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The polygonal method of solving the differential equation $y' = h(t, y, y, y')$

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§ 1. The purpose of the following considerations is a strictly Eulerian approach to the problem of solving the differential equation

$$y' = h(t, y, y, y')\,, \tag{1,1}$$

the right-hand member containing the derivative of an unknown function.

This equation will be solved in a complete Banach space with a homogeneous norm, since then the case of n differential equations for n real functions can be treated without complicated calculations.

The polygonal method presented here does not involve the solving of (1,1) with respect to the derivative y', and can be described in the following manner:

Let

$$t_j = t_j(n) = j\cdot\frac{\delta}{n} \quad (j = 0, 1, 2, \ldots) \tag{1,2}$$

be a sequence of equally spaced points t_j in the interval I': $0 \leqslant t < a$, n being a positive integer and $0 < \delta = \text{const}$. We shall define successively rectilinear segments of the polygonal line $y_n(t)$ in the intervals

$$\Delta_j\colon \quad t_{j-1} \leqslant t < t_j \quad (j = 1, 2, 3, \ldots)\,. \tag{1,3}$$

First we define a slope $y_n'(0)$ at the initial point $t = 0$. To this end we assume that the right-hand member $h(t, u, v, w)$ of (1,1) is defined in some domain (cf. § 3, assumptions H), we compute d_{n+1} (the slope) with the aid of the formula

$$d_{n+1} = h(0, 0, 0, d_n)\,, \quad d_1 = 0 \quad (n = 1, 2, 3, \ldots) \tag{1,4}$$

and we define the first rectilinear segment for $t \in \Delta_1(n)$ by the differential equation

$$y_n'(t) = h(0, 0, 0, d_n) \quad \text{for} \quad t \in \Delta_1(n)\,, \quad y_n(0) = 0\,, \tag{1,5}$$

with a constant right-hand member.

Suppose that the segment $y_n(t)$ in the interval $\Delta_j(n)$ (j—fixed, $j \geqslant 1$) is given. Then we define the segment $y_n(t)$ in the next interval $\Delta_{j+1}(n)$ as a solution of the differential equation with a constant right-hand member

$$(1,6) \qquad y'_n(t) = h\big(t_j, y_n(t_j), y_n(t_{j-1}), y'_n(t_{j-1})\big) \quad \text{for} \quad t \in \Delta_{j+1}(n),$$

joining the segment $y_n(t)$, $t \in \Delta_{j+1}(n)$, with the preceding one so as to obtain the continuous function $y_n(t)$ for $0 \leqslant t < t_{j+1}$.

In the subsequent intervals (1,3) we proceed in a similar manner. Accordingly, the value $y_n(t)$ in the next interval $\Delta_{j+1}(n)$ is always determined by the values $y_n(t)$ and $y'_n(t)$ at two points of the preceding interval $\Delta_j(n)$.

So we obtain the sequence of polygonal lines $y_n(t)$, as n tend to infinity, the slopes at the initial point $t = 0$ being defined in a special way (cf. (1,4) and (1,5)).

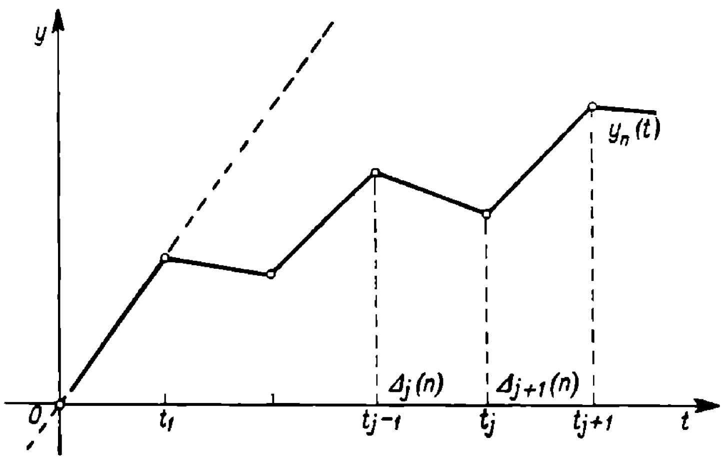

Fig. 1. The construction of the rectilinear approximation $y_n(t)$

Hence this is a difference method of the Eulerian type. Because of its connection it may be regarded as a consequence of another difference method (that of curvilinear polygons $x_n(t)$), treated previously (cf. Z. Kowalski [3]), where the values of $x_n(t)$ were determined by known values at all (∞ many) points of the preceding interval $\Delta_j(n)$. This can be seen by connecting (1,4), (1,5) and (1,6) with the corresponding formula

$$(1,7) \qquad d_{n+1} = h(0, 0, 0, d_n), \quad d_1 = 0 \quad (n = 1, 2, 3, \dots),$$

$$(1,8) \qquad x'_n(t) = h\big(t, x_n(t), 0, d_n\big) \quad \text{for} \quad t \in \Delta_1(n),$$

$$(1,9) \qquad x'_n(t) = h\left(t, x_n(t), x_n\left(t - \frac{\delta}{n}\right), x'_n\left(t - \frac{\delta}{n}\right)\right) \quad \text{for} \quad \frac{\delta}{n} \leqslant t < a,$$

dealt with in the method of curvilinear polygons (cf. [3]).

In this connection the rectilinear polygons $y_n(t)$ can be recognized as an approximation to curvilinear polygons $x_n(t)$.

The rectilinear approximations $y_n(t)$ may deviate to a greater extent from the solution of (1,1) than the curvilinear approximations $x_n(t)$, nothwithstanding the fact that they may be of practical interest in numerical computations.

It is worth noting that equation (1,1) can be solved with the aid of a special iterative method, which also dispenses with the solution of (1,1) with respect to the derivative y', the successive approximations $u_n(t)$ being defined by

$$(1,10) \qquad u_n'(t) = h\big(t, u_n(t), u_{n-1}(t), u_{n-1}'(t)\big)\,, \qquad u_0(t) \equiv 0\,,$$

for $0 \leqslant t < a$ (cf. Z. Kowalski [2]).

In this paper we solve the problem of location of the functions $y_n(t)$, $y_n'(t)$ and of the solution, we evaluate the length of the interval in question and prove the almost uniform convergence to the unique solution of (1,1). In addition, we derive two error estimates for $y_n(t)$, the second with the aid of the values $\|y_n(t_j)-y_n(t_{j-1})\|$ and $\|y_n'(t_j)-y_n'(t_{j-1})\|$, the number of points t_j being finite.

Also in that case the most difficult situation arises when the uniform convergence of the (right) derivatives of $y_n(t)$ is to be proved. However it could be proved (cf. the first lines of the proof of theorem 2), with the aid of precise estimations obtained by means of theorems on differential inequalities.

Moreover in that case the method of differential inequalities leads to non-local theorems.

§ 2. We shall make use of the following well-known theorems (cf. T. Ważewski [6]):

THEOREM A. *Suppose that the real-valued function* $l(t), l(t_0) = 0$, *continuous in the interval* J,

$$J:\quad t_0 \leqslant t < a \qquad (a \leqslant +\infty)\,,$$

possesses the right upper derivative $\bar{D}_+ l(t)$ *and satisfies the inequality*

$$(2,1) \qquad \bar{D}_+ l(t) \leqslant F\big(t, l(t)\big) \quad \text{for} \quad t \in J\,,$$

where $F(t,z)$ *is a real-valued function of real variables* (t,z), *continuous for* $t_0 \leqslant t < a$, $-\infty < z < +\infty$.

Suppose that the greatest solution $z(t)$ *of the equation*

$$(2,2) \qquad z' = F(t,z)\,,$$

satisfying the condition

$$(2,3)\qquad l(t_0) \leqslant z(t_0)\,,$$

exists in the interval J.

Under these assumptions

$$(2,4)\qquad l(t) \leqslant z(t) \quad \textit{for} \quad t \in J\,.$$

Remark 1. If $l(t)$ satisfies the inequality $\bar{D}_+ l(t) \geqslant F\big(t, l(t)\big)$ for $t \in J$, $z(t)$ is the lower integral of equation (2,2) for $t \in J$, and $l(t_0) \geqslant z(t_0)$, then $l(t) \geqslant z(t)$ for $t \in J$.

THEOREM B. *Suppose that a function* $\varphi(t)$ *defined for* $t \in J$, J: $t_0 \leqslant t < a$, *with values in a complete Banach space with a homogeneous norm, possesses a derivative* $\varphi'(t)$ *for* $t \in J$.

Under these assumptions

$$(2,5)\qquad \bar{D}_+\|\varphi(t)\| \leqslant \|\varphi'(t)\| \quad \textit{for} \quad t \in J\,.$$

(Cf. T. Ważewski [4]).

§ 3. Throughout the rest of the paper we shall use the following assumptions H:

ASSUMPTIONS H. 1) Assume that the function $h(t, u, v, w)$ is defined and continuous for $(t, u, v, w) \in \omega$, where

$$(3,1)\qquad \omega\colon\quad 0 \leqslant t < a\,,\quad \|u\| < b\,,\quad \|v\| < b\,,\quad \|w\| < c$$
$$(a \leqslant +\infty\,,\; b \leqslant +\infty\,,\; c \leqslant +\infty)\,.$$

2) The values of $h(t, u, v, w)$ are in the Banach space B with a homogeneous norm:

$$h(t, u, v, w) \in B \quad \text{for} \quad (t, u, v, w) \in \omega\,.$$

3) The Lipschitz condition

$$(3,2)\qquad \|h(\bar{t}, \bar{u}, \bar{v}, \bar{w}) - h(t, u, v, w)\|$$
$$\leqslant K\cdot|\bar{t}-t| + M\cdot\|\bar{u}-u\| + N\cdot\|\bar{v}-v\| + L\cdot\|\bar{w}-w\|$$

holds for $(\bar{t}, \bar{u}, \bar{v}, \bar{w}) \in \omega$, $(t, u, v, w) \in \omega$, with arbitrary constants K, M, N, and the constant L satisfying

$$(3,3)\qquad 0 \leqslant L < 1\,.$$

We suppose also that

$$(3,4)\qquad \|h(0, 0, 0, 0)\| \leqslant P \quad (P < +\infty)\,,$$

where the constant P satisfies the condition

$$(3,5)\qquad P < \tfrac{1}{2}c\cdot(1-L)\,.$$

Denote by $s(t)$ the solution of the non-homogeneous linear differential equation

$$s'(t) = \frac{M+N}{1-L} \cdot s(t) + \frac{\bar{h}(t)+P}{1-L}, \tag{3,6}$$

satisfying the initial condition $s(0) = 0$, and let

$$\bar{h}(t) = \max_{0 \leqslant t' \leqslant t} \|h(t', 0, 0, 0)\| . \tag{3,7}$$

Let I' be the greatest interval contained in the interval I: $0 \leqslant t < a$:

$$I': \quad 0 \leqslant t < \alpha \qquad (\alpha \leqslant a) , \tag{3,8}$$

such that

$$s(t) < b , \quad s'(t) < c \quad \text{for} \quad t \in I' . \tag{3,9}$$

The existence of the interval I' follows from the theorem on continuation of the solutions of differential equations.

Denote by Δ_j the intervals

$$\Delta_j: \quad t_{j-1} \leqslant t < t_j \qquad (j = 1, 2, 3, \ldots) , \tag{3,10}$$

with end-points

$$t_j = t_j(n) = j \cdot \frac{\delta}{n} \qquad (j = 0, 1, 2, \ldots) , \tag{3,11}$$

where n is an arbitrary fixed positive integer and

$$\delta = \begin{cases} \alpha \text{ if } \alpha < +\infty , \\ \text{arbitrary number if } \alpha = +\infty . \end{cases} \tag{3,12}$$

§ 4. Now we shall prove lemma 1, connected with some estimations for a sequence d_n, and lemma 2, on functions $z_n(t)$, $\tau_0 \leqslant t < +\infty$, simplifying the proof of lemma 3.

LEMMA 1. *Let us suppose that the function $h(t, u, v, w)$ fulfils assumtions* H.

Under these assumptions:

1° *All terms of a sequence*

$$d_{n+1} = h(0, 0, 0, d_n) , \quad d_1 = 0 \qquad (n = 1, 2, 3, \ldots) , \tag{4,1}$$

are defined, the sequence d_n converges and a limit $d = \lim_{n\to+\infty} d_n$ is a unique solution of the equation $d = h(0, 0, 0, d)$.

2° *The inequalities*

$$\|d_p - d_q\| \leqslant 2 \cdot L^{n-1} \cdot \frac{P}{1-L}, \quad \|d_n\| \leqslant \frac{P}{1-L} \qquad (n = 1, 2, 3, \ldots) , \tag{4,2}$$

are satisfied for arbitrary positive integers n, p, q, such that $p \geqslant n$, $q \geqslant n$.

3° *The error estimates of the form*

$$\|d_p - d\| \leqslant 2 \cdot L^{n-1} \cdot \frac{P}{1-L}, \tag{4,3}$$

hold for $p \geqslant n$ $(n = 1, 2, 3, \ldots)$.

Proof. From $\|d_1\| < c$ it follows that the term d_2, $d_2 = h(0, 0, 0, d_1)$, can be defined.

We shall prove that d_1 and d_2 satisfy the conditions

$$\|d_1 - d_2\| \leqslant L^0 \cdot P, \quad \|d_2\| \leqslant \frac{P}{1-L}. \tag{4,4}$$

In fact, from (3,4) it follows that $\|d_1 - d_2\| = \|h(0, 0, 0, 0)\| \leqslant P = L^0 \cdot P$ and $\|d_2\| \leqslant L^0 \cdot P \leqslant P \cdot \sum_{j=0}^{\infty} L^j = P/(1-L)$. On the other hand (3,5) implies $\|d_2\| \leqslant c$, whence the next term d_3, $d_3 = h(0, 0, 0, d_2)$, can be defined.

Proceeding by induction, assume that all terms d_j $(j = 2, 3, \ldots, n;$ $n \geqslant 2)$, are defined and satisfy the inequalities

$$\|d_{j-1} - d_j\| \leqslant L^{j-2} \cdot P, \quad \|d_j\| \leqslant P/(1-L) \quad (j = 2, 3, \ldots, n). \tag{4,5}$$

We shall prove that the next term d_{n+1} can be defined, and d_j $(j = 2, 3, \ldots, n+1)$ satisfy the inequalities

$$\|d_{j-1} - d_j\| \leqslant L^{j-2} \cdot P, \quad \|d_j\| \leqslant P/(1-L) \quad (j = 2, 3, \ldots, n+1). \tag{4,6}$$

In fact, from the second formula of (4,5) and from (3,5) we obtain $\|d_n\| < c$, whence the term $d_{n+1} = h(0, 0, 0, d_n)$ can be defined. Moreover, (4,1), the Lipschitz condition and (4,5) imply that

$$\begin{aligned} \|d_n - d_{n+1}\| &= \|h(0, 0, 0, d_{n-1}) - h(0, 0, 0, d_n)\| \\ &\leqslant L \cdot \|d_{n-1} - d_n\| \leqslant L^{n-1} \cdot P, \end{aligned} \tag{4,7}$$

and according to (4,7) and (4,5):

$$\|d_{n+1}\| = \Big\|\sum_{j=2}^{n+1} (d_{j-1} - d_j)\Big\| \leqslant P \cdot \sum_{j=2}^{n+1} L^{j-2} \leqslant P/(1-L);$$

this means that condition (4,6) holds for $j = 2, 3, \ldots, n+1$.

By finite induction all terms of the sequence d_n are defined and satisfy (4,5) for arbitrary n.

Hence the series $d_1 + (d_2 - d_1) + (d_3 - d_2) + \ldots$ as well as the sequence $d_1, d_2, d_3, \ldots$ converge to a limit $d = \lim_{n \to +\infty} d_n$, satisfying the equation $d = h(0, 0, 0, d)$ and the inequality $\|d\| \leqslant P/(1-L) < c$.

The uniqueness results immediately from the Lipschitz condition, since the equality $\bar{d} = h(0, 0, 0, \bar{d})$ implies that $\|\bar{d}-d\| = \|h(0, 0, 0, \bar{d})-h(0, 0, 0, d)\| \leqslant L\cdot\|\bar{d}-d\|$, and $\bar{d} = d$, in view of (3,3).

This completes the proof of part 1°.

Now we shall prove part 2°. For that purpose observe first that from the second formula of (4,5) follows

$$(4,8)\qquad \|d_p-d_q\| \leqslant 2\cdot P/(1-L) = 2\cdot L^0\cdot P/(1-L) \qquad \text{for} \qquad p\geqslant 1\,,\ q\geqslant 1\,.$$

Suppose that for $p\geqslant n$, $q\geqslant n$, we have

$$(4,9)\qquad \|d_p-d_q\| \leqslant 2\cdot L^{n-1}\cdot P/(1-L)\,.$$

We shall prove that for $p\geqslant n+1$, $q\geqslant n+1$:

$$(4,10)\qquad \|d_p-d_q\| \leqslant 2\cdot L^{n}\cdot P/(1-L)\,.$$

In fact, from $p\geqslant n+1$, $q\geqslant n+1$ follows $p-1\geqslant n$, $q-1\geqslant n$, whence (4,9) implies

$$\begin{aligned}\|d_p-d_q\| &= \|h(0, 0, 0, d_{p-1})-h(0, 0, 0, d_{q-1})\| \\ &\leqslant L\cdot\|d_{p-1}-d_{q-1}\| \leqslant 2\cdot L^{n}\cdot P/(1-L)\,,\end{aligned}$$

for $p\geqslant n+1$, $q\geqslant n+1$.

By finite induction the last estimation holds for all n, which completes the proof of (4,2).

From (4,2) we obtain (4,3) as $q\to+\infty$. This completes the proof of lemma 1.

LEMMA 2. *Suppose that the real-valued function $z_n(t)$ fulfils the following assumptions in the intervals θ_j:*

$$(4,11)\qquad \begin{aligned}&\theta_j:\quad \tau_{j-1}\leqslant t<\tau_j \qquad (j=1,2,3,\dots)\,,\\ &\tau_j=\tau_j(n)=\tau_0+j\cdot\frac{\delta}{n} \qquad (j=0,1,2,\dots)\,.\end{aligned}$$

1° *The function $z_n(t)$, defined in the interval θ_1, as well as the derivative $z'_n(t)$ are increasing functions in θ_1 and satisfy conditions $z_n(\tau_0)=0$ and*

$$(4,12)\qquad z_n(t)\geqslant 0\,,\quad z'_n(t)\geqslant 0 \qquad \textit{for} \qquad t\in\theta_1\,,$$

$$(4,13)\qquad z'_n(t)\geqslant M\cdot z_n(t)+N\cdot z_n(t)+L\cdot z'_n(t)+f_n \qquad \textit{for} \qquad t\in\theta_1\,,$$

where

$$(4,14)\qquad f_n=\text{const}\geqslant 0\,.$$

2° *The function $z_n(t)$ is the solution $\zeta=z_n(t)$ of the linear nonhomogeneous equation*

$$(4,15)\qquad \zeta'(t)=M\cdot\zeta(t)+N\cdot z_n\Big(t-\frac{\delta}{n}\Big)+L\cdot z'_n\Big(t-\frac{\delta}{n}\Big)+f_n \qquad \textit{for} \quad t\in\theta_{j+1}(n) \quad (j=1,2,\dots)$$

and satisfies the initial condition

(4,16) $$\zeta(\tau_j) = z_n(\tau_j) = 0 \quad (j = 1, 2, 3, \ldots).$$

Thus, (4,15) *becomes*

(4,17) $$z_n'(t) \equiv M\cdot z_n(t) + N\cdot z_n\left(t-\frac{\delta}{n}\right) + L\cdot z_n'\left(t-\frac{\delta}{n}\right) + f_n\,, \qquad t\in\theta_{j+1}(n) \quad (j = 1, 2, 3, \ldots)$$

identically for $t\in\theta_{j+1}(n)$ $(j = 1, 2, 3, \ldots)$.

Under these assumptions the function $z_n(t)$ *and its derivative* $z_n'(t)$ *are increasing for* $t\in\theta_j(n)$ $(j = 1, 2, 3, \ldots)$ *and satisfy the conditions*

(4,18) $$z_n\left(t-\frac{\delta}{n}\right) \geqslant z_n(t) \geqslant 0\,, \qquad z_n'\left(t-\frac{\delta}{n}\right) \geqslant z_n'(t) \geqslant 0 \quad \textit{for} \quad t\in\theta_{j+1}(n) \quad (j = 1, 2, 3, \ldots),$$

(4,19) $$z_n'(t) \geqslant M\cdot z_n(t) + N\cdot z_n(t) + L\cdot z_n'(t) + f_n\,, \quad t\in\theta_j(n) \;\; (j = 1, 2, 3, \ldots)$$

and

(4,20) $$0 \leqslant z_n(t) < z_n(\tau_1 - 0)\,, \quad 0 \leqslant z_n'(t) < z_n'(\tau_1 - 0) \quad \textit{for} \quad \tau_0 \leqslant t < +\infty\,.$$

Remark 2. Inequality (4,18) can be interpreted as follows: If we translate the graph of the segment $z_n(t)$, $t\in\theta_j$, parallel to the t-axis so as to obtain the graph of $z_n\left(t-\frac{\delta}{n}\right)$ for $t\in\theta_{j+1}$, then the segment $z_n(t)$ for $t\in\theta_{j+1}$ is under the segment $z_n\left(t-\frac{\delta}{n}\right)$ for $t\in\theta_{j+1}$. Consequently the function $z_n(t)$ satisfies the inequality $0\leqslant z_n(t) < z_n(\tau_1-0)$ in the whole interval $\tau_0 \leqslant t < +\infty$. A similar result can be rewritten in the case of derivative $z_n'(t)$.

Proof. According to the definition the function $z_n(t)$ is completely determined in the whole interval $\tau_0 \leqslant t < +\infty$ by its values in the interval θ_1 and initial conditions (4,16).

We shall verify first that $z_n(t)$ and $z_n'(t)$ are increasing functions for $t\in\theta_j$ $(j = 1, 2, 3, \ldots)$. In fact, $z_n(t)$ and $z_n'(t)$ are increasing for $t\in\theta_1$ by assumption. Proceeding by induction suppose that they are increasing for $t\in\theta_j$. Then $z_n\left(t-\frac{\delta}{n}\right)$ and $z_n'\left(t-\frac{\delta}{n}\right)$ are increasing for $t\in\theta_{j+1}$; solving (4,15) we verify that $\zeta = z_n(t)$ is an increasing function for $t\in\theta_{j+1}$; therefore $z_n'(t)$ is increasing for $t\in\theta_{j+1}$ because of (4,17). Hence $z_n(t)$ and $z_n'(t)$ are increasing in all intervals θ_j $(j = 1, 2, 3, \ldots)$ because of the principle of finite induction.

We shall now prove the inequalities (4,18) and (4,19). We observe first that the function $z_n(t)$ satisfies by assumption (4,12) and (4,13) in the interval θ_1.

Now we proceed by induction. We assume that for some integer j the function $z_n(t)$ fulfils the inequalities

$$(\mathrm{A}_j) \qquad z_n(t) \geqslant 0\,, \quad z'_n(t) \geqslant 0 \quad \text{for} \quad t \in \theta_j\,,$$

$$(\mathrm{B}_j) \qquad z'_n(t) \geqslant M\cdot z_n(t) + N\cdot z_n(t) + L\cdot z'_n(t) + f_n \quad \text{for} \quad t \in \theta_j\,,$$

in the interval θ_j.

We shall prove that the function $z_n(t)$, $t \in \theta_{j+1}$, obtained from the non-homogeneous linear equation (4,15) and the initial condition (4,16), satisfies the inequalities

$$(\mathrm{A}_{j+1}) \qquad \begin{aligned} &z_n\left(t-\frac{\delta}{n}\right) \geqslant z_n(t) \geqslant 0\,,\\ &z'_n\left(t-\frac{\delta}{n}\right) \geqslant z'_n(t) \geqslant 0 \quad \text{for} \quad t \in \theta_{j+1}\,,\end{aligned}$$

$$(\mathrm{B}_{j+1}) \qquad z'_n(t) \geqslant M\cdot z_n(t) + N\cdot z_n(t) + L\cdot z'_n(t) + f_n\,, \quad t \in \theta_{j+1}\,,$$

in the interval θ_{j+1}.

a) First we show that

$$(4,21) \qquad z_n\left(t-\frac{\delta}{n}\right) \geqslant z_n(t) \quad \text{for} \quad t \in \theta_{j+1}\,.$$

In fact, the function $z_n\left(t-\frac{\delta}{n}\right)$, $t \in \theta_{j+1}$, satisfies the differential inequality

$$z'_n\left(t-\frac{\delta}{n}\right) \geqslant M\cdot z_n\left(t-\frac{\delta}{n}\right) + N\cdot z_n\left(t-\frac{\delta}{n}\right) + L\cdot z'_n\left(t-\frac{\delta}{n}\right) + f_n \quad \text{for} \quad t \in \theta_{j+1}$$

(because of the induction assumption (B_j)), the function $z_n(t)$, $t \in \theta_{j+1}$, satisfies the differential equation (4,15) (because of the definition), and both functions satisfy the same initial condition $z_n\left(\tau_j-\frac{\delta}{n}\right) = z_n(\tau_j) = 0$ (cf. (4,16)); hence the theorem A on differential inequalities implies that $z_n\left(t-\frac{\delta}{n}\right) \geqslant z_n(t)$ for $t \in \theta_{j+1}$. This completes the proof of part a).

b) We show that

$$(4,22) \qquad z_n(t) \geqslant 0 \quad \text{for} \quad t \in \theta_{j+1}\,.$$

In fact, let us substitute the function $\gamma(t) \equiv 0$, $t \in \theta_{j+1}$, in the differential equation (4,15) in place of $\zeta(t)$; then we obtain the differential inequality

$$(4,23) \qquad \gamma'(t) \equiv 0 \leqslant M\cdot 0 + N\cdot z_n\left(t-\frac{\delta}{n}\right) + L\cdot z'_n\left(t-\frac{\delta}{n}\right) + f_n \quad \text{for} \quad t \in \theta_{j+1}\,,$$

because inequalities (A_j) are fulfilled.

So the function $\gamma(t) \equiv 0$, $t \in \theta_{j+1}$, decreases with respect to the differential equation (4,15), the function $z_n(t)$, $t \in \theta_{j+1}$, satisfies equation (4,15), and the initial values are equal: $\gamma(\tau_j) = z_n(\tau_j) = 0$; hence $z_n(t) \geqslant \gamma(t) \equiv 0$ for $t \in \theta_{j+1}$. This completes the proof of part b).

c) We show that

$$z_n'\left(t-\frac{\delta}{n}\right) \geqslant z_n'(t) \quad \text{for} \quad t \in \theta_{j+1}. \tag{4,24}$$

In fact, (4,17), (4,21) and (B_j) imply

$$\begin{aligned} z_n'(t) &= M\cdot z_n(t) + N\cdot z_n\left(t-\frac{\delta}{n}\right) + L\cdot z_n'\left(t-\frac{\delta}{n}\right) + f_n \\ &\leqslant M\cdot z_n\left(t-\frac{\delta}{n}\right) + N\cdot z_n\left(t-\frac{\delta}{n}\right) + L\cdot z_n'\left(t-\frac{\delta}{n}\right) + f_n \leqslant z_n'\left(t-\frac{\delta}{n}\right) \\ &\qquad \text{for} \quad t \in \theta_{j+1}. \end{aligned}$$

This completes the proof of part c).

d) Now we shall prove that $z_n'(t) \geqslant 0$ for $t \in \theta_{j+1}$. In fact, because of (4,17), (4,22) and (A_j) we have

$$z_n'(t) = M\cdot z_n(t) + N\cdot z_n\left(t-\frac{\delta}{n}\right) + L\cdot z_n'\left(t-\frac{\delta}{n}\right) + f_n \geqslant 0 \quad \text{for} \quad t \in \theta_{j+1}.$$

This completes the proof of part d).

e) We show that $z_n(t)$, $t \in \theta_{j+1}$, satisfies the inequality (B_{j+1}). In fact, (4,17), (4,21) and (4,24) imply that

$$\begin{aligned} z_n'(t) &= M\cdot z_n(t) + N\cdot z_n\left(t-\frac{\delta}{n}\right) + L\cdot z_n'\left(t-\frac{\delta}{n}\right) + f_n \\ &\geqslant M\cdot z_n(t) + N\cdot z_n(t) + L\cdot z_n'(t) + f_n \quad \text{for} \quad t \in \theta_{j+1}. \end{aligned}$$

This completes the proof of part e).

Thus we have obtained the desired inequalities (A_{j+1}) and (B_{j+1}). These inequalities are true for all $j = 1, 2, 3, \ldots$, because of the principle of finite induction, and this concludes the proof of (4,18) and (4,19).

Inequalities (4,20) follow immediately from (4,19) and remark 2 on the translation of the graph of the function $z_n(t)$, $t \in \theta_j$, and its derivative $z_n'(t)$, $t \in \theta_j$.

This concludes the proof of lemma 2.

§ 5. Now we shall prove a lemma connected with the properties of the functions $s_n(t)$. Using these functions we shall obtain lemma 5 and 6 needed in the proof of theorem 2 (cf. fig. 2).

LEMMA 3. *Suppose that the functions $s_n(t)$ satisfy the following conditions in the interval $0 \leqslant t < +\infty$:*

1° *$s_n(t) = S_n(t)$ for $t \in \Delta_2(n)$, where $S_n(t)$ denotes the solution of the linear non-homogeneous differential equation with constant coefficients:*

$$(5,1)\qquad S_n'(t) = \frac{M+N}{1-L}\cdot S_n(t) + \frac{1}{1-L}\cdot C_n\left(\frac{\delta}{n}\right) \quad \text{for} \quad \frac{\delta}{n} \leqslant t < +\infty ,$$

satisfying the initial condition $S_n\left(\frac{\delta}{n}\right) = 0$, and

$$(5,2)\qquad \begin{gathered} C_n(t) = \gamma\cdot t + \gamma_n, \qquad \gamma = (M+N)\cdot s'(\tau) + K , \\ \gamma_n = 2\cdot L^n \cdot \frac{P}{1-L} + (M+N)\cdot s'(\tau)\cdot \frac{\delta}{n} . \end{gathered}$$

In these formulas $s(t)$ is the solution of linear equation (3,6), the number τ being fixed.

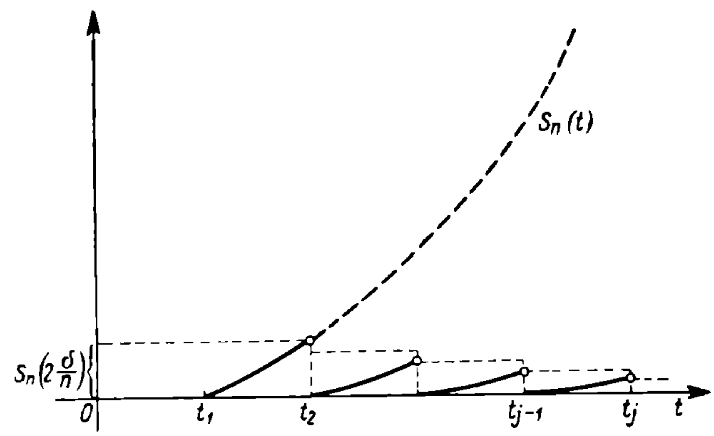

Fig. 2. The graph of the functions $S_n(t)$ and $s_n(t)$ (cf. lemma 3)

2° *$s_n(t)$, $t \in \Delta_{j+1}(n)$ $(j \geqslant 2)$, is the solution $\sigma = s_n(t)$ of the linear non-homogeneous equation*

$$(5,3)\quad \sigma'(t) = M\cdot\sigma(t) + N\cdot s_n\left(t-\frac{\delta}{n}\right) + L\cdot s_n'\left(t-\frac{\delta}{n}\right) + C_n\left(\frac{\delta}{n}\right) \quad \text{for} \quad t \in \Delta_{j+1} ,$$

and fulfils the initial condition

$$(5,4)\qquad \sigma(t_j) = s_n(t_j) = 0 \qquad (j = 1, 2, 3, \ldots) .$$

Thus (5,3) becomes

$$(5,5)\ s_n'(t) \equiv M\cdot s_n(t) + N\cdot s_n\left(t-\frac{\delta}{n}\right) + L\cdot s_n'\left(t-\frac{\delta}{n}\right) + C_n\left(\frac{\delta}{n}\right) \quad \text{for} \quad t \in \Delta_{j+1}(n) ,$$

identically for $t \in \Delta_{j+1}(n)$.

Under these assumptions the functions $s_n(t)$ satisfy the conditions of monotonicity

$$(5,6)\quad S_p\left(2\cdot\frac{\delta}{p}\right)\leqslant S_n\left(2\cdot\frac{\delta}{n}\right),\quad S'_p\left(2\cdot\frac{\delta}{p}\right)\leqslant S'_n\left(2\cdot\frac{\delta}{n}\right),\quad C_p\left(\frac{\delta}{p}\right)\leqslant C_n\left(\frac{\delta}{n}\right)\ \textit{for}\ p\geqslant n,$$

the conditions of uniform boundedness

$$(5,7)\quad 0\leqslant s_n(t)<S_n\left(2\cdot\frac{\delta}{n}\right),\quad 0\leqslant s'_n(t)<S'_n\left(2\cdot\frac{\delta}{n}\right)\quad \textit{for}\quad \frac{\delta}{n}\leqslant t<+\infty,$$

and the conditions of convergence

$$(5,8)\quad S_n\left(2\cdot\frac{\delta}{n}\right)\to 0,\quad S'_n\left(2\cdot\frac{\delta}{n}\right)\to 0,\quad C_n\left(\frac{\delta}{n}\right)\to 0,\quad \textit{as}\quad n\to+\infty,$$

$$(5,9)\quad s_n(t)\Rightarrow 0,\quad s'_n(t)\Rightarrow 0,\quad \textit{as}\quad n\to+\infty,\quad 0\leqslant t<+\infty.$$

The sign $\Rightarrow$ *denotes almost uniform convergence in the interval* $0\leqslant t<+\infty$, *i.e. uniform convergence in every closed interval, bounded and contained in the interval* $0\leqslant t<+\infty$.

Remark 3. It is worth noting that the unknown function $s_n(t)$ occurring in equation (5,3) appears only at those places where the letter σ is printed. Consequently, equation (5,3) is a linear non-homogeneous equation, since $N\cdot s_n\left(t-\frac{\delta}{n}\right)+L\cdot s'_n\left(t-\frac{\delta}{n}\right)+C_n\left(\frac{\delta}{n}\right)$ is a known function for $t\in\Delta_{j+1}(n)$.

Proof. We verify first the conditions of monotonicity (5,6). In fact, from definition (5,2) we have

$$(5,10)\qquad C_p\left(\frac{\delta}{p}\right)\leqslant C_n\left(\frac{\delta}{n}\right)\quad\text{for}\quad p\geqslant n,$$

and

$$(5,11)\qquad C_n\left(\frac{\delta}{n}\right)\to 0,\quad\text{as}\quad n\to+\infty.$$

The differential equation (5,1) is linear and can be solved in quadratures, which gives

$$(5,12)\quad S_n(t)=\frac{1}{1-L}\cdot C_n\left(\frac{\delta}{n}\right)\cdot\left(e^{k(t-t_1)}-1\right)\quad\text{for}\quad\frac{\delta}{n}\leqslant t<+\infty,\quad k=\frac{M+N}{1-L},$$

where $t_1=t_1(n)=\frac{\delta}{n}$, whence (5,12) and (5,10) imply

$$(5,13)\qquad S_p\left(2\cdot\frac{\delta}{p}\right)\leqslant S_n\left(2\cdot\frac{\delta}{n}\right)\quad\text{for}\quad p\geqslant n.$$

On the other hand, according to (5,1), (5,13) and (5,10) we obtain

$$S'_p\left(2\cdot\frac{\delta}{p}\right) \leqslant S'_n\left(2\cdot\frac{\delta}{n}\right) \quad \text{for} \quad p \geqslant n\,,$$

which means that the conditions of monotonicity (5,6) are satisfied.

We shall now deal with the conditions of uniform boundedness (5,7). To this end we observe first that the functions $s_n(t)$ satisfy all assumptions of lemma 2. In fact, according to (5,12) we have $s_n(t) \geqslant 0$ for $t \in \Delta_2(n)$, whence $s'_n(t) \geqslant 0$ as can be seen from (5,1), i.e. $s_n(t)$ is an increasing function for $t \in \Delta_2(n)$. Therefore it follows from (5,1) that the derivative $s'_n(t)$ is also an increasing function for $t \in \Delta_2(n)$. In addition, (5,1) implies that

$$s'_n(t) \geqslant M\cdot s_n(t) + N\cdot s_n(t) + L\cdot s'_n(t) + C_n\left(\frac{\delta}{n}\right) \quad \text{for} \quad t \in \Delta_2(n)\,,$$

whence the functions $s_n(t)$ satisfy assumptions 1° of lemma 2. The functions $s_n(t)$ satisfy also assumptions 2° of lemma 2 in view of (5,3) and (5,4); thus by formula (4,20) of lemma 2 we obtain the conditions of the uniform boundedness (5,7).

The conditions of convergence (5,9) follow immediately from (5,7) and (5,8); consequently, all that remains to be proved is that relations (5,8) are satisfied.

The third part of (5,8) is identical with (5,11); therefore place $t = 2\cdot\frac{\delta}{n}$ in the formula (5,12) and (5,1). Then we obtain

$$\text{(5,14)} \quad \begin{aligned} &S_n\left(2\cdot\frac{\delta}{n}\right) = \frac{1}{1-L}\cdot C_n\left(\frac{\delta}{n}\right)\cdot(e^{k\delta/n}-1) \to 0\,, \quad \text{as} \quad n\to+\infty\,,\\ &S'_n\left(2\cdot\frac{\delta}{n}\right) = k\cdot S_n\left(2\cdot\frac{\delta}{n}\right) + \frac{1}{1-L}\cdot C_n\left(\frac{\delta}{n}\right) \to 0\,, \text{ as } n\to+\infty\,,\ k=\frac{M+N}{1-L}\,, \end{aligned}$$

which means that the conditions of convergence (5,8) are fulfilled.

This completes the proof of lemma 3.

Remark 4. It can be seen from definitions (5,1) and (5,2) that the functions $S_n(t)$, $s_n(t)$ and $C_n(t)$ depend on τ, since γ and γ_n depend on τ.

Remark 5. The functions $s_n(t)$ satisfy all assumptions of lemma 2; therefore in particular

$$\text{(5,15)} \quad \begin{aligned} &s_n\left(t-\frac{\delta}{n}\right) \geqslant s_n(t) \geqslant 0\,,\\ &s'_n\left(t-\frac{\delta}{n}\right) \geqslant s'_n(t) \geqslant 0 \quad \text{for} \quad t\in\Delta_{j+1}(n) \quad (j=1,2,3,\ldots)\,. \end{aligned}$$

Inequalities (5,15) are connected with a certain characteristic property of the difference method (1,2) (cf. § 8, lemma 5).

§ 6. Now we shall prove a theorem connected with the existence and location of the functions $y_n(t)$ and of derivatives $y_n'(t)$ in some interval common for all $y_n(t)$.

THEOREM 1. *Suppose that the function $h(t, u, v, w)$ fulfils assumptions* H.

Under these assumptions the continuous functions $y_n(t)$ satisfying conditions

$$(6,1)\quad \begin{aligned} &y_n'(t) = h(0, 0, 0, d_n) \quad \text{for} \quad t \in \Delta_1(n) \quad y_n(0) = 0, \\ &y_n'(t) = h\big(t_j, y_n(t_j), y_n(t_{j-1}), y_n'(t_{j-1})\big) \\ &\qquad\qquad \text{for} \quad t \in \Delta_{j+1}(n) \quad (j = 1, 2, 3, \ldots), \\ &d_{n+1} = h(0, 0, 0, d_n), \quad d_1 = 0 \quad (n = 1, 2, 3, \ldots), \end{aligned}$$

are defined in a common interval I': $0 \leqslant t < a$ and satisfy the inequalities:

$$(6,2)\quad \|y_n(t)\| \leqslant s(t), \quad \|y_n'(t)\| \leqslant s'(t) \quad \text{for} \quad t \in I',$$

where $s(t)$, $t \in I'$, denotes the solution of the non-homogeneous linear equation (3,6) *and* $s(0) = 0$.

Proof. We observe first that $s(t)$ and $s'(t)$ are increasing functions for $t \in I'$. In fact, solving (3,6) we obtain $s(t) \geqslant 0$ for $t \in I'$, whence $s'(t) \geqslant 0$ in the interval I', i.e. $s(t)$ is increasing for $t \in I'$. In view of (3,6) this means that the derivative $s'(t)$ is also an increasing function for $t \in I'$.

We shall now verify that the function $y_n(t)$ is defined in the first interval $\Delta_1(n)$ and satisfies the inequalities

$$(6,3)\quad \|y_n(t)\| \leqslant s(t), \quad \|y_n'(t)\| \leqslant s'(t) \quad \text{for} \quad t \in \Delta_1(n).$$

In fact, (4,2) and (3,5) imply that $\|d_n\| \leqslant P/(1-L) < c$; therefore $h(0, 0, 0, d_n)$ exists and $y_n(t)$ can be defined for $t \in \Delta_1(n)$. Furthermore, using successively (6,1), (4,2), (3,7) and (3,6) we obtain

$$(6,4)\quad \begin{aligned} \|y_n'(t)\| &= \|h(0, 0, 0, d_n)\| = \|d_{n+1}\| \\ &\leqslant \frac{P}{1-L} \leqslant \frac{P+\bar{h}(0)}{1-L} = s'(0) \leqslant s'(t), \end{aligned}$$

for $t \in \Delta_1(n)$, since $s(t)$ and its derivative $s'(t)$ are increasing functions for $t \in I'$.

Thus, the function $\|y_n(t)\|$ satisfies the differential inequality (cf. theorem B):

$$(6,5)\quad \bar{D}_+\|y_n(t)\| \leqslant s'(t) \quad \text{for} \quad t \in \Delta_1(n),$$

the function $\sigma = s(t)$, $t \in I'$, satisfies the differential equation

$$(6,6)\quad \sigma'(t) = s'(t) \quad \text{for} \quad t \in I',$$

with an initial condition $\sigma(0)=s(0)=0$, and the initial values are equal: $s(0)=\|y_n(0)\|=0$; therefore theorem A implies that

$$\|y_n(t)\| \leqslant s(t) \quad \text{for} \quad t \in \Delta_1(n)\,. \tag{6,7}$$

Hence it can be seen from (6,7) and (6,4) that the desired inequalities (6,3) hold for $t \in \Delta_1(n)$.

It may be mentioned that, in virtue of (6,7), we obtain, as $t \to t_1 - 0$,

$$\|y_n(t_1)\| \leqslant s(t_1)\,, \quad \text{where} \quad t_1(n) \leqslant a\,. \tag{6,8}$$

Proceeding by induction let us suppose that the function $y_n(t)$ is defined in the interval $\Delta_j(n)$, $t_j(n) < a$ $(j \geqslant 1)$, and fulfils the conditions

$$\|y_n(t)\| \leqslant s(t)\,, \quad \|y'_n(t)\| \leqslant s'(t) \quad \text{for} \quad t \in \Delta_j(n)\,, \tag{6,9}$$

i.e. in particular

$$\|y_n(t_j)\| \leqslant s(t_j)\,. \tag{6,10}$$

We prove that the function $y_n(t)$ can be defined in the next interval $\Delta_{j+1}(n)$, the inequalities

$$\|y_n(t)\| \leqslant s(t)\,, \quad \|y'_n(t)\| \leqslant s'(t) \quad \text{for} \quad t \in \Delta_{j+1}(n)\,, \tag{6,11}$$

hold for $t \in \Delta_{j+1}(n)$, and if $t_{j+1}(n) < a$ then

$$\|y_n(t_{j+1})\| \leqslant s(t_{j+1})\,. \tag{6,12}$$

In fact, assumption $t_j(n) < a$ and (6,10) show that $h\big(t_j, y_n(t_j), y_n(t_{j-1}), y'_n(t_{j-1})\big)$ exists; therefore $y_n(t)$ is defined for $t \in \Delta_{j+1}(n)$.

According to (6,1), (3,7), (6,10), (6,9) and monotonicity of the functions $s(t)$ and $s'(t)$ we have

$$\begin{aligned}\|y'_n(t)\| &\leqslant \|h\big(t_j, y_n(t_j), y_n(t_{j-1}), y'_n(t_{j-1})\big) - h(t_j, 0, 0, 0)\| + \\ &\quad + \|h(t_j, 0, 0, 0)\| \\ &\leqslant M\cdot\|y_n(t_j)\| + N\cdot\|y_n(t_{j-1})\| + L\cdot\|y'_n(t_{j-1})\| + \bar{h}(t) \\ &\leqslant M\cdot s(t_j) + N\cdot s(t_{j-1}) + L\cdot s'(t_{j-1}) + \bar{h}(t) \\ &\leqslant M\cdot s(t) + N\cdot s(t) + L\cdot s'(t) + \bar{h}(t) + P = s'(t) \text{ for } t \in \Delta_{j+1}(n)\,.\end{aligned} \tag{6.13}$$

Hence the function $\|y_n(t)\|$ satisfies the differential inequality (cf. theorem B):

$$\bar{D}_+\|y_n(t)\| \leqslant s'(t) \quad \text{for} \quad t \in \Delta_{j+1}(n)\,, \tag{6,14}$$

the function $\sigma = s(t)$, $t \in \Delta_{j+1}(n)$, $\sigma(t_j) = s(t_j)$, satisfies the differential equation

$$\sigma'(t) = s'(t) \quad \text{for} \quad t \in \Delta_{j+1}(n)\,, \tag{6,15}$$

and the initial values for $t = t_j(n)$ satisfy condition (6,10); hence from theorem A and from (6,13) we obtain

$$\|y_n(t)\| \leqslant s(t) \quad \text{for} \quad t \in \Delta_{j+1}(n)\,. \tag{6,16}$$

Consequently, as may be seen from (6,16) and (6,13), the desired inequalities (6,11) hold for $t \in \Delta_{j+1}(n)$.

Now, if $t_{j+1}(n) = a$, then $y_n(t)$ is defined in the whole interval I'; in the remaining case, when $t_{j+1}(n) < a$, we obtain from (6,16)

$$\|y_n(t_{j+1})\| \leqslant s(t_{j+1})\,. \tag{6,17}$$

So the function $y_n(t)$ is defined in the whole interval I': $0 \leqslant t < a$, $(a \leqslant +\infty)$ because of the principle of finite induction, and satisfies conditions (6,2).

This completes the proof of theorem 1.

§ 7. We shall now give the precise location of the functions $y_p(t)$ and their derivatives $y'_p(t)$ for $p \geqslant n$, in an arbitrary prescribed interval θ:

$$\theta: \quad 0 \leqslant t < \tau \quad (\tau < a)\,,$$

bounded and contained in the interval I' (cf. fig. 3).

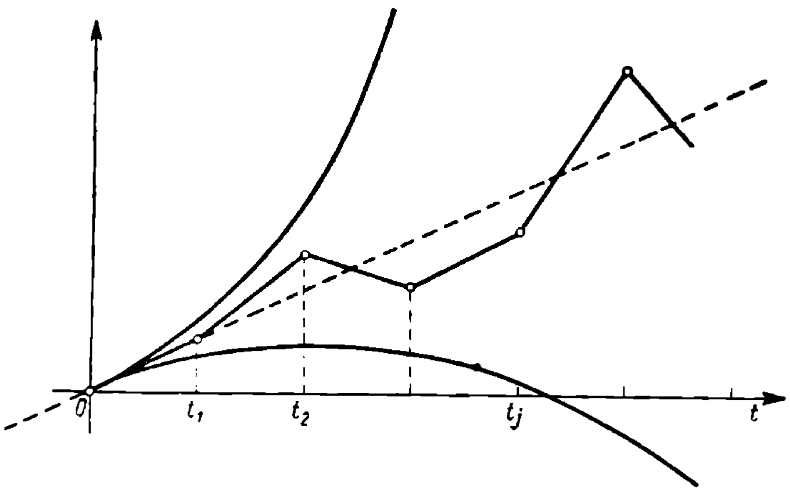

Fig. 3. The precise location of $y_n(t)$ with the aid of the function $R_n(t)$ (cf. lemma 4)

LEMMA 4. *Suppose that the function $h(t, u, v, w)$ satisfies assumptions* H *and consider the sequence $y_n(t)$ defined by* (6,1).

Under these assumptions the functions $y_n(t)$ fulfil the inequalities

$$\begin{aligned} &\|y_p(t) - t\cdot y'_p(0)\| \leqslant R_n(t)\,, \\ &\|y'_p(t) - y'_p(0)\| \leqslant R'_n(t) \quad \textit{for} \quad t \in \theta\,, \quad p \geqslant n\,, \end{aligned} \tag{7,1}$$

where $R_n(t)$, $R_n(0) = 0$, *denotes the solution of the linear equation*

$$R'_n(t) = \frac{M+N}{1-L} \cdot R_n(t) + \frac{1}{1-L} \cdot C_n(t)\,, \tag{7.2}$$

$C_n(t)$ *is defined by* (5,2) *and* $s(t)$ *is the solution of* (3,6).

Proof. We shall observe first that $R_n(t)$ and $R'_n(t)$ are increasing functions in the interval θ. In fact, solving (7,2) we obtain $R_n(t) \geqslant 0$, whence $R'_n(t) \geqslant 0$ for $t \in \theta$ because of (7,2), i.e. $R_n(t)$ is an increasing function in the interval θ. Therefore it follows from (7,2) that the derivative $R'_n(t)$ is also an increasing function for $t \in \theta$.

We shall verify inequalities (7,1) successively in the intervals

$$\Delta_j(p) \cdot \theta \qquad (j = 1, 2, 3, \ldots)\,,$$

choosing n so as to obtain $\delta/n < \tau$, and we begin with the proof of (7,1) in the first interval $\Delta_1(p)$: $0 \leqslant t < \delta/p$, $p \geqslant n$.

All functions $y_p(t)$ $(p = 1, 2, 3, \ldots)$ are defined in the common interval I' because of theorem 1, whence for $t \in \Delta_1(p)$ it follows from (6,1), (7,2) and (5,2) that

$$\begin{aligned} \|y'_p(t) - y'_p(0)\| &= \|h(0, 0, 0, d_p) - h(0, 0, 0, d_p)\| = 0 \\ &\leqslant M \cdot R_n(t) + N \cdot R_n(t) + L \cdot R'_n(t) + C_n(t) = R'_n(t) \\ &\qquad\qquad \text{for} \quad t \in \Delta_1(p)\,. \end{aligned} \tag{7,3}$$

Thus, the function $\|y_p(t) - t \cdot y'_p(0)\|$ satisfies the differential inequality (cf. theorem B)

$$\bar{D}_+\|y_p(t) - t \cdot y'_p(0)\| \leqslant R'_n(t) \qquad \text{for} \quad t \in \Delta_1(p)\,, \quad p \geqslant n\,, \tag{7,4}$$

the function $\varrho = R_n(t)$ satisfies the differential equation

$$\varrho'(t) = R'_n(t) \qquad \text{for} \quad t \in \theta\,, \tag{7,5}$$

and the initial values are equal: $R_n(0) = \|y_p(0) - 0 \cdot y'_p(0)\| = 0$; hence from the theorem A and from (7,3) we obtain

$$\begin{aligned} &\|y_p(t) - t \cdot y'_p(0)\| \leqslant R_n(t)\,, \\ &\|y'_p(t) - y'_p(0)\| \leqslant R'_n(t) \qquad \text{for} \quad t \in \Delta_1(p)\,, \quad p \geqslant n\,. \end{aligned} \tag{7,6}$$

Proceeding by induction assume that the conditions

$$\begin{aligned} &\|y_p(t) - t \cdot y'_p(0)\| \leqslant R_n(t)\,, \\ &\|y'_p(t) - y'_p(0)\| \leqslant R'_n(t) \qquad \text{for} \quad t \in \Delta_j(p)\,, \quad p \geqslant n \quad (j \geqslant 1)\,, \end{aligned} \tag{7,7}$$

are fulfilled in the interval $\Delta_j(p)$ $(j \geqslant 1)$.

We shall prove that

$$(7,8)\qquad \begin{aligned} &\|y_p(t)-t\cdot y_p'(0)\| \leqslant R_n(t)\,, \\ &\|y_p'(t)-y_p'(0)\| \leqslant R_n'(t) \quad \text{for} \quad t \in \Delta_{j+1}(p)\,, \quad p \geqslant n\,, \end{aligned}$$

hold in the next interval $\Delta_{j+1}(p)$, $p \geqslant n$.

In fact, let us write

$$(7,9)\qquad \mu(t) = y_p(t)-t\cdot y_p'(0)\,, \qquad \nu(t) = y_p\left(t-\frac{\delta}{p}\right)-\left(t-\frac{\delta}{p}\right)\cdot y_p'(0)\,,$$
$$\text{for} \quad t \in \Delta_{j+1}(p)\,,\ p \geqslant n\,.$$

According to (6,1) and the Lipschitz condition we obtain

$$\begin{aligned}
(7,10)\qquad &\|y_p'(t)-y_p'(0)\| = \|h(t_j, y_p(t_j), y_p(t_{j-1}), y_p'(t_{j-1}))-h(0,0,0,d_p)\| \\
&\leqslant K\cdot t_j + M\cdot\|y_p(t_j)\| + N\cdot\|y_p(t_{j-1})\| + L\cdot\|y_p'(t_{j-1})-d_p\| \\
&\leqslant K\cdot t + M\cdot\|y_p(t_j)+\mu(t)-\mu(t)\| + N\cdot\|y_p(t_{j-1})+\nu(t)-\nu(t)\| + \\
&\quad + L\cdot\|y_p'(t_{j-1})+\nu'(t)-\nu'(t)-d_p\| \\
&\leqslant M\cdot\|\mu(t)\| + N\cdot\|\nu(t)\| + L\cdot\|\nu'(t)\| + \\
&\quad + M\cdot\|y_p(t)-y_p(t_j)\| + N\cdot\left\|y_p\left(t-\frac{\delta}{p}\right)-y_p(t_{j-1})\right\| + L\cdot\left\|y_p'\left(t-\frac{\delta}{p}\right)-y_p'(t_{j-1})\right\| + \\
&\quad + [(M+N)\cdot\|y_p'(0)\|+K]\cdot t + L\cdot\|y_p'(0)-d_p\| \quad \text{for} \quad t \in \Delta_{j+1}(p)\,, \quad p \geqslant n\,.
\end{aligned}$$

But

$$(7,11)\qquad \begin{aligned} &\|\nu(t)\| \leqslant R_n\left(t-\frac{\delta}{p}\right) \leqslant R_n(t)\,, \\ &\|\nu'(t)\| \leqslant R_n'\left(t-\frac{\delta}{p}\right) \leqslant R_n'(t) \quad \text{for} \quad t \in \Delta_{j+1}(p)\,, \quad p \geqslant n\,, \end{aligned}$$

because of (7,7), whence from (7,10), (7,11), (4,2) and (5,2) it follows that

$$(7,12)\qquad \|\mu'(t)\| \leqslant M\cdot\|\mu(t)\| + N\cdot R_n(t) + L\cdot R_n'(t) + C_n(t)$$
$$\text{for} \quad t \in \Delta_{j+1}(p)\,, \quad p \geqslant n\,.$$

Thus, the function $\|\mu(t)\|$ satisfies the differential inequality (cf. theorem B)

$$(7,13)\qquad \bar{D}_+\|\mu(t)\| \leqslant M\cdot\|\mu(t)\| + N\cdot R_n(t) + L\cdot R_n'(t) + C_n(t)\,,$$

for $t \in \Delta_{j+1}(p)$, $p \geqslant n$, the function $\varrho = R_n(t)$ satisfies the differential equation (cf. (7,2))

$$(7,14)\qquad \varrho'(t) = M\cdot\varrho(t) + N\cdot R_n(t) + L\cdot R_n'(t) + C_n(t) \quad \text{for} \quad t \in \theta\,,$$

and the initial values for $t=t_j(p)$ fulfil the condition $\|\mu(t_j)\| \leqslant R_n(t_j)$ because of (7,7). Therefore from theorem A it follows that

$$(7,15) \qquad \|\mu(t)\| \leqslant R_n(t) \quad \text{for} \quad t \in \Delta_{j+1}(p)\,, \quad p \geqslant n\,.$$

In addition, (7,12), (7,15) and equation (7,14) imply that

$$\begin{aligned}\|\mu'(t)\| &\leqslant M\cdot\|\mu(t)\| + N\cdot R_n(t) + L\cdot R_n'(t) + C_n(t)\\ &\leqslant M\cdot R_n(t) + N\cdot R_n(t) + L\cdot R_n'(t) + C_n(t) = R_n'(t)\\ &\qquad\qquad \text{for} \quad t \in \Delta_{j+1}(p)\,,\ p \geqslant n\,,\end{aligned}$$

which completes the proof of inequality (7,8).

By (7,6), (7,7), (7,8) and by finite induction relations (7,1) are satisfied in the whole interval θ.

This completes the proof of lemma 4.

§ 8. We shall now give the precise location of the rectilinear segment $y_n(t)$, $t \in \Delta_{j+1}(n)$, with respect to the continuation

$$y(t) = y_n(t_{j-1}) + (t-t_{j-1})\cdot y_n'(t_{j-1})\,, \qquad t \in \Delta_{j+1}(n)\,,$$

of the rectilinear segment $y_n(t)$, $t \in \Delta_j(n)$.

In this connection it seems to be worth observing that there is a certain characteristic property of the difference method (6,1), namely estimates (8,1) are decreasing when the interval $\Delta_j(n)$ is replaced by the next interval $\Delta_{j+1}(n)$ (cf. property (5,15) of the functions $s_n(t)$).

LEMMA 5. *Suppose that the function $h(t,u,v,w)$ satisfies assumptions* H *and consider the sequence $y_n(t)$ defined by* (6,1).

Under these assumptions

$$(8,1) \qquad \begin{aligned}&\|y_n(t) - y_n(t_{j-1}) - (t-t_{j-1})\cdot y_n'(t_{j-1})\| \leqslant s_n(t)\,,\\ &\|y_n'(t) - y_n'(t_{j-1})\| \leqslant s_n'(t) \quad \textit{for} \quad t \in \Delta_{j+1}(n)\cdot\theta \quad (j=1,2,3,\ldots)\,.\end{aligned}$$

Proof. We shall verify (8,1) successively in the intervals

$$\Delta_{j+1}(n)\cdot\theta \qquad (j=1,2,3,\ldots)\,,$$

choosing n so as to obtain $2\cdot\dfrac{\delta}{n} < \tau$.

We begin with the proof of (8,1) in the interval $\Delta_2(n)$. In fact, using successively (6,1), (6,2), (4,2) and (5,2) we obtain

$$\begin{aligned}(8,2) \qquad &\|y_n'(t) - y_n'(0)\| = \big\|h\big(t_1, y_n(t_1), y_n(0), y_n'(0)\big) - h(0,0,0,d_n)\big\|\\ &\leqslant K\cdot t_1 + M\cdot\|y_n(t_1)\| + L\cdot\|y_n'(0) - d_n\|\\ &\leqslant M\cdot\|y_n(t_1) + y_n(t) - t\cdot y_n'(0) - y_n(t) + t\cdot y_n'(0)\| + K\cdot\frac{\delta}{n} + L\cdot\|d_{n+1} - d_n\|\end{aligned}$$

$$\leqslant M\cdot\|y_n(t)-t\cdot y_n'(0)\|+M\cdot\|y_n(t)-y_n(t_1)\|+M\cdot t\cdot\|y_n'(0)\|+K\cdot\frac{\delta}{n}+2\cdot L^n\cdot\frac{P}{1-L}$$

$$\leqslant M\cdot\|y_n(t)-t\cdot y_n'(0)\|+N\cdot s_n(t)+L\cdot s_n'(t)+C_n\left(\frac{\delta}{n}\right)\quad \text{for}\quad t\in\Delta_2(n)\,.$$

Thus, the function $\mu(t)=\|y_n(t)-t\cdot y_n'(0)\|$ satisfies the differential inequality

$$\bar{D}_+\mu(t)\leqslant M\cdot\mu(t)+N\cdot s_n(t)+L\cdot s_n'(t)+C_n\left(\frac{\delta}{n}\right)\quad \text{for}\quad t\in\Delta_2(n)\,, \tag{8,3}$$

the function $\sigma=s_n(t)$ satisfies the differential equation (cf. (5,1))

$$\sigma'(t)=M\cdot\sigma(t)+N\cdot s_n(t)+L\cdot s_n'(t)+C_n\left(\frac{\delta}{n}\right)\quad \text{for}\quad \frac{\delta}{n}\leqslant t<+\infty\,, \tag{8,4}$$

and the initial values for $t=\delta/n$ are equal: $\mu(\delta/n)=s_n(\delta/n)=0$; therefore from theorem A we obtain

$$\|y_n(t)-t\cdot y_n'(0)\|\leqslant s_n(t)\quad \text{for}\quad t\in\Delta_2(n)\,. \tag{8,5}$$

In addition, (8,2), (8,5) and the differential equation (8,4) imply

$$\|y_n'(t)-y_n'(0)\|\leqslant M\cdot\|y_n(t)-t\cdot y_n'(0)\|+N\cdot s_n(t)+L\cdot s_n'(t)+C_n\left(\frac{\delta}{n}\right)$$

$$\leqslant M\cdot s_n(t)+N\cdot s_n(t)+L\cdot s_n'(t)+C_n\left(\frac{\delta}{n}\right)=s_n'(t)\ \text{for}\ t\in\Delta_2(n)\,,$$

which completes the proof of (8,1) in the interval $\Delta_2(n)$.

Proceeding by induction assume that

$$\begin{aligned}&\|y_n(t)-y_n(t_{j-1})-(t-t_{j-1})\cdot y_n'(t_{j-1})\|\leqslant s_n(t)\,,\\&\|y_n'(t)-y_n'(t_{j-1})\|\leqslant s_n'(t)\quad \text{for}\quad t\in\Delta_{j+1}(n)\quad (j\text{—fixed}\,,\ j\geqslant 1)\,.\end{aligned} \tag{8,6}$$

We prove that

$$\begin{aligned}&\|y_n(t)-y_n(t_j)-(t-t_j)\cdot y_n'(t_j)\|\leqslant s_n(t)\,,\\&\|y_n'(t)-y_n'(t_j)\|\leqslant s_n'(t)\quad \text{for}\quad t\in\Delta_{j+2}(n)\,.\end{aligned} \tag{8,7}$$

To this end let us write

$$\begin{aligned}&a(t)=y_n(t)-y_n(t_j)-(t-t_j)\cdot y_n'(t_j)\,,\\&b(t)=y_n\left(t-\frac{\delta}{n}\right)-y_n(t_{j-1})-\left(t-\frac{\delta}{n}-t_{j-1}\right)\cdot y_n'(t_{j-1})\quad \text{for}\quad t\in\Delta_{j+2}(n)\,.\end{aligned} \tag{8,8}$$

Then by (6,1) and (8,8) we have

$$(8,9) \quad \|y_n'(t)-y_n'(t_j)\|$$

$$= \left\|h\big(t_{j+1}, y_n(t_{j+1}), y_n(t_j), y_n'(t_j)\big) - h\big(t_j, y_n(t_j), y_n(t_{j-1}), y_n'(t_{j-1})\big)\right\|$$

$$\leqslant K\cdot\frac{\delta}{n} + M\cdot\|y_n(t_{j+1})-y_n(t_j)\| + N\cdot\|y_n(t_j)-y_n(t_{j-1})\| + L\cdot\|y_n'(t_j)-y_n'(t_{j-1})\|$$

$$\leqslant M\cdot\|y_n(t_{j+1})+a(t)-a(t)-y_n(t_j)\| + N\cdot\|y_n(t_j)+b(t)-b(t)-y_n(t_{j-1})\| +$$

$$+L\cdot\|y_n'(t_j)+b'(t)-b'(t)-y_n'(t_{j-1})\| + K\cdot\frac{\delta}{n}$$

$$\leqslant M\cdot\|a(t)\| + N\cdot\|b(t)\| + L\cdot\|b'(t)\| + M\cdot\|y_n(t)-y_n(t_{j+1})\| +$$

$$+M\cdot(t-t_j)\cdot\|y_n'(t_j)\| + N\cdot\left\|y_n\Big(t-\frac{\delta}{n}\Big)-y_n(t_j)\right\| + N\cdot\Big(t-\frac{\delta}{n}-t_{j-1}\Big)\cdot\|y_n'(t_{j-1})\| +$$

$$+L\cdot\left\|y_n'\Big(t-\frac{\delta}{n}\Big)-y_n'(t_j)\right\| + K\frac{\delta}{n}.$$

From the induction assumption (8,6) it follows that

$$(8,10) \quad \|b(t)\| \leqslant s_n\Big(t-\frac{\delta}{n}\Big), \quad \|b'(t)\| \leqslant s_n'\Big(t-\frac{\delta}{n}\Big) \quad \text{for} \quad t \in \Delta_{j+2}(n),$$

whence according to (8,9), (8,10) and (5,2) we obtain

$$(8,11) \quad \|a'(t)\| \leqslant M\cdot\|a(t)\| + N\cdot s_n\Big(t-\frac{\delta}{n}\Big) + L\cdot s_n'\Big(t-\frac{\delta}{n}\Big) + C_n\Big(\frac{\delta}{n}\Big) \quad \text{for} \quad t \in \Delta_{j+2}(n).$$

Thus, the function $\|a(t)\|$ satisfies the inequality (cf. theorem B)

$$(8,12) \quad \bar{D}_+\|a(t)\| \leqslant M\cdot\|a(t)\| + N\cdot s_n\Big(t-\frac{\delta}{n}\Big) + L\cdot s_n'\Big(t-\frac{\delta}{n}\Big) + C_n\Big(\frac{\delta}{n}\Big) \quad \text{for} \quad t \in \Delta_{j+2}(n),$$

the function $\sigma = s_n(t)$ satisfies the differential equation (5,3) for $t \in \Delta_{j+2}(n)$, and the initial values fulfil the condition $\|a(t_{j+1})\| \leqslant s_n(t_{j+1})$ because of (8,6), therefore from theorem A we obtain

$$(8,13) \quad \|a(t)\| \leqslant s_n(t) \quad \text{for} \quad t \in \Delta_{j+2}(n).$$

In addition, (8,11), (8,13) and equation (5,3) imply

$$\|a'(t)\| \leqslant M\cdot\|a(t)\| + N\cdot s_n\Big(t-\frac{\delta}{n}\Big) + L\cdot s_n'\Big(t-\frac{\delta}{n}\Big) + C_n\Big(\frac{\delta}{n}\Big)$$

$$\leqslant M\cdot s_n(t) + N\cdot s_n\Big(t-\frac{\delta}{n}\Big) + L\cdot s_n'\Big(t-\frac{\delta}{n}\Big) + C_n\Big(\frac{\delta}{n}\Big) = s_n'(t) \quad \text{for } t \in \Delta_{j+2}(n),$$

which completes the proof of relations (8,7).

According to the principle of finite induction inequalities (8,1) are fulfilled in the whole interval θ, and this completes the proof of lemma 5.

We shall now give the estimates for $y_n(t)-y_n(t_{j-1})$ and $y'_n(t)-y'_n(t_{j-1})$, $t\in\Delta_{j+1}(n)$, with the aid of the functions $s_n(t)$ considered in lemma 3.

LEMMA 6. *Let us suppose that the function $h(t,u,v,w)$ satisfies assumptions* H *and consider the sequence $y_n(t)$ defined by* (6,1).

Under these assumptions

$$(8,14)\qquad \begin{aligned} &\|y_n(t)-y_n(t_{j-1})\|\leqslant s_n(t)+(t-t_{j-1})\cdot s'(\tau)\,,\\ &\|y'_n(t)-y'_n(t_{j-1})\|\leqslant s'_n(t)\quad \textit{for}\quad t\in\Delta_{j+1}(n)\cdot\theta\quad (j=1,2,3,\ldots)\,,\end{aligned}$$

and in particular

$$(8,15)\qquad \begin{aligned} &\|y_n(t)-y_n(t_{j-1})\|\leqslant S_n\Big(2\cdot\frac{\delta}{n}\Big)+2\cdot\frac{\delta}{n}\cdot s'(\tau)\,,\\ &\|y'_n(t)-y'_n(t_{j-1})\|\leqslant S'_n\Big(2\cdot\frac{\delta}{n}\Big)\quad \textit{for}\quad t\in\Delta_{j+1}(n)\cdot\theta\quad (j=1,2,3,\ldots)\,.\end{aligned}$$

Proof. Inequality (8,1) implies the second line of the (8,14). But

$$\begin{aligned}\|y_n(t)-y_n(t_{j-1})\| &\leqslant \|y_n(t)-y_n(t_{j-1})-(t-t_{j-1})\cdot y'_n(t_{j-1})\|+(t-t_{j-1})\cdot\|y'_n(t_{j-1})\|\\ &\leqslant s_n(t)+(t-t_{j-1})\cdot s'(\tau)\,,\end{aligned}$$

for $t\in\Delta_{j+1}(n)\cdot\theta$ $(j=1,2,3,\ldots)$ and this completes the proof of (8,14).

Relations (8,15) follow immediately from (8,14) and the conditions of uniform boundedness (5,7).

This completes the proof of lemma 6.

LEMMA 7. *Suppose that the function $h(t,u,v,w)$ satisfies assumptions* H *and consider the sequence $y_n(t)$ defined by* (6,1).

Under these assumptions

$$(8,16)\qquad \begin{aligned} &\|y_p(t)-y_q(t)\|\leqslant R_n(t)\,,\\ &\|y'_p(t)-y'_q(t)\|\leqslant R'_n(t)\quad \textit{for}\quad t\in\Delta_1(q)\,,\quad p\geqslant q\geqslant n\,,\end{aligned}$$

where $R_n(t)$, $t\in\theta$, $R_n(0)=0$, is the solution of the linear equation (7,2).

Proof. We shall verify (8,16) successively in the intervals

$$(8,17)\qquad \Delta_j(p)\cdot\Delta_1(q)\quad (j=1,2,3,\ldots)\,,$$

choosing n so as to obtain $\delta/n<\tau$.

We prove first that (8,16) hold in the interval $\Delta_1(p)$: $0\leqslant t<\delta/p$. In fact, (6,1), (4,2) and definition (5,2) of $C_n(t)$ imply

$$(8,18)\qquad \begin{aligned}\|y'_p(t)-y'_q(t)\| &= \|h(0,0,0,d_p)-h(0,0,0,d_q)\|\leqslant L\cdot\|d_p-d_q\|\\ &\leqslant M\cdot R_n(t)+N\cdot R_n(t)+L\cdot R'_n(t)+C_n(t)=R'_n(t)\quad \text{for}\quad t\in\Delta_1(p)\,.\end{aligned}$$

Thus, the function $\|y_p(t)-y_q(t)\|$ satisfies the differential inequality (cf. theorem B)

$$\bar{D}_+\|y_p(t)-y_q(t)\| \leqslant R'_n(t) \quad \text{for} \quad t \in \Delta_1(p)\,, \tag{8,19}$$

the function $\varrho = R_n(t)$ satisfies the differential equation (cf. (7.2))

$$\varrho'(t) = M\cdot\varrho(t)+N\cdot R_n(t)+L\cdot R'_n(t)+C_n(t) \quad \text{for} \quad t \in \theta\,, \tag{8,20}$$

and the initial values are equal, $\|y_p(0)-y_q(0)\| = R_n(0) = 0$, whence from theorem A we obtain

$$\|y_p(t)-y_q(t)\| \leqslant R_n(t) \quad \text{for} \quad t \in \Delta_1(p)\,. \tag{8,21}$$

So relations (8,16) hold in the first interval $\Delta_1(p)$ because of (8,21) and (8,18).

Proceeding by induction assume that

$$\begin{aligned} &\|y_p(t)-y_q(t)\| \leqslant R_n(t)\,,\\ &\|y'_p(t)-y'_q(t)\| \leqslant R'_n(t) \quad \text{for} \quad t \in \Delta_j(p)\,. \end{aligned} \tag{8,22}$$

We shall prove that

$$\begin{aligned} &\|y_p(t)-y_q(t)\| \leqslant R_n(t)\,,\\ &\|y'_p(t)-y'_q(t)\| \leqslant R'_n(t) \quad \text{for} \quad t \in \Delta_{j+1}(p)\,. \end{aligned} \tag{8,23}$$

In fact, if we write

$$\begin{aligned} &\bar{a}(t) = y_p(t)-t\cdot y'_p(0)\,,\\ &\bar{b}(t) = y_p\Big(t-\frac{\delta}{p}\Big)-\Big(t-\frac{\delta}{p}\Big)\cdot y'_p(0) \quad \text{for} \quad t \in \Delta_{j+1}(p)\,, \end{aligned} \tag{8,24}$$

then from (6,1) and (4,2) follows

$$\begin{aligned} (8,25)\quad &\|y'_p(t)-y'_q(t)\|\\ &= \Big\|h\big(t_j, y_p(t_j), y_p(t_{j-1}), y'_p(t_{j-1})\big)-h(0, 0, 0, d_q)\Big\|\\ &\leqslant K\cdot t_j+M\cdot\|y_p(t_j)\|+N\cdot\|y_p(t_{j-1})\|+L\cdot\|y'_p(t_{j-1})-d_q\|\\ &\leqslant K\cdot t+M\cdot\|y_p(t_j)+\bar{a}(t)-\bar{a}(t)\|+N\cdot\|y_p(t_{j-1})+\bar{b}(t)-\bar{b}(t)\|+\\ &\quad +L\cdot\|y'_p(t_{j-1})+\bar{b}'(t)-\bar{b}'(t)-d_q\|\\ &\leqslant M\cdot\|\bar{a}(t)\|+N\cdot\|\bar{b}(t)\|+L\cdot\|\bar{b}'(t)\|+\\ &\quad +M\cdot\|y_p(t)-y_p(t_j)\|+N\cdot\Big\|y_p\Big(t-\frac{\delta}{p}\Big)-y_p(t_{j-1})\Big\|+L\cdot\Big\|y'_p\Big(t-\frac{\delta}{p}\Big)-y'_p(t_{j-1})\Big\|+\\ &\quad +[(M+N)\cdot\|y'_p(0)\|+K]\cdot t+L\cdot\|y'_p(0)-d_q\| \quad \text{for} \quad t \in \Delta_{j+1}(p)\,. \end{aligned}$$

But the induction assumption (8,22), the monotonicity of $R_n(t)$, $R'_n(t)$ and (7,1) imply that

$$\begin{aligned} &\|\bar{b}(t)\| \leqslant R_n\left(t-\frac{\delta}{p}\right) \leqslant R_n(t)\,, \\ (8,26)\qquad &\|\bar{b}'(t)\| \leqslant R'_n\left(t-\frac{\delta}{p}\right) \leqslant R'_n(t) \quad \text{for} \quad t \in \Delta_{j+1}(p)\,, \\ &\|\bar{a}(t)\| \leqslant R_n(t) \quad \text{for} \quad t \in \theta; \end{aligned}$$

therefore according to (8,25), (8,26) and (5,2) we have

$$\begin{aligned} (8,27)\qquad &\|y'_p(t)-y'_q(t)\| \\ &\leqslant M\cdot R_n(t)+N\cdot R_n(t)+L\cdot R'_n(t)+C_n(t) = R'_n(t) \quad \text{for} \quad t \in \Delta_{j+1}(p)\,. \end{aligned}$$

Thus, the function $\|y_p(t)-y_q(t)\|$ satisfies the differential inequality (cf. theorem B)

$$(8,28)\qquad \bar{D}_+\|y_q(t)-y_p(t)\| \leqslant R'_n(t) \quad \text{for} \quad t \in \Delta_{j+1}(p)\,,$$

the function $\varrho = R_n(t)$ satisfies the differential equation (8,20) and the initial values for $t = t_j(p)$ fulfil the condition $\|y_p(t_j)-y_q(t_j)\| \leqslant R_n(t_j)$ in view of (8,22), whence theorem A implies

$$(8,29)\qquad \|y_p(t)-y_q(t)\| \leqslant R_n(t) \quad \text{for} \quad t \in \Delta_{j+1}(p)\,.$$

Thus we have obtained the desired inequalities (8,23) in the interval $\Delta_{j+1}(p)$ since (8,29) and (8,27) hold.

By induction relations (8,16) are fulfilled in the whole interval $\Delta_1(q)$, and this completes the proof of lemma 7.

§ 9. Now we shall give a theorem connected with the existence and uniqueness of the solution.

THEOREM 2. *Let us suppose that the right-hand member of the differential equation*

$$(9,1)\qquad y' = h(t, y, y, y')\,,$$

satisfies assumptions H.

Under these assumptions:

1° *The sequence of functions $y_n(t)$ defined by the formula*

$$\begin{aligned} &y'_n(t) = h(0, 0, 0, d_n) \quad \text{for} \quad t \in \Delta_1(n)\,, \quad y_n(0) = 0\,, \\ (9,2)\ &y'_n(t) = h\big(t_j, y_n(t_j), y_n(t_{j-1}), y'_n(t_{j-1})\big) \quad \text{for} \quad t \in \Delta_{j+1}(n) \quad (j = 1, 2, 3, \ldots)\,, \\ &d_{n+1} = h(0, 0, 0, d_n)\,, \quad d_1 = 0 \quad (n = 1, 2, 3, \ldots)\,, \end{aligned}$$

converges almost uniformly in the interval I': $0 \leqslant t < a$ to the unique solution $y = \varphi(t)$, $t \in I'$, of equation (9,1), *satisfying the initial condition $\varphi(0) = 0$.*

2° *The error estimates of the form*

$$(9,3)\qquad \begin{aligned} &\|y_q(t)-\varphi(t)\| \leqslant k_n(t)\,,\\ &\|y_q'(t)-\varphi'(t)\| \leqslant k_n'(t) \quad \textit{for} \quad t\in\theta\,, \quad q\geqslant n\,, \end{aligned}$$

hold in an arbitrarily prescribed interval θ:

$$(9,4)\qquad \theta:\quad 0\leqslant t<\tau \quad (\tau<a)\,,$$

bounded and contained in the interval I'.

Here the real-valued function $k_n(t)$, $t\in\theta$, *is the solution of the linear equation with constant coefficients*

$$(9,5)\qquad k_n'(t)=\frac{M+N}{1-L}\cdot k_n(t)+\frac{b_n}{1-L}\,,$$

and satisfies the initial condition $k_n(0)=0$. *The constant* $b_n\geqslant 0$ *is defined by*

$$(9,6)\qquad b_n=2\cdot N\cdot\left[S_n\left(2\cdot\frac{\delta}{n}\right)+2\cdot\frac{\delta}{n}\cdot s'(\tau)\right]+2\cdot L\cdot S_n'\left(2\cdot\frac{\delta}{n}\right)+C_n\left(\frac{\delta}{n}\right),$$

can be computed from (5,2), (5,14) *and* (3,6), *and fulfils the condition*

$$(9,7)\qquad b_n\to 0\,,\quad \textit{as}\quad n\to+\infty\,.$$

Proof. a) We shall verify that the sequence $y_n(t)$ and the sequence of derivatives $y_n'(t)$ satisfy the Cauchy criterion for the almost uniform convergence in the interval I'.

For this purpose denote by θ: $0\leqslant t<\tau$ an interval contained in the interval I' and assume that $\delta/n<\tau$.

We prove first that

$$(9,8)\qquad \begin{aligned} &\|y_p(t)-y_q(t)\| \leqslant k_n(t)\,,\\ &\|y_p'(t)-y_q(t)\| \leqslant k_n'(t) \quad \text{for} \quad t\in\Delta_1(q)\,, \quad p\geqslant q\geqslant n\,, \end{aligned}$$

in the interval $\Delta_1(q)$: $0\leqslant t<\delta/q$. In order to see this, it is sufficient to verify that

$$(9,9)\qquad \begin{aligned} &\|y_p(t)-y_q(t)\| \leqslant R_n(t)\,,\\ &\|y_p'(t)-y_q'(t)\| \leqslant R_n'(t) \quad \text{for} \quad t\in\Delta_1(q)\,, \quad p\geqslant q\geqslant n\,, \end{aligned}$$

and

$$(9,10)\qquad R_n(t)\leqslant k_n(t),\quad R_n'(t)\leqslant k_n'(t) \quad \text{for} \quad t\in\Delta_1(n)\,.$$

Inequalities (8,16) imply the relations (9,9), and (9,10) can be proved as follows:

From (5,2) and (9,6) we see that

$$(9,11)\qquad C_n(t)\leqslant C_n\left(\frac{\delta}{n}\right)<b_n \quad \text{for} \quad t\in\Delta_1(n)\,.$$

Thus, the function $R_n(t)$ satisfies the differential inequality (cf. (7,2) and (9,11))

$$R_n'(t) \leqslant \frac{M+N}{1-L}\cdot R_n(t) + \frac{b_n}{1-L} \quad \text{for} \quad t \in \Delta_1(n)\,, \tag{9,12}$$

the function $k_n(t)$ satisfies the differential equation (9,5) and the initial values are equal, $R_n(0) = k_n(0) = 0$, whence theorem A implies

$$R_n(t) \leqslant k_n(t) \quad \text{for} \quad t \in \Delta_1(n)\,. \tag{9,13}$$

In addition, as can be seen from (9,12), (9,13) and equation (9,5),

$$R_n'(t) \leqslant \frac{M+N}{1-L}\cdot R_n(t) + \frac{b_n}{1-L} \leqslant \frac{M+N}{1-L}\cdot k_n(t) + \frac{b_n}{1-L} = k_n'(t) \quad \text{for} \quad t \in \Delta_1(n)\,,$$

which means that relations (9,10) hold.

This completes the proof of inequalities (9,8) in the interval $\Delta_1(q)$.

Now we shall deal with the interval $\delta/n \leqslant t < \tau$; namely we prove that

$$\begin{aligned} &\|y_p(t) - y_q(t)\| \leqslant k_n(t)\,, \\ &\|y_p'(t) - y_q'(t)\| \leqslant k_n'(t)\,, \quad \text{for} \quad \frac{\delta}{q} \leqslant t < \tau\,, \quad p \geqslant q \geqslant n\,. \end{aligned} \tag{9,14}$$

In fact, if t is in the intervals

$$\begin{aligned} &\frac{\delta}{q} \leqslant t < \tau\,, \\ \Delta_{j+1}(p)\colon \quad & t_j(p) \leqslant t < t_{j+1}(p)\,, \\ \Delta_{i+1}(q)\colon \quad & t_i(q) \leqslant t < t_{i+1}(q)\,, \end{aligned} \tag{9,15}$$

then

$$\begin{aligned} \|y_p'(t) - y_q'(t)\| &= \Big\|h\big(t_j, y_p(t_j), y_p(t_{j-1}), y_p'(t_{j-1})\big) - h\big(t_i, y_q(t_i), y_q(t_{i-1}), y_q'(t_{i-1})\big)\Big\| \\ &\leqslant K\cdot\frac{\delta}{n} + M\cdot\|y_p(t_j) - y_q(t_i)\| + N\cdot\|y_p(t_{j-1}) - y_q(t_{i-1})\| + \\ &\quad + L\cdot\|y_p'(t_{j-1}) - y_q'(t_{i-1})\|\,. \end{aligned} \tag{9,16}$$

But

$$\begin{aligned} &\|y_p(t_j) - y_q(t_i)\| \leqslant \|y_p(t_j) - y_p(t)\| + \|y_p(t) - y_q(t)\| + \|y_q(t) - y_q(t_i)\|\,, \\ &\|y_p(t_{j-1}) - y_q(t_{i-1})\| \leqslant \|y_p(t_{j-1}) - y_p(t)\| + \|y_p(t) - y_q(t)\| + \\ &\qquad\qquad + \|y_q(t) - y_q(t_{i-1})\|\,, \\ &\|y_p'(t_{j-1}) - y_q'(t_{i-1})\| \leqslant \|y_p'(t_{j-1}) - y_p'(t)\| + \|y_p'(t) - y_q'(t)\| + \\ &\qquad\qquad + \|y_q'(t) - y_q'(t_{i-1})\|\,, \end{aligned} \tag{9,17}$$

whence according to (9,17) and (9,16) we have

$$(9,18)\qquad (1-L)\cdot\|y'_p(t)-y'_q(t)\| \leqslant (M+N)\cdot\|y_p(t)-y_q(t)\| + \\ + M\cdot\big(\|y_p(t)-y_p(t_j)\|+\|y_q(t)-y_q(t_i)\|\big) + \\ + N\cdot\big(\|y_p(t)-y_p(t_{j-1})\|+\|y_q(t)-y_q(t_{i-1})\|\big) + \\ + L\cdot\big(\|y'_p(t)-y'_p(t_{j-1})\|+\|y'_q(t)-y'_q(t_{i-1})\|\big)\,,$$

for t in the intervals (9,15).

Hence, (9,15), (8,15), the conditions of monotonicity (5,6) and definition (9,6) of the constant b_n imply

$$(1-L)\cdot\|y'_p(t)-y'_q(t)\| \leqslant (M+N)\cdot\|y_p(t)-y_q(t)\| + \\ + 2\cdot M\cdot s'(\tau)\cdot\frac{\delta}{n} + 2\cdot N\cdot\left[S_n\left(2\cdot\frac{\delta}{n}\right) + 2\cdot\frac{\delta}{n}\cdot s'(\tau)\right] + 2\cdot L\cdot S'_n\left(2\cdot\frac{\delta}{n}\right),$$

which means that

$$(9,19)\qquad (1-L)\cdot\|y'_p(t)-y'_q(t)\| \leqslant (M+N)\cdot\|y_p(t)-y_q(t)\| + b_n\,,$$

for $\delta/q \leqslant t < \tau$, $p \geqslant q \geqslant n$.

So the function $\|y_p(t)-y_q(t)\|$ satisfies the differential inequality (cf. theorem B)

$$(9,20)\qquad \bar{D}_+\|y_p(t)-y_q(t)\| \leqslant \frac{M+N}{1-L}\cdot\|y_p(t)-y_q(t)\| + \frac{b_n}{1-L},$$

for $\delta/q \leqslant t < \tau$, $p \geqslant q \geqslant n$, the function $k_n(t)$ satisfies the differential equation (9,5) for $t \in \theta$, and the initial values for $t = \delta/q$ fulfil condition

$$(9,21)\qquad \left\|y_p\left(\frac{\delta}{q}\right) - y_q\left(\frac{\delta}{q}\right)\right\| \leqslant k_n\left(\frac{\delta}{q}\right),$$

as can be seen from (9,8). Hence from theorem A we obtain

$$(9,22)\qquad \|y_p(t)-y_q(t)\| \leqslant k_n(t) \quad \text{for} \quad \frac{\delta}{q} \leqslant t < \tau\,,\ p \geqslant q \geqslant n\,.$$

In addition, from (9,19), (9,22) and (9,5) it follows that

$$\|y'_p(t)-y'_q(t)\| \leqslant \frac{M+N}{1-L}\cdot\|y_p(t)-y_q(t)\| + \frac{b_n}{1-L} \\ \leqslant \frac{M+N}{1-L}\cdot k_n(t) + \frac{b_n}{1-L} = k'_n(t) \quad \text{for} \quad \frac{\delta}{q} \leqslant t < \tau,\ p \geqslant q \geqslant n\,,$$

which completes the proof of (9,14).

Thus, according to (9,14) and (9,9) we have

$$(9,23)\qquad \begin{aligned} &\|y_p(t)-y_q(t)\| \leqslant k_n(t)\,,\\ &\|y'_p(t)-y'_q(t)\| \leqslant k'_n(t) \quad \text{for} \quad t\in\theta\,,\ p\geqslant q\geqslant n\,, \end{aligned}$$

in the whole interval θ.

On the other hand $k_n(t)$, $t\in\theta$, is the solution of the linear equation (9,5) with constant coefficients, $k_n(0)=0$, and the term b_n tends to zero, as $n\to+\infty$, because of (9,7); therefore

$$(9,24)\qquad k_n(t)\Rightarrow 0\,,\quad k'_n(t)\Rightarrow 0\,,\quad \text{as}\quad n\to+\infty\,,\quad t\in\theta\,,$$

(cf. for example E. Kamke [1], p. 145).

Formulas (9,24) and (9,23) imply that the sequence $y_n(t)$ and the sequence of derivatives $y'_n(t)$ satisfy the Cauchy criterion for uniform convergence in the interval θ; consequently, they are almost uniformly convergent in the interval I'.

b) Denote by $\varphi(t)$ the limit function of the almost uniformly convergent sequence $y_n(t)$. The sequence of (right) derivatives $y'_n(t)$ is also almost uniformly convergent; therefore (cf. T. Ważewski [5], theorem 3);

$$(9,25)\qquad y_n(t)\Rightarrow\varphi(t)\,,\quad y'_n(t)\Rightarrow\varphi'(t)\,,\quad \text{as}\quad n\to+\infty\,,\quad t\in I'\,.$$

We prove that $\varphi(t)$, $t\in I'$, is the solution of equation (9,1) and satisfies the initial condition $\varphi(0)=0$.

It is sufficient to show that

$$(9,26)\qquad \Big\|y'_n(t)-h\big(t, y_n(t), y_n(t), y'_n(t)\big)\Big\| \leqslant b_n \quad \text{for} \quad t\in\theta\,,$$

since from (9,26), (9,25) and (9,7) follows, as $n\to+\infty$, the identity

$$(9,27)\qquad \varphi'(t)\equiv h\big(t, \varphi(t), \varphi(t), \varphi'(t)\big) \quad \text{for} \quad t\in\theta\,,$$

in an arbitrary prescribed interval θ, contained in the interval I'.

We shall verify (9,26) successively in the intervals

$$\Delta_j(n)\cdot\theta \qquad (j=1,2,3,\ldots)\,,$$

and we begin with the proof of the inequality

$$(9,28)\qquad \Big\|y'_n(t)-h\big(t, y_n(t), y_n(t), y'_n(t)\big)\Big\| \leqslant b_n \quad \text{for} \quad t\in\Delta_1(n)\,.$$

In fact, if $t \in \Delta_1(n)$, then

$$(9.29)\quad \left\|y_n'(t) - h\big(t, y_n(t), y_n(t), y_n'(t)\big)\right\|$$

$$= \left\|h(0, 0, 0, d_n) - h\big(t, y_n(t), y_n(t), y_n'(t)\big)\right\|$$

$$\leqslant K\cdot t + M\cdot \|y_n(t)\| + N\cdot \|y_n(t)\| + L\cdot \|y_n'(t) - d_n\|$$

$$= [(M+N)\cdot \|y_n'(0)\| + K]\cdot t + L\cdot \|d_{n+1} - d_n\| \leqslant b_n \quad \text{for} \quad t \in \Delta_1(n),$$

because of (9,2), (4,2), (5,2) and (9,6).

This completes the proof of (9,28) in the first interval $\Delta_1(n)$.

Proceeding by induction, assume that (9,26) holds in the interval $\Delta_j(n)$. We shall prove that (9,26) is fulfilled in the next interval $\Delta_{j+1}(n)$.

In fact, using successively (9,2), (8,15), (9,6) and (5,2), we obtain

$$(9,30)\quad \left\|y_n'(t) - h\big(t, y_n(t), y_n(t), y_n'(t)\big)\right\|$$

$$= \left\|h\big(t_j, y_n(t_j), y_n(t_{j-1}), y_n'(t_{j-1})\big) - h\big(t, y_n(t), y_n(t), y_n'(t)\big)\right\|$$

$$\leqslant K\cdot \frac{\delta}{n} + M\cdot \|y_n(t) - y_n(t_j)\| + N\cdot \|y_n(t) - y_n(t_{j-1})\| + L\cdot \|y_n'(t) - y_n'(t_{j-1})\|$$

$$\leqslant K\cdot \frac{\delta}{n} + M\cdot \frac{\delta}{n}\cdot s'(\tau) + N\cdot \left[S_n\left(2\cdot \frac{\delta}{n}\right) + 2\cdot \frac{\delta}{n}\cdot s'(\tau)\right] + L\cdot S_n'\left(2\cdot \frac{\delta}{n}\right) \leqslant b_n$$

$$\text{for} \quad t \in \Delta_{j+1}(n),$$

which means that inequality (9,26) holds in the next interval $\Delta_{j+1}(n)$.

By induction (9,26) is satisfied in the whole interval θ, whence identity (9,27) holds.

In addition $\varphi(0) = 0$, since $y_n(0) = 0$ $(n = 1, 2, 3, \ldots)$, and this completes the proof of existence.

c) Now the uniqueness of the solution will be proved. To this end assume that $\psi(t)$, $t \in I'$, is the solution of (9,1) satisfying the initial condition $\psi(0) = 0$. Then we have

$$(9,31)\quad \|\varphi'(t) - \psi'(t)\|$$

$$= \left\|h\big(t, \varphi(t), \varphi(t), \varphi'(t)\big) - h\big(t, \psi(t), \psi(t), \psi'(t)\big)\right\|$$

$$\leqslant (M+N)\cdot \|\varphi(t) - \psi(t)\| + L\cdot \|\varphi'(t) - \psi'(t)\| \quad \text{for} \quad t \in I',$$

and

$$(9,32)\quad \bar{D}_+\|\varphi(t) - \psi(t)\| \leqslant \frac{M+N}{1-L}\cdot \|\varphi(t) - \psi(t)\| \quad \text{for} \quad t \in I',$$

because of theorem B.

Thus, the function $\|\varphi(t)-\psi(t)\|$ satisfies the differential inequality (9,32), the function $\xi(t)\equiv 0$, $t\in I'$, is the solution (also the greatest solution) of the equation

$$\xi'(t)=\frac{M+N}{1-L}\cdot\xi(t) \quad \text{for} \quad t\in I', \tag{9,33}$$

satisfying condition $\xi(0)=0$, and the initial values are equal, $\|\varphi(0)-$ $-\psi(0)\|=\xi(0)=0$; therefore from theorem A we obtain

$$\|\varphi(t)-\psi(t)\|\leqslant\xi(t)\equiv 0 \quad \text{for} \quad t\in I', \tag{9,34}$$

which implies $\varphi(t)\equiv\psi(t)$ for $t\in I'$.

This completes the proof of uniqueness.

d) The error estimates (9,3) follows from (9,23), as $p\to+\infty$.

This completes the proof of theorem 2.

§ 10. Now we shall derive another error estimate for the approximation $y_n(t)$. In this connection we shall evaluate the difference $\|y_n(t)-\varphi(t)\|$ with the aid of the differences $\|y_n(t_j)-y_n(t_{j-1})\|$ and $\|y_n'(t_j)-y_n'(t_{j-1})\|$, which are considered only at a finite number of points t_j in an interval θ.

THEOREM 3. *Suppose that $h(t,u,v,w)$ satisfies assumptions* H *and that the function $y_n(t)$ is constructed for some n with the aid of equations* (9,2) *in the interval* (9,4) *bounded and contained in the interval I'.*

Assume in addition that

$$\begin{aligned}\varepsilon&=\max_j\|y_n(t_j)-y_n(t_{j-1})\|,\\ \eta&=\max_j\|y_n'(t_j)-y_n'(t_{j-1})\|,\\ \sigma'&=\max_j\|y_n'(t_j)\|,\end{aligned} \tag{10,1}$$

for all values j such that $t_j\in\theta$ and $t_{j-1}\in\theta$; thus, there is a finite number of points t_j, all being defined by (3,11).

Under these assumptions the error estimates are provided by

$$\begin{aligned}&\|y_n(t)-\varphi(t)\|\leqslant x(t),\\ &\|y_n'(t)-\varphi'(t)\|\leqslant x'(t) \quad \text{for} \quad t\in\theta.\end{aligned} \tag{10,2}$$

Here $x(t)=2\cdot s(t)$ for $t\in\Delta_1(n)$, $s(t)$ being defined by (3,6) *and condition $s(0)=0$; for $\frac{\delta}{n}\leqslant t<\tau$ the function $x(t)$ is the solution of the linear equation with constant coefficients*

$$x'(t)=\frac{M+N}{1-L}\cdot x(t)+\frac{\varkappa}{1-L}, \tag{10,3}$$

and satisfies the initial condition $x(\delta/n) = 2\cdot s(\delta/n)$, *the constant* $\varkappa \geqslant 0$ *being defined by*

$$\varkappa = N\cdot\varepsilon + L\cdot\eta + [(M+N)\cdot\sigma' + K]\cdot\frac{\delta}{n}. \tag{10,4}$$

Proof. Let us observe that from (6,2) it follows

$$\|\varphi(t)\| \leqslant s(t), \quad \|\varphi'(t)\| \leqslant s'(t) \quad \text{for} \quad t\in I', \tag{10,5}$$

as $n\to+\infty$, as well as

$$\begin{aligned} &\|y_n(t)-\varphi(t)\| \leqslant \|y_n(t)\| + \|\varphi(t)\| \leqslant 2\cdot s(t) = x(t), \\ &\|y_n'(t)-\varphi'(t)\| \leqslant \|y_n'(t)\| + \|\varphi'(t)\| \leqslant 2\cdot s'(t) = x'(t) \quad \text{for} \quad t\in\Delta_1(n), \end{aligned} \tag{10,6}$$

i.e. relations (10,2) hold for $t\in\Delta_1(n)$.

For $t\in\Delta_{j+1}(n)\cdot\theta$ we obtain

$$\begin{aligned} &(10,7) \quad \|y_n'(t)-\varphi'(t)\| \\ &\leqslant \big\|h\big(t_j, y_n(t_j), y_n(t_{j-1}), y_n'(t_{j-1})\big) - h\big(t_j, y_n(t_j), y_n(t_j), y_n'(t_j)\big)\big\| + \\ &\quad + \big\|h\big(t_j, y_n(t_j), y_n(t_j), y_n'(t_j)\big) - h\big(t, \varphi(t), \varphi(t), \varphi'(t)\big)\big\| \\ &\leqslant N\cdot\|y_n(t_j)-y_n(t_{j-1})\| + L\cdot\|y_n'(t_j)-y_n'(t_{j-1})\| + \\ &\quad + K\cdot(t-t_j) + (M+N)\cdot\|y_n(t_j)-\varphi(t)\| + L\cdot\|y_n'(t_j)-\varphi'(t)\| \quad \text{for} \quad t\in\Delta_{j+1}(n)\cdot\theta. \end{aligned}$$

However,

$$\begin{aligned} &y_n(t) = y_n(t_j) + (t-t_j)\cdot y_n'(t_j), \\ &y_n'(t) = y_n'(t_j) \quad \text{for} \quad t\in\Delta_{j+1}(n)\cdot\theta \quad (j=1,2,3,\ldots); \end{aligned} \tag{10,8}$$

accordingly, (10,8) and (10,7) imply

$$\begin{aligned} \|y_n'(t)-\varphi'(t)\| \leqslant (M+N)\cdot\|y_n(t)-\varphi(t)\| &+ \\ + [(M+N)\cdot\|y_n'(t_j)\| + K]\cdot(t-t_j) &+ \\ + N\cdot\|y_n(t_j)-y_n(t_{j-1})\| + L\cdot\|y_n'(t_j)-y_n'(t_{j-1})\|&, \end{aligned} \tag{10,9}$$

for $t\in\Delta_{j+1}(n)\cdot\theta$ $(j=1,2,3,\ldots)$, whence

$$\|y_n'(t)-\varphi'(t)\| \leqslant \frac{M+N}{1-L}\cdot\|y_n(t)-\varphi(t)\| + \frac{\varkappa}{1-L} \quad \text{for} \quad \frac{\delta}{n}\leqslant t<\tau. \tag{10,10}$$

Thus, the function $\|y_n(t)-\varphi(t)\|$ satisfies the differential inequality (cf. theorem B):

$$\bar{D}_+\|y_n(t)-\varphi(t)\| \leqslant \frac{M+N}{1-L}\cdot\|y_n(t)-\varphi(t)\| + \frac{\varkappa}{1-L} \quad \text{for} \quad \frac{\delta}{n}\leqslant t<\tau, \tag{10,11}$$

the function $x(t)$ satisfies the differential equation (10,3) and the initial values for $t = \delta/n$ fulfil the condition $\|y_n(\delta/n) - \varphi(\delta/n)\| \leqslant x(\delta/n)$ because of (10,6). Hence from theorem A we obtain

$$\|y_n(t) - \varphi(t)\| \leqslant x(t) \quad \text{for} \quad \frac{\delta}{n} \leqslant t < \tau . \tag{10,12}$$

Furthermore (10,10), (10,12) and (10,3) imply

$$\begin{aligned} \|y'_n(t) - \varphi'(t)\| &\leqslant \frac{M+N}{1-L} \cdot \|y_n(t) - \varphi(t)\| + \frac{\varkappa}{1-L} \\ &\leqslant \frac{M+N}{1-L} \cdot x(t) + \frac{\varkappa}{1-L} = x'(t) \quad \text{for} \quad \frac{\delta}{n} \leqslant t < \tau , \end{aligned} \tag{10,13}$$

which completes the proof of relations (10,2).

This completes also the proof of theorem 3.

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