} \sum_{k=0}^{n-1} h \circ T^k\right| < \infty\right\}.$$

Finally, we shall describe the functions h appearing in Theorem 4 in terms of $\mathscr{T}_X$. For simplicity, we consider only the ergodic case.

COROLLARY 3. *Suppose that T is conservative, ergodic, and measure-*

preserving. Let h be a finite function. Then the following conditions are equivalent:

(i) $\lim_{n\to\infty} D_n(h, g) = 0$ *on* X;

(ii) *there exists an increasing sequence* $(v_n)_{n\geqslant 0}$ *of non-negative elements of* $\mathscr{T}_X$ *such that*

$$\lim_{n\to\infty} \int_X |h-(v_n-v_n\circ T)|\, d\mu = 0.$$

Proof. (i) $\Rightarrow$ (ii) (cf. also [3], Theorem 2.4, for integrable h). Suppose that (i) holds. By Theorem 4 we have $h = u+v-v\circ T$, where $u\in L_1$, $\int u = 0$, and $v\in \mathscr{T}_X$, $v\geqslant 0$. Let h_n and H_n ($n\geqslant 0$) be the sequences defined in Section 3, but associated with u. Then $v_n = H_n^+ + v$ is an increasing sequence of non-negative elements of $\mathscr{T}_X$ and

$$h-(v_n-v_n\circ T) = u-(H_n^+ - H_n^+\circ T) = h_{n+1}^+ - h_{n+1}^-.$$

By the assumptions on T and on u, and by the results of Rost (see [8], Theorem 4),

$$\lim_n\downarrow \int h_{n+1}^+ = 0 \quad \text{and} \quad \lim_n\downarrow \int h_{n+1}^- = 0,$$

which yields (ii).

(ii) $\Rightarrow$ (i). By (ii), there exists an N_1 such that $z_N = h-(v_N-v_N\circ T)$ is integrable for $N\geqslant N_1$. Now,

$$D_n(h, g) = D_n(z_N, g)+\Big(v_N/\sum_{k=0}^{n} g\circ T^k\Big)-\Big(v_N\circ T^{n+1}/\sum_{k=0}^{n} g\circ T^k\Big).$$

By the Hopf ergodic theorem and by the Proposition,

$$\lim_n D_n(h, g) = \int z_N/\int g \quad \text{on } X \text{ for } N\geqslant N_1.$$

This yields (i), since $\left|\int z_N\right| \leqslant \int |z_N| \to 0$ as $N\to\infty$ by hypothesis.

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