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THE PERFORMANCE OF A DIGITAL FSK SYSTEM WITH ACTUAL DISCRIMINATOR: TIME DISTORTIONS EFFECTS

1. Introduction. The error performance of a digital FSK system is studied in the presence of additive Gaussian noise. We take into consideration the distortion effects due to band limitations and the influence of the spectrum shift in the carrier current channel.

In previous investigations ([3], [4], [6], [7], [11], [13]) the ideal frequency discriminator as a demodulator was considered, and consequently the time distortions were neglected. In this paper the FSK system with actual discriminator is analyzed, which enables us to study such distortions. In particular, we obtain the formula for the error probability as the function of the parameters of time distortions.

The time distortions are the useful parameter characterizing the system performance in measurement practice, but the direct formula between the error probability and the time distortions has not been known till now.

2. The system. In Fig. 1 we show the essential parts of an FSK data transmission system with actual frequency discriminator (see [1] and [3]).

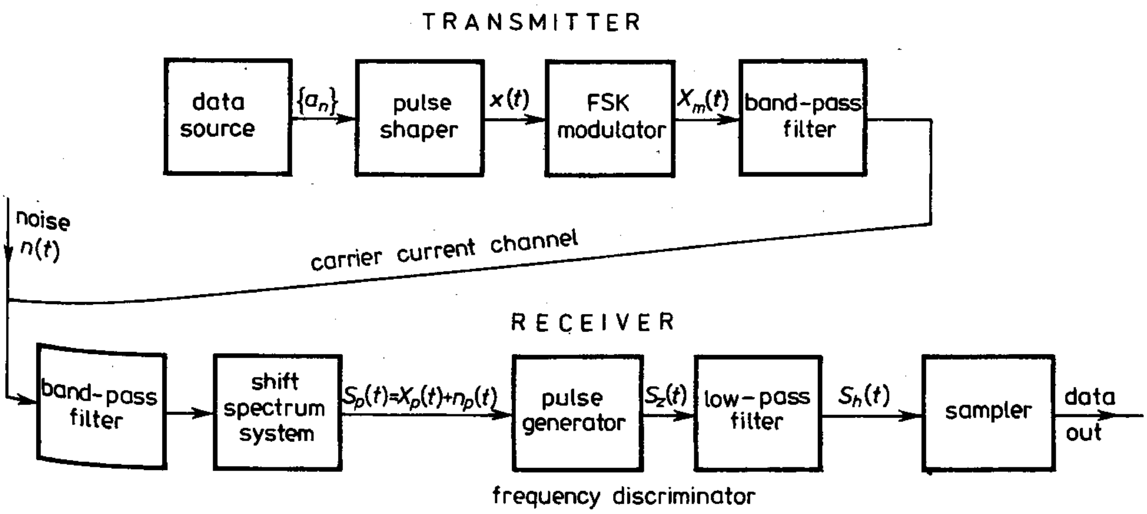

Fig. 1. The block diagram of the digital FSK system

The data source is a digital one producing every T seconds one of two symbols: a mark (the logical value 1) and a space (the logical value 0), which are equally probable. We can assume that each digit is independent of the values of past or future digits.

The pulse shaper starts producing a rectangular signalling pulse of duration T at the moment the data source generates one digit. The height of this pulse depends on the value of the digit generated.

If $s(t)$ is the function representing the rectangular signalling pulse of unit amplitude, and a_n is the value of the digit generated by the data source at time $(n-1)T$ ([1]), then the input to the FSK modulator at time t is given by

$$x(t) = \sum_{n=1}^{\infty} b_n s(t-(n-1)T), \tag{2.1}$$

where $b_n = -1$ if $a_n = 1$, and $b_n = 1$ if $a_n = 0$, i.e., $b_n = -2a_n+1$.

The function $x(t)$ is called a *modulating signal.*

The output of the FSK modulator is then given by the following expression:

$$X_m(t) = A\cos[\omega_{cp}t + \Delta\omega \int_0^t x(z)dz + \psi_0], \tag{2.2}$$

where A is the constant amplitude of the FSK wave, $\Delta\omega$ is a constant of proportionality (it relates amplitudes to frequency shifts), ω_{cp} is the unmodulated carrier frequency, and ψ_0 is the initial phase of the modulator at time $t = 0$.

The modulated signal $X_m(t)$ passes through the sending band-pass filter.

The transmitted signal is applied to the carrier current channel, which shifts the signal spectrum on the pulsation axis ω by the value δ_ω. The noise interfering with the signal transmission in the carrier current channel is assumed to be additive and white Gaussian.

The combined signal and noise enter a receiver which consists of a band-pass filter, a spectrum shift system on the pulsation axis ω by the value $k\omega_{cp}$, a frequency discriminator and a sampler. The frequency discriminator is a zero-crossing type demodulator. It generates a pulse of fixed length z_0 and height a, at each zero-crossing (at which the time derivative at the carrier wave is positive) and integrates the pulse train in a low-pass filter. The combined signal and noise finally are synchronously sampled at discrete sampling instants, and on the basis of each sample produce a mark or a space symbol.

3. Distortion effects due to the channel ([2]). In the analysis we made the assumption that the data source generates from $-\infty$ to 0 a sequence of

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([1]) Assuming the data source starts transmitting at time zero.

([2]) In this analysis, the *channel* means the combination of the carrier current channel, the receiving filter and the sending filter.

identical digits for which each of digits equals a_0 and starts transmitting at time zero the sequence $\{a_n\}_{n=1}^{N}$ of N digits. Let $\{a_n\}_{n=0}^{N}$, or shortly $\{a_n\}$, be a sequence generated by the data source, arbitrarily chosen but fixed.

Then the FSK modulated signal $S_m(t)$, by (2.2), can be written in the form

$$X_m(t) = \begin{cases} A\cos[(\omega_{cp}+b_{l+1}\Delta\omega)t+\psi_{l+1}] & \text{for } lT < t \leqslant (l+1)T, \\ A\cos[(\omega_{cp}+b_0\Delta\omega)t+\psi_0] & \text{for } t \leqslant 0, \end{cases} \tag{3.1}$$

where

$$l = 0, 1, 2, \ldots, N-1, \qquad \sum_{i=1}^{l} b_i \overset{\text{df}}{=} 0 \text{ for } l = 0,$$

$$\psi_{l+1} = \psi_0 + \Delta\omega T \sum_{i=1}^{l} b_i - \Delta\omega l T b_{l+1}.$$

We assume that both the sending and the receiving filters are combinations of two filters: a low-pass filter and an upper-pass filter with impulse responses given by the formulas

$$h_1(t) = \alpha'_1 \exp(-\alpha'_1 t) 1(t), \tag{3.2}$$

$$h_2(t) = \delta_{\mathrm{DIR}}(t) - \alpha'_2 \exp(-\alpha'_2 t) 1(t), \tag{3.3}$$

respectively, where $\alpha'_1, \alpha'_2 > 0$, $1(t) = \chi_{(0,\infty)}(t)$. Then each of the impulse responses $d(t)$ and $r(t)$ of the sending filter and the receiving filter, respectively, is a convolution of the functions given by (3.2) and (3.3). Thus the impulse response of the combined filters is equal to

$$d(t)*r(t) = \beta'_1 t\exp(-\alpha'_1 t) + \beta'_2 t\exp(-\alpha'_2 t) + \beta'_3 \exp(-\alpha'_1 t) + \beta'_4 \exp(-\alpha'_2 t), \tag{3.4}$$

where

$$\beta'_1 = \left(\alpha'_1 + \frac{\alpha'_1\alpha'_2}{\alpha'_1 - \alpha'_2}\right)^2, \qquad \beta'_2 = \left(\frac{\alpha'_1\alpha'_2}{\alpha'_1 - \alpha'_2}\right)^2,$$

$$\beta'_3 = 2\left(\alpha'_1 + \frac{\alpha'_1\alpha'_2}{\alpha'_1 - \alpha'_2}\right)\frac{\alpha'_1\alpha'_2}{(\alpha'_1 - \alpha'_2)^2}, \qquad \beta'_4 = -\beta'_3.$$

The FSK modulated signal passes through the sending filter, the carrier current channel, the receiving filter, the shift spectrum system, and finally reaches the discriminator. Thus we obtain the following final form for the signal $X_p(t)$ at the input of the discriminator (as the convolution of the FSK modulated signal given by (3.1) and the impulse responses of the above four systems ([3]):

$$(3.5)\quad X_p(t) = \sum_{i=1}^{2} [(p_{2in}t+p_{1in})\exp(-\alpha_i t)]+c_n\cos(\omega_n t+\psi_n)$$
$$-s_n\sin(\omega_n t+\psi_n) \quad \text{for } (n-1)T \leqslant t < nT,$$

where

$$\omega_n = \omega_c+b_n\Delta\omega, \qquad \omega_c = \omega_{cp}+\delta_\omega+k\omega_{cp},$$
$$\beta_i = \beta_i' + k\omega_{cp}, \qquad \alpha_i = \alpha_i'+k\omega_{cp},$$
$$p_{2in} = \beta_i q_{2in}, \qquad p_{1in} = \beta_{i+2}q_{2in}-\beta_i q_{1in},$$
$$q_{1in} = \sum_{l=1}^{n} \exp[\alpha_i(l-1)T]\Bigg\{(l-1)T\Bigg(\frac{\cos[\omega_{l-1}(l-1)T+\psi_{l-1}+\varphi_{i,l-1}]}{\sqrt{a_i^2+\omega_{l-1}^2}}$$
$$-\frac{\cos[\omega_l(l-1)T+\psi_l+\varphi_{il}]}{\sqrt{\alpha_i^2+\omega_l^2}}\Bigg)-\frac{\cos[\omega_{l-1}(l-1)T+\psi_{l-1}+2\varphi_{i,l-1}]}{\alpha_i^2+\omega_{l-1}^2}$$
$$-\frac{\cos[\omega_l(l-1)T+\psi_l+2\varphi_{il}]}{\alpha_i^2+\omega_l^2}\Bigg\},$$
$$q_{2in} = \sum_{l=1}^{n} \exp[\alpha_i(l-1)T]\Bigg(\frac{\cos[\omega_{l-1}(l-1)T+\psi_{l-1}+\varphi_{i,l-1}]}{\sqrt{\alpha_i^2+\omega_{l-1}^2}}$$
$$-\frac{\cos[\omega_l(l-1)T+\psi_l+\varphi_{il}]}{\sqrt{\alpha_i^2+\omega_l^2}}\Bigg),$$
$$\cos\varphi_{il} = \frac{\alpha_i}{\sqrt{\alpha_i^2+\omega_l^2}}, \qquad \sin\varphi_{il} = -\frac{\omega_l}{\sqrt{\alpha_i^2+\omega_l^2}},$$
$$c_n = \sum_{i=1}^{2}\left(\beta_{i+2}\frac{\cos\varphi_{in}}{\sqrt{\alpha_i^2+\omega_n^2}}+\beta_i\frac{\cos 2\varphi_{in}}{\alpha_i^2+\omega_n^2}\right),$$
$$s_n = \sum_{i=1}^{2}\left(\beta_{i+2}\frac{\sin\varphi_{in}}{\sqrt{\alpha_i^2+\omega_n^2}}+\beta_i\frac{\sin 2\varphi_{in}}{\alpha_i^2+\omega_n^2}\right).$$

The formula (3.5) for $X_p(t)$ clearly shows how the past information on digits influences the value of the signal $X_p(t)$.

([3]) We use the following equalities:

$$\int xe^{ax}\cos(bx+c)dx = \frac{1}{\sqrt{a^2+b^2}}xe^{ax}\cos(bx+c+\varphi)-\frac{1}{a^2+b^2}e^{ax}\cos(bx+c+2\varphi),$$
$$\int e^{ax}\cos(bx+c)\,dx = e^{ax}\frac{1}{\sqrt{a^2+b^2}}\cos(bx+c+\varphi),$$

where $\sin\varphi = -b/\sqrt{a^2+b^2}$ and $\cos\varphi = a/\sqrt{a^2+b^2}$.

Now we attempt to analyze the zero-crossings of $X_p(t)$.

Note that $X_p(t)$, as the signal of carrier frequency, may be represented in the form

$$X_p(t) = A_p(t)\cos[\omega_i t + \psi_{l+1} + \varphi_p(t)], \tag{3.6}$$

where $lT \leqslant t < (l+1)T$, $i = a_{l+1}$, $\omega_i = \omega_c + b_{l+1}\Delta\omega$, $A_p(t)$ is the envelope of the signal, and $\psi_{l+1} + \varphi_p(t)$ is the phase of the signal.

$A_p(t)$ and $\varphi_p(t)$ depend on the channel characteristics and on the entire data sequence transmitted by the data source. Hence in the channel with intersymbol interferences we may compute $A_p(t)$ and $\varphi_p(t)$ only by a digital computer.

Let t_n be the n-th zero-crossing of $X_p(t)$, i.e., t_n represents the value of t such that

$$\omega_i t_n + \psi_l + \varphi_p(t_n) = -\pi/2 + 2m_n\pi, \tag{3.7}$$

where each of the integers l and m_n depends on n. Hence we obtain ([4])

$$t_n = \frac{-\pi/2 + 2m_n\pi - \psi_l - \varphi_{pn}}{\omega_i}, \tag{3.8}$$

where $\varphi_{pn} = \varphi_p(t_n)$.

In the next section we attempt to analyze the zero-crossings of the combined signal and noise.

4. Additive Gaussian noise. Since the additive Gaussian noise $n(t)$ has been band limited by the receiving filter, at the input of the discriminator we can assume that it is a narrow-band Gaussian noise with zero mean and variance σ_s^2. Thus the noise at the input of the discriminator can be represented by

$$n_p(t) = B_p(t)\cos[\omega_i t + \alpha_p(t)], \tag{4.1}$$

where $lT \leqslant t < (l+1)T$, $i = a_{l+1}$, and $B_p(t)$ and $\alpha_p(t)$ are the envelope and the phase of the noise, respectively. We made the assumption that the signal-to-noise ratio at any instant t, defined as the signal power divided by the noise average power, i.e., $A_p^2(t)/(2\sigma_s^2)$, is large. Then the input to the frequency discriminator after adding the noise is

$$S'_p(t) = \mathrm{E}_p(t)[\cos\omega_i t + \Psi_p(t)], \tag{4.2}$$

([4]) In the absence of the transmitting filter and the receiving filter, for the zero-crossings from the l-th interval $((l-1)T,\ lT]$ we have the equality $\varphi_{pl} = 0$ and the integers m_n in (3.8) are all integers from the closed interval on the endpoints:

$$\operatorname{entier}\left(\frac{(l-1)T\omega_i + \psi_l}{2\pi} + \frac{5}{4}\right) \quad \text{and} \quad \operatorname{entier}\left(\frac{lT\omega_i + \psi_l}{2\pi} + \frac{1}{4}\right).$$

If the above filters are present, then the zero-crossings, lying near the above-described ones if the filters are ignored, can be easily found by a digital computer from (3.5).

where $lT \leqslant t < (l+1)T$, $\Psi_p(t)$ has a Gaussian distribution with mean $\psi_l + \varphi_p(t)$ and variance $\sigma_s^2/A_p^2(t)$ (see [5] and [12]). Let Y_n be the n-th zero-crossing of $S_p(t)$. It is easy to establish the equality

(4.3) $$\omega_i Y_n + \Psi_p(Y_n) = -\pi/2 + 2m_n\pi.$$

Hence Y_n is a random variable which has a Gaussian distribution with mean t_n and standard deviation σ_n equal to

(4.4) $$\sigma_n = \frac{\sigma_s}{A_p(t_n)\omega_i}.$$

Setting

$$A_s^2 = \overline{A_p^2(t)} = \lim_{\theta\to\infty} \frac{1}{\theta} \int_{-\theta/2}^{\theta/2} A_p^2(t)dt,$$

we can write σ_n in the form

(4.5) $$\sigma_n = \frac{A_s}{A_p(t_n)\omega_i\sqrt{2S/N}},$$

where S/N, called the *signal-to-noise ratio*, is a ratio of the average signal power to the average noise power at the input of the discriminator, i.e., $S/N = A_s^2/(2\sigma_s^2)$. Consequently, each of the random variables ε_n defined as

(4.6) $$\varepsilon_n = Y_n - t_n$$

has a Gaussian distribution with zero mean and standard deviation σ_n given by (4.5) with $i = a_{l+1}$. Obviously, we can assume that the random variables $\{\varepsilon_n\}$ are all independent of each other.

5. Actual frequency discriminator. Using the variables $\{\varepsilon_n\}$ defined in (4.6), we can write both the signal and the noise at the output of the pulse generator of the discriminator as follows:

(5.1) $$S_z(t) = \sum_{n=1}^{N} ag(t-t_n-\varepsilon_n),$$

where a is the amplitude, $g(t)$ is a rectangular pulse of duration z_0, i.e., $g(t)$ equals 1 for $0 \leqslant t \leqslant z_0$, and 0 otherwise.

Next, the output of the discriminator is given by the expression

(5.2) $$S_h(t) = S_z(t) * h(t),$$

where $h(t)$ is the impulse response of the low-pass filter of the discriminator. We now attempt to analyze the expectation of $S_h(t)$ and the standard deviation of $S_h(t)$.

5.1. *Expectation of the signal at the output of the discriminator.* From (5.1) and (5.2) we conclude that

(5.3) $$S_h(t) = \sum_{n=1}^{N} a \int_{-\infty}^{\infty} g(t-t_n-\varepsilon_n-u)h(u)du.$$

For simplicity of the notation, let the function g_σ be defined by

$$(5.4) \qquad g_\sigma(w) = \int_{-\infty}^{\infty} g(w-\varepsilon-u)h(u)du,$$

where $w \in R$, $\sigma > 0$, ε is a Gaussian random variable with zero mean and variance σ^2. Then the expectation of $S_h(t)$, denoted by $X(t)$, can be written, by (5.3), as follows:

$$(5.5) \qquad X(t) = a \sum_{n=1}^{N} \mathrm{E} g_{\sigma_n}(t-t_n).$$

It is easy to check the equality

$$(5.6) \qquad \mathrm{E}g(w-\varepsilon) = \phi\left(\frac{w}{\sigma}\right) - \phi\left(\frac{w-z_0}{\sigma}\right),$$

where

$$\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy.$$

Consider the function h defined by

$$(5.7) \qquad h(t) = \begin{cases} \alpha^2 t e^{-\alpha t} & \text{for } t \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then from (5.4) and (5.6) we obtain

$$(5.8) \qquad \mathrm{E}g_\sigma(w) = A_\sigma(w) - A_\sigma(w-z_0),$$

where

$$A_\sigma(w) = \phi\left(\frac{w}{\sigma}\right) - \frac{\sigma\alpha}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2\sigma^2}\right) - [\alpha(w-\sigma^2\alpha)+1] \exp\left[-\alpha\left(w-\frac{\sigma^2\alpha}{2}\right)\right] \phi\left(\frac{w-\sigma^2\alpha}{\sigma}\right).$$

Substituting this into (5.5) we see that the expectation of $S_h(t)$ is

$$(5.9) \qquad X(t) = a \sum_{n=1}^{N} [A_{\sigma_n}(t-t_n) - A_{\sigma_n}(t-t_n-z_0)].$$

From (5.9) we notice that the value of $X(t)$ at time t, given $\{a_n\}$ and S/N, depends on the initial phase ψ_0. Let $\overline{X(t)}$ be the average expectation which is obtained by averaging $X(t)$ with respect to ψ_0. If we assume that ψ_0 is uniformly distributed in the interval between 0 and 2π, then

$$(5.10) \qquad \overline{X(t)} = \frac{1}{2\pi} \int_0^{2\pi} X(t) d\psi_0.$$

First we compute $\overline{X(t)}$, denoted by A_i, when each of the digits generated by the data source has the value i. Note that, even in this case, $X(t)$ changes in time because of the integration of the pulse train in the low-pass filter of the discriminator. From (5.5) and (5.10) we obtain

$$A_i = a\left[\int_0^{2\pi} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{E}g(t-t_n-\varepsilon_n)\,d\psi_0\right] * h(t), \tag{5.11}$$

where, in the formula (3.8) for t_n, $\psi_l = \psi_0$, $m_n = n$, each of $\varphi_{pn}/\omega_i = \Delta\psi$ and $\sigma_n = \sigma$ is independent of n (for ease of calculation we assume that in this case the data source generates digits from $-\infty$ to ∞).

The integral in (5.11) is equal to

$$\int_0^{2\pi} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{t-t_n-z_0}^{t-t_n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx\, d\psi_0$$

$$= \frac{\omega_i}{2\pi} \sum_{n=-\infty}^{\infty} \int_{nT_i+\Delta\eta}^{(n+1)T_i+\Delta\eta} \int_{t-u-z_0}^{t-u} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx\, du = \frac{z_0}{T_i},$$

where $\Delta\eta = (-\pi/2+\Delta\psi)/\omega_i$ and $T_i = 2\pi/\omega_i$. Substituting this into (5.11) we obtain

$$A_i = \frac{az_0}{T_i}. \tag{5.12}$$

In general, given $\{a_n\}$, $\overline{X(t)}$ with a very good approximation can be written in the form

$$X(t) = p_\alpha(t-lT)A_{a_l} + [1-p_\alpha(t-lT)]\,A_{a_{l+1}}, \tag{5.13}$$

where $t \in (lT, (l+1)T]$, and

$$p_\alpha(x) = \int_{-\infty}^{0} h(x-u)\,du = (1+\alpha x)\exp(-\alpha x) \quad \text{for } x > 0.$$

Let $\sigma(X(t))$ be the standard deviation of $X(t)$ (from $\overline{X(t)}$) defined as follows:

$$\sigma(X(t)) = \sqrt{\int_0^{2\pi} \frac{1}{2\pi}(X(t)-\overline{X(t)})^2\, d\psi_0}. \tag{5.14}$$

As may be seen in Fig. 2, $X(t)$ has a uniform distribution (an approximation of course). Thus

$$\max_{\psi_0}(X(t)-\overline{X(t)}) = \sqrt{3}\,\sigma(X(t)). \tag{5.15}$$

5.2. *The variance of the signal at the output of the discriminator.* Let $\sigma^2(t)$ denote the variance of the signal at the output of the discriminator, i.e., $\sigma^2(t) = \sigma^2(S_h(t))$. From (5.3) and (5.4) we obtain

$$\sigma^2(t) = a \sum_{n=1}^{N} \left[\mathrm{E}g^2_{\sigma_n}(t-t_n) - (\mathrm{E}g_{\sigma_n}(t-t_n))^2\right]. \tag{5.16}$$

By (5.8), it is sufficient to find the first term in the brackets of (5.16).

It is easy to check that for $0 < z < z_0$ the following equalities hold:

$$E[g(w-\varepsilon-u)g(w-\varepsilon-u-z)] = \phi\left(\frac{u-w-z_0}{\sigma}\right)-\phi\left(\frac{u-w+z}{\sigma}\right),$$

(5.17)
$$E[g(w-\varepsilon-u)g(w-\varepsilon-u+z)] = \phi\left(\frac{u-w+z_0-z}{\sigma}\right)-\phi\left(\frac{u-w}{\sigma}\right).$$

Consequently, after easy computations we get

$$\begin{aligned}
Eg_\sigma^2(w) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[g(w-\varepsilon-u)g(w-\varepsilon-z)]h(u)h(z)\,du\,dz \\
&= \int_0^{z_0}\int_{-\infty}^{\infty}\left[\phi\left(\frac{u-w+z_0}{\sigma}\right)-\phi\left(\frac{u-w+z}{\sigma}\right)\right]h(u)h(u+z)\,du\,dz \\
&\quad+\int_0^{z_0}\int_{-\infty}^{\infty}\left[\phi\left(\frac{u-w+z_0-z}{\sigma}\right)-\phi\left(\frac{u-w}{\sigma}\right)\right]h(u)h(u-z)\,du\,dz \\
&= \int_0^{z_0}\int_{-\infty}^{\infty}\int_{u-w+z}^{u-w+z_0}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma^2}\right)h(u)h(u+z)\,dy\,du\,dz \\
&\quad+\int_0^{z_0}\int_{-\infty}^{\infty}\int_{u-w}^{u-w+z_0-z}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma^2}\right)h(u)h(u-z)\,dy\,du\,dz \\
&= \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma^2}\right)\left\{\int_{y+w-z_0}^{y+w} h(u)\left[\int_{y+w-u-z_0}^{y+w-u} h(u+z)\,dz\right]du\right\}dy \\
&= \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma^2}\right)\left\{\left[\int_{y+w-z_0}^{y+w} h(u)\,du\right]\left[\int_{y+w-z_0}^{y+w} h(x)\,dx\right]\right\}dy \\
&= \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma^2}\right)\{[H(y+w)-H(y+w-z_0)]^2\}\,dy,
\end{aligned}$$

where $H(y)$ is the function defined by the formula

$$H(y) = \int_{-\infty}^{y} h(u)\,du.$$

For $h(t)$, given by (5.7), $H(y)$ is equal to $1-(1+\alpha y)\exp(-\alpha y)$ for $y > 0$, and 0 otherwise. Hence we obtain the following final form for $Eg_\sigma^2(w)$:

(5.18)
$$Eg_\sigma^2(w) = B_\sigma(w)-B_\sigma(w-z_0)-2G_\sigma(w),$$

where

$$\begin{aligned}
B_\sigma(w) = \exp(-2\alpha w+2\sigma^2\alpha^2)&\left\{\sigma\alpha[\alpha(w-2\sigma^2\alpha)+2]f\left(\frac{w-2\sigma^2\alpha}{\sigma}\right)\right. \\
&\left.+[1+2\alpha(w-2\sigma^2\alpha)+\alpha^2(w-2\sigma^2\alpha)^2+\sigma^2\alpha^2]\phi\left(\frac{w-2\sigma^2\alpha}{\sigma}\right)\right\} \\
&+\exp(-\alpha w)\left\{-2\sigma\alpha f\left(\frac{w-\sigma^2\alpha}{\sigma}\right)\right.
\end{aligned}$$

$$-2[\alpha(w-\sigma^2\alpha)+1]\phi\left(\frac{w-\sigma^2\alpha}{\sigma}\right)\Big\}+\phi\left(\frac{w}{\sigma}\right),$$

$$G_\sigma(w) = \exp(-2\alpha w+2\sigma^2\alpha^2+\alpha z_0)\Big\{\sigma\alpha[\alpha(w-2\sigma^2\alpha)+2]f\left(\frac{w-z_0-2\sigma^2\alpha}{\sigma}\right)$$

$$+[\sigma^2\alpha^2+1-\alpha z_0+(w-2\sigma^2\alpha)(2\alpha-\alpha^2 z_0)$$

$$+\alpha^2(w-2\sigma^2\alpha)^2]\phi\left(\frac{w-z_0-2\sigma^2\alpha}{\sigma}\right)\Big\}$$

$$+\exp\left(-\alpha w+\frac{\sigma^2\alpha^2}{2}\right)\Big\{-\sigma\alpha[1+\exp(\alpha z_0)]f\left(\frac{w-z_0-\sigma^2\alpha}{\sigma}\right)$$

$$+[\alpha z_0\exp(\alpha z_0)-\exp(\alpha z_0)-1-\alpha][1+\exp(\alpha z_0)]$$

$$\times(w-\sigma^2\alpha)\phi\left(\frac{w-z_0-\sigma^2\alpha}{\sigma}\right)\Big\}+\phi\left(\frac{w-z_0}{\sigma}\right),$$

$$f(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right).$$

Substituting (5.8) and (5.18) into (5.16) the following final form for $\sigma^2(t)$ is obtained:

$$(5.19)\qquad \sigma^2(t) = \sum_{n=1}^{N}\{B_{\sigma_n}(t-t_n)-B_{\sigma_n}(t-t_n-z_0)-2G_{\sigma_n}(t-t_n)$$
$$-(A_{\sigma_n}(t-t_n)-A_{\sigma_n}(t-t_n))^2\}.$$

In order to obtain numerical results, the expressions obtained in this section are evaluated for the special case. The FSK system was operated at 1200 bits per second.

Both the sending filter and the receiving filter are a combination of a high-pass filter and a low-pass filter, each of them has a slope 6 dB per octave and the limit frequencies 1100 Hz and 2300 Hz, respectively. The low-pass filter of the discriminator has the slope 12 dB per octave and the limit frequency 800 Hz. It is assumed that the data source generates from $-\infty$ to 0 the 0 digit and starts transmitting at time zero the sequence $\{1, 0, 0, 0, 1, 0, \ldots\}$.

Figs. 2a and 2b show curves for the expectation $X(t)$, the average expectation $\overline{X(t)}$ and the standard deviation $\sigma(t)$ of the signal $S_h(t)$ at the output of the discriminator for S/N equaling to 7 dB. The curves are plotted for bits with sign change in comparison with the past bit.

In Fig. 3 the standard deviation $\sigma(t)$, which is averaged over the initial phase, is plotted as a function of the signal-to-noise ratio S/N.

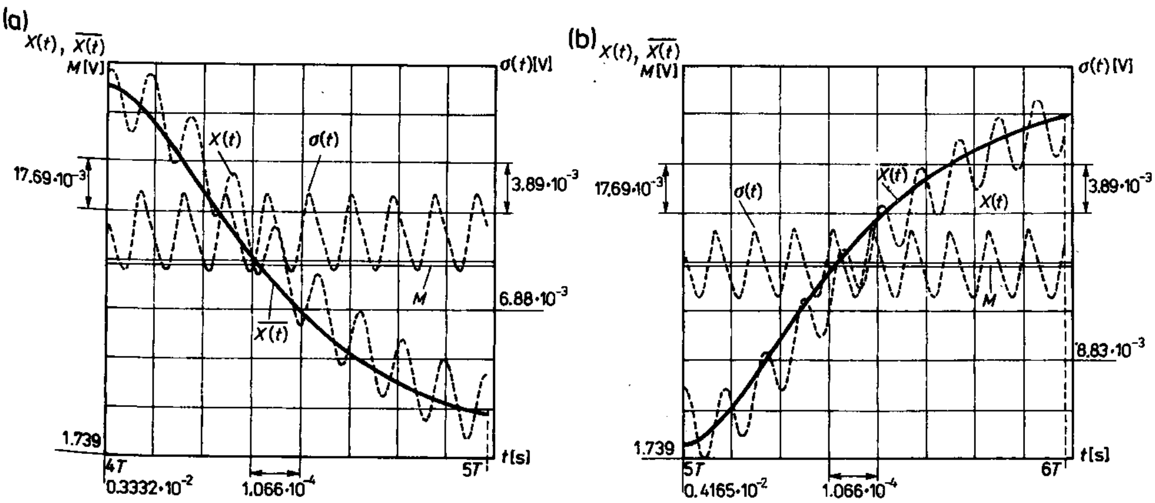

Fig. 2. The expectation, the average expectation and the standard deviation of the signal at the output of the discriminator when $S/N = 7$dB, $\delta_\omega = 0$rad/s, M is a threshold, the sequence of digits generated by the data source from time zero is equal to $\{1, 0, 0, 0, 1, 0, 0, 0, 1, \ldots\}$
(a) for 5-bit, (b) for 6-bit

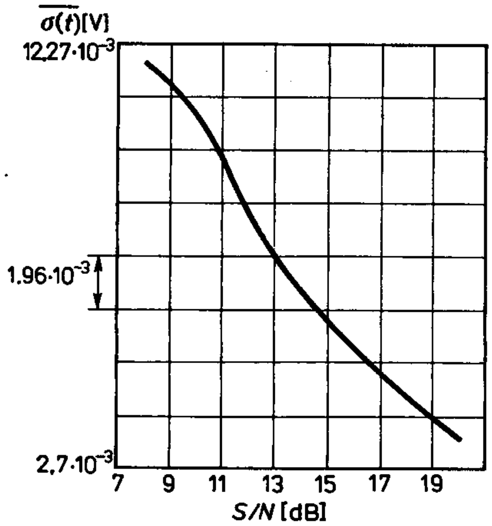

Fig. 3. Dependence of the standard deviation of the signal at the output of the discriminator on the signal-to-noise ratio S/N

6. The error probability as a function of the signal-to-noise ratio. Let M be a decision threshold. Since the discriminator is adjusted to demodulate the signals, which are not distorted by a pulsation shift δ_ω, we have

$$M = \frac{az_0}{2\pi}(\omega_c - \delta_\omega).$$

The detection of the digit a_n is clearly based on the output $S_h(t)$ of the sampler at time $t = T_n = nT + T/2 + \Delta T$, where ΔT is a constant time delay of the transmission. ΔT is produced by the discriminator. That is,

we decide $a_n = 0$ if $S_h(T_n) \geqslant M$,

we decide $a_n = 1$ if $S_h(T_n) < M$.

The probabilities of a false detection, given ψ_0, $\{a_n\}$ and $a_n = 1$ or $a_n = 0$, are determined by the following expressions, respectively:

$$\begin{aligned}(6.1)\quad &\Pr[\text{error} \mid \psi_0, \{a_n\}, a_n = 1] = \Pr[S_h(T_n) \geqslant M \mid \psi_0, \{a_n\}, a_n = 1],\\ &\Pr[\text{error} \mid \psi_0, \{a_n\}, a_n = 0] = \Pr[S_h(T_n) < M \mid \psi_0, \{a_n\}, a_n = 0].\end{aligned}$$

Consider the signal $S_h(t)$ in the form

$$S_h(t) = X(t) + \xi(t), \tag{6.2}$$

where $\xi(t)$ is a function depending on the noise. Note that the spectrum of the wide-band signal $S_z(t)$ at the input of the low-pass filter of the discriminator is far more wide than the band-pass of this filter. Then, following Levin [5] (p. 301), for each t the distribution of the random variable $S_h(t)$, and consequently $\xi(t)$, can be considered to be approximately equal to the Gaussian distribution.

Thus, the error probabilities given by (6.1) may be rewritten in the form

$$\begin{aligned}(6.3)\quad &\Pr[\text{error} \mid \psi_0, \{a_n\}, a_n = 1] = \int_M^\infty \frac{1}{\sigma(T_n)\sqrt{2\pi}} \exp\left[-\frac{(x - X(T_n))^2}{2\sigma^2(T_n)}\right] dx,\\ &\Pr[\text{error}] \mid \psi_0, \{a_n\}, a_n = 0] = \int_{-\infty}^M \frac{1}{\sigma(T_n)\sqrt{2\pi}} \exp\left[-\frac{(x - X(T_n))^2}{2\sigma^2(T_n)}\right] dx,\end{aligned}$$

and, defining $\operatorname{erfc}(x) = 1 - \phi(x)$, they are equal to

$$\Pr[\text{error} \mid \psi_0, \{a_n\}] = \operatorname{erfc}\left[\frac{|X(T_n) - M|}{\sigma(T_n)}\right]. \tag{6.4}$$

By (4.5), the probability of a false detection expressed by the formula (6.4) is a function of the signal-to-noise ratio S/N defined as previously.

Substituting $X(T_n)$ and $\sigma(T_n)$ from the formulas (5.9) and (5.19), respectively, into the expression (6.4), we see how past information digits and the initial phase ψ_0 (5) influence the probability of a false detection. Averaging the expression (6.4) over all ψ_0 and $\{a_n\}$ we obtain the error probability.

Averaging over all data sequences and initial phases is a more formidable task and is not of much practical interest. We find an upper bound of the error probability.

The numerator of (6.4) can be written as follows:

$$X(T_n) - M = [X(T_n) - \overline{X(T_n)}] + [\overline{X(T_n)} - A_{a_n}] + [A_{a_n} - M].$$

Then we obtain

$$\begin{aligned}(6.5)\quad &\Pr[\text{error} \mid \psi_0, \{a_n\}, a_n = i]\\ &= \operatorname{erfc}\left\{\frac{(az_0/2\pi)[\Delta\omega + (1-2i)\delta_\omega] - |\overline{X(T_n)} - A_i| + (1-2i)(X(T_n) - \overline{X(T_n)})}{\sigma(T_n)}\right\}.\end{aligned}$$

(5) Note that in the FSK system with ideal discriminator the probability of a false detection is independent of the initial phase (see [7], p. 873).

Note that in (6.5) only $(X(T_n)-\overline{X(T_n)})$ and $\overline{\sigma(T_n)}$ depend on ψ_0. By a numerical search we see that the upper bound of the error probability over ψ_0 is one order of magnitude greater than the error probability averaged over ψ_0 (this is an approximation of course), and it is equal to

(6.6) $$\sup_{\psi_0} \Pr[\text{error} \mid \{a_n\}, a_n = i]$$
$$= \operatorname{erfc}\left\{\frac{(az_0/2\pi)[\Delta\omega+(1-2i)\delta_\omega]-|X(T_n)-A_i|-\sup_{\psi_0}|X(T_n)-\overline{X(T_n)}|}{\overline{\sigma(T_n)}}\right\}.$$

Thus, the upper bound of the error probability with respect to ψ_0 and $\{a_n\}$ can now be calculated by the formula

(6.7) $$\sup_{\psi_0,\{a_n\}} \Pr[\text{error} \mid a_n = i]$$
$$= \operatorname{erfc}\left\{\frac{(az_0/2\pi)[\Delta\omega+(1-2i)\delta_\omega]-\sup_{\{a_n\}}|\overline{X(T_n)}-A_i|-\sup_{\psi_0}|X(T_n)-\overline{X(T_n)}|}{\overline{\sigma(T_n)}}\right\}.$$

The above expression enables us to calculate the upper bound of the error probability given by the expression

$$\sup_{\psi_0,\{a_n\}} \Pr[\text{error}] \leqslant \sup_{\psi_0,\{a_n\}} \Pr[\text{error} \mid a_n = 0] + \sup_{\psi_0,\{a_n\}} \Pr[\text{error} \mid a_n = 1]$$

and, consequently, the average error probability can also be approximated.

6.1. *Concluding comments.* In this section we illustrate important conclusions from the obtained formulas on the error probability. In order to obtain

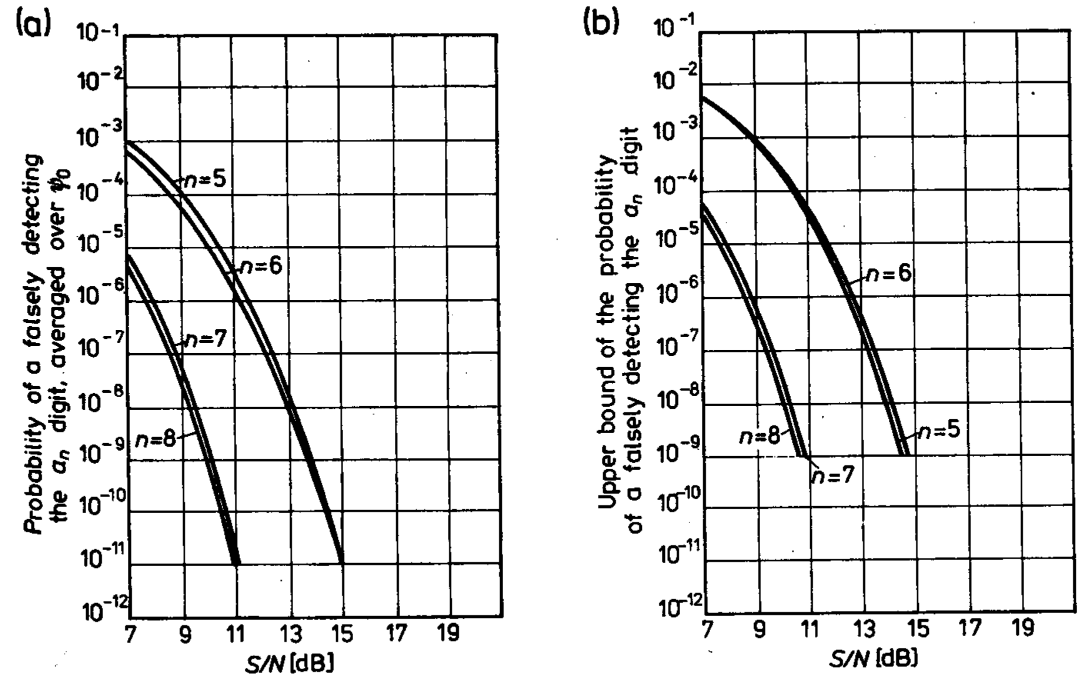

Fig. 4. Probability of falsely detecting the a_n digit for $\{a_n\} = \{1, 0, 0, 0, 1, 0, 0, 0, 1, \ldots\}$ (a) the average with respect to ψ_0, (b) the upper bound with respect to ψ_0

numerical results, the expressions for the error probability averaged over ψ_0 and the upper bound of the error probability are evaluated for the special case described previously. It is assumed that the data source generates from $-\infty$ to 0 the 0 digit, and starts transmitting at time zero the sequence $\{1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, \ldots\}$. The results, as functions of the signal-to-noise ratio S/N, are plotted in Figs. 4a and 4b.

As may be seen in Figs. 4a and 4b, the probability of falsely detecting the a_n digit depends not only on the past digit a_{n-1} but also on the digit a_{n-2}. At the change of the logical value, the probability of a false detection increases by at least two orders of magnitude.

In Figs. 5a and 5b the probability of a false detection of the 0 digit, the 1 digit and the error probability, given $\delta_\omega = 0$ rad/s and $\delta_\omega = 120\pi$ rad/s, are plotted. For the zero spectrum shift δ_ω, the difference between the probabilities of a false detection of the 0 digit and the 1 digit is small and approximately equals one order of magnitude. This difference increases fast with the spectrum shift δ_ω increasing.

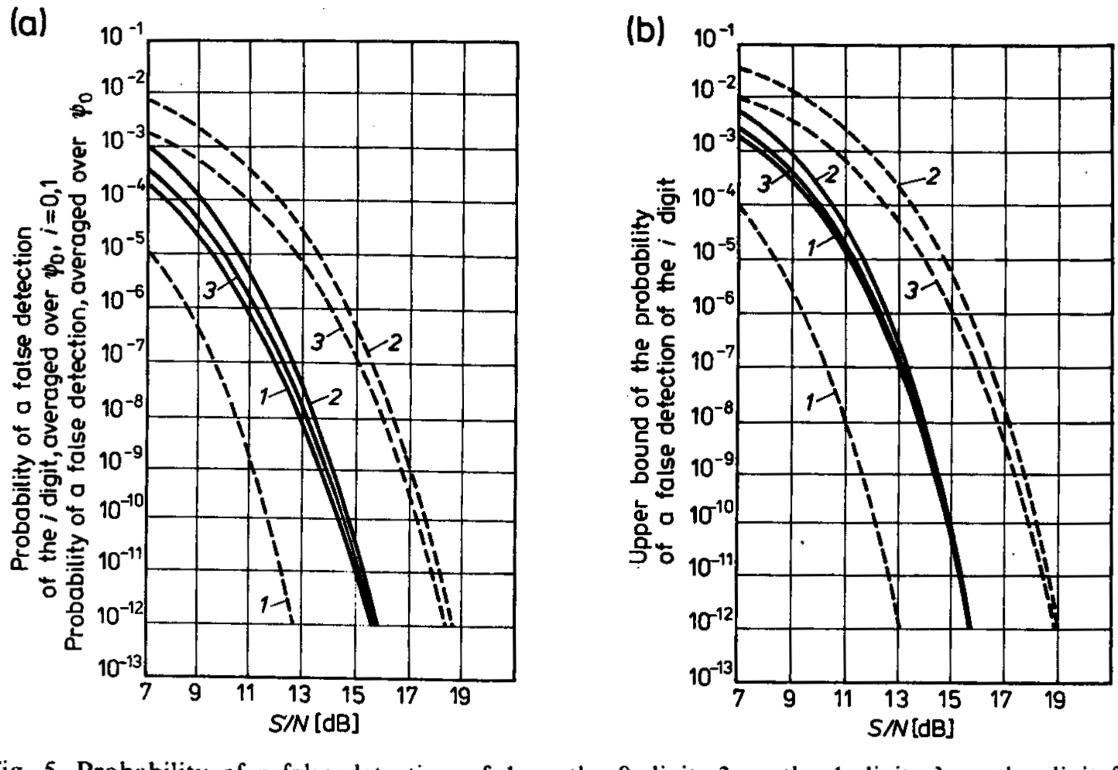

Fig. 5. Probability of a false detection of *1* — the 0 digit, *2* — the 1 digit, *3* — the digit for $\{a_n\} = \{1, 0, 0, 0, 1, 0, 0, 0, 1, \ldots\}$
(a) the average with respect to ψ_0, (b) the upper bound with respect to ψ_0
——— $\delta_\omega = 0$ rad/s, ----- $\delta_\omega = 120\pi$ rad/s

As may be seen in Figs. 6a and 6b, the spectrum shift δ_ω influences strongly the performance of the system. Taking into account the result obtained in the next section and comparing the error probability for several values of δ_ω, plotted in Fig. 6b, and the experimental results included in [9], we see that the theoretical results obtained in this paper and the experimental results are conformable.

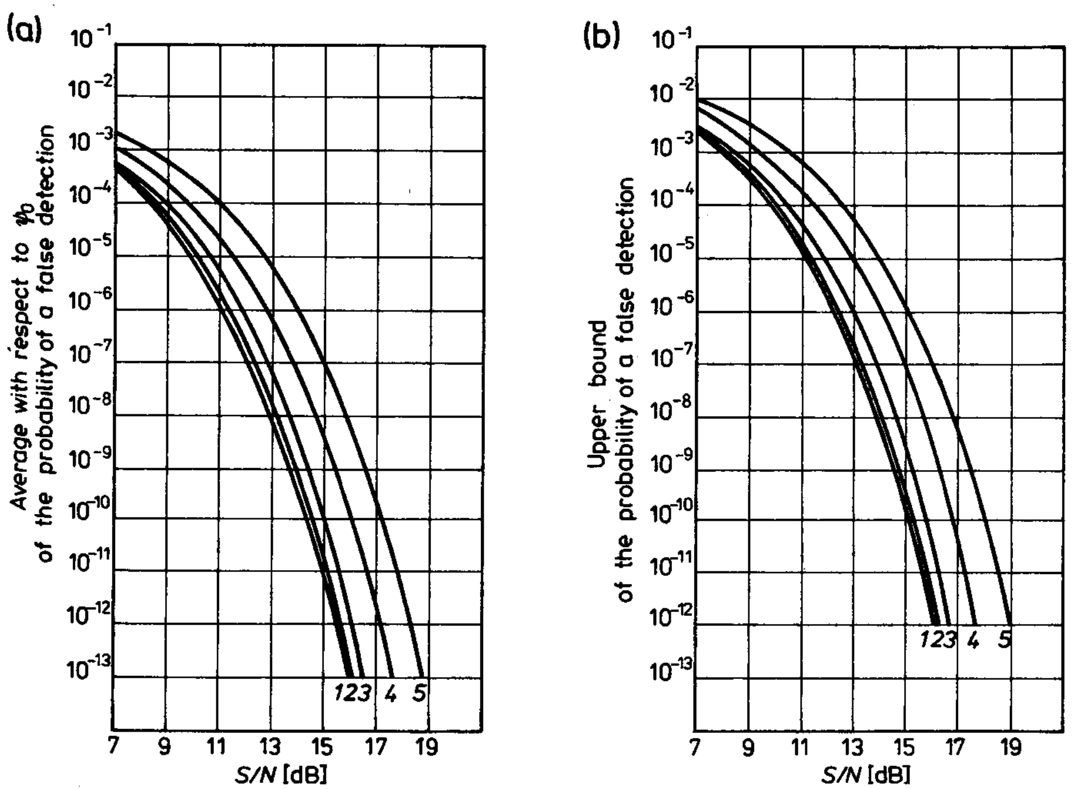

Fig. 6. Influence of the spectrum shift δ_ω on the probability of a false detection. (a) the average with respect to ψ_0, (b) the upper bound with respect to ψ_0 for $\{a_n\} = \{1, 0, 0, 0, 1, 0, 0, 0, 1, \ldots\}$ and for δ_ω = 0, 20, 40, 80, 120 rad/s (curves *1*, *2*, *3*, *4*, *5*, respectively)

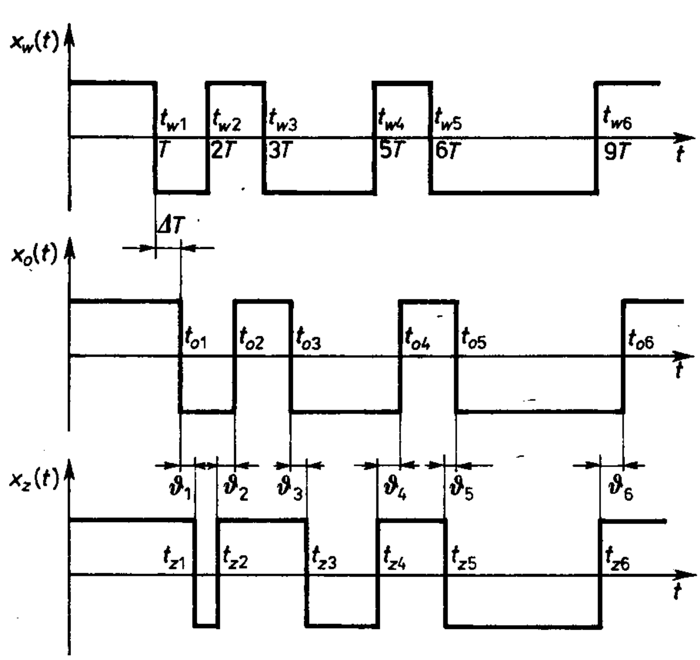

Fig. 7. Time distortion effects

$x_w(t)$ — the standard signal, $x_0(t)$ — the reference standard signal, $x_z(t)$ — the real signal, $\vartheta_1, \ldots, \vartheta_6$ — elementary time deflections corresponding to elementary time distortions, ΔT — the delay of the transmission

7. Time distortions at the output of the discriminator. The error probability.

7.1. *Introduction.* First we give some basic definitions (see [8] and [14]). Let $x_0(t)$ and $x_z(t)$, called the *reference standard signal* and the *real signal*, respectively, be signals generated in the receiver consisting of a train of pulses, each pulse having the amplitude either A or 0 depending on whether it represents a mark or a space. Such a signal generated by the data source is called the *standard signal* and denoted by $x_w(t)$. The time moments at which the signal changes the value of the generated digit are called *characteristic moments* of the signal. The difference between the k-th characteristic moments t_{zk} and t_{0k} of the signals $x_z(t)$ and $x_0(t)$, respectively, is called the *k-th elementary time deflection* ϑ_k of the real signal $x_z(t)$ with respect to the signal $x_0(t)$, i.e., $\vartheta_k = t_{zk} - t_{0k}$. The elementary time deflections divided by T are called *elementary time distortions* of the signal $x_z(t)$ with respect to the signal $x_0(t)$.

Next, we analyze various time distortions at the output discriminator.

7.2. *Bias distortions.* Let $\bar{x}_z(t)$, given δ_ω, be the real signal reproduced from the average expectation $\overline{X(t)}$ of the signal at the output of the discriminator. If $\delta_\omega = 0$, the signal $\bar{x}_z(t)$ is called the *reference standard signal* and denoted by $x_0(t)$; in this case $\overline{X(t)}$ is denoted by $\overline{X^0(t)}$. The discriminator introduces a delay of transmission by the constant value ΔT. Thus the signal $x_0(t)$ is the shift of the standard signal $x_w(t)$ on the time axis by the value ΔT.

The elementary time distortions of the signal $\bar{x}_z(t)$ with respect to the signal x_0 bělong to the class of bias distortions (see [8]–[10]). These distortions follow from the spectrum shift δ_ω.

Let δ be a constant satisfying the following two conditions:

$$\operatorname{sgn}\delta = \operatorname{sgn}\delta_\omega \quad \text{and} \quad |\delta| = |\Delta T - D_M|,$$

where D_M and ΔT represent the values of t such that $\overline{X(D_M)} = M$ and $\overline{X^0(\Delta T)} = M$ (see Fig. 7). Then the bias distortions are a random variable which equals δ_s or $-\delta_s$ with equal probability, where $\delta_s = \delta/T$.

In Fig. 8 we show a comparison of the signals $x_w(t)$, $x_0(t)$ and $\bar{x}_z(t)$, and consequently the time deflections $\bar{x}_z(t)$ with respect to $x_0(t)$, corresponding to the bias distortions. It is easy to establish that for the parameters D_M and ΔT the equalities

$$p_\alpha(\Delta T) = \tfrac{1}{2}, \qquad p_\alpha(D_M) = (\Delta\omega + \delta_\omega)/(2\Delta\omega)$$

hold, which for $h(t)$ given by (5.7) imply

$$(7.1) \qquad (1+\alpha\Delta T)\exp(-\alpha\Delta T) = \tfrac{1}{2}, \qquad (1+\alpha D_M)\exp(-\alpha D_M) = \frac{\Delta\omega+\delta_\omega}{2\Delta\omega}.$$

The values ΔT and D_M from the equations (7.1) have been found by numerical computations by a digital computer. Hence an approximation is now found. Note that for $|\delta|$ the following approximate equality holds:

$$(7.2) \qquad |\delta| = \left(\overline{X(t)} - \overline{X^0(t)}\right)\left|\frac{d\overline{X(t)}}{dt}\bigg|_{t=\Delta T}\right|^{-1}.$$

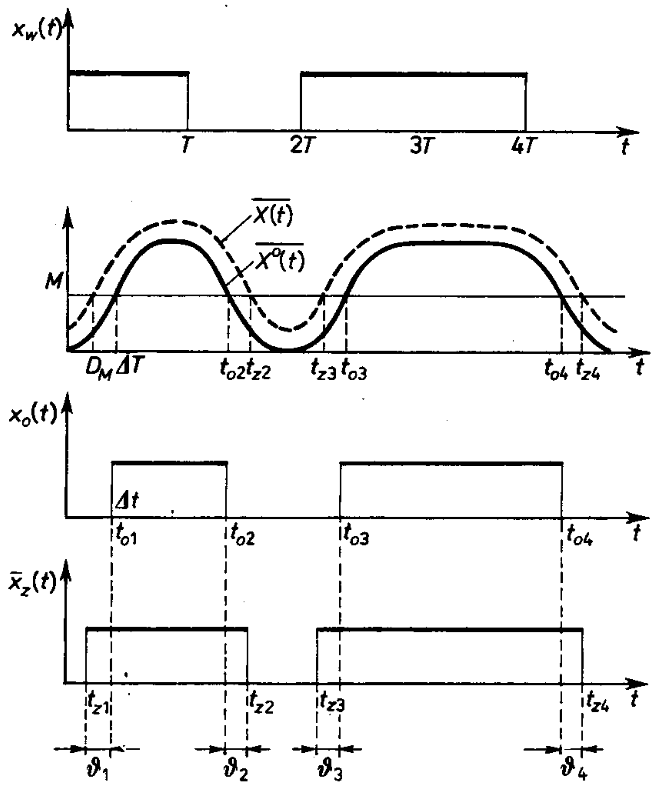

Fig. 8. Comparison of the signals: the standard signal $x_w(t)$, the reference standard signal $x_0(t)$, and the real signal $\bar{x}_z(t)$

$\vartheta_1, \ldots, \vartheta_4$ — time deflections corresponding to bias distortions at the output of the discriminator

From this it follows that, for $h(t)$ in (5.7), $\delta = \delta_\omega / C_1$ and

$$\delta_s = \delta_\omega / (C_1 T), \tag{7.3}$$

where $C_1 = |2\exp(-\alpha \Delta T) - 1| \alpha \Delta \omega$.

The formula (7.3) enables us to compute with a very good approximation the value of δ_s. In Fig. 9 the absolute value of δ_s is plotted as a function of δ_ω.

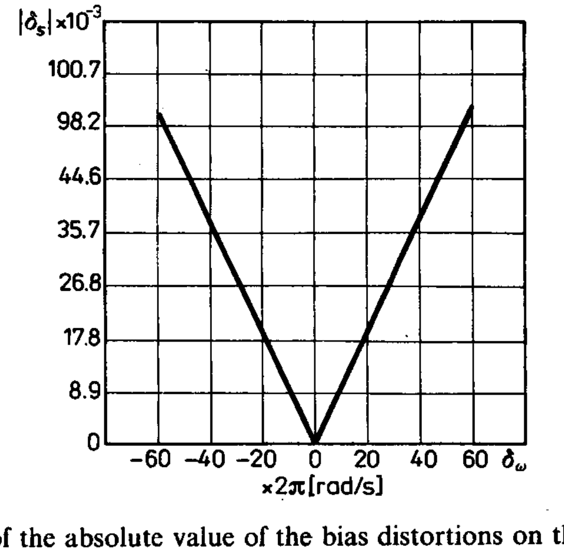

Fig. 9. Dependence of the absolute value of the bias distortions on the spectrum shift in the carrier current channel

7.3. *Dynamic time distortions.* The elementary time distortions of the signal $x_z(t)$, reproduced from the signal $X(t)$, with respect to the signal $\bar{x}_z(t)$ are called *dynamic time distortions.* These distortions are produced by the actual frequency discriminator, as the effect of integrating the pulse train by the low-pass filter. Similarly as in paper [12] (pp. 358 and 359) we see that the dynamic time distortions have a uniform distribution, which is symmetric about zero with the standard deviation $\sigma_d = \sigma(X(t))/(C_1 T)$.

7.4. *Stochastic time distortions.* The elementary time distortions of the signal reproduced from the signal $S_h(t)$ with respect to the signal $x_z(t)$ are called *stochastic time distortions* and may be approximated by the Gaussian truncated distribution (similarly as in [12] (pp. 358 and 359) with the standard deviation equal to $\sigma_p = \overline{\sigma(T_n)}/(C_1 T)$. Such distortions follow from the presence of the Gaussian noise in the carrier current channel.

7.5. *Error probability.* Thus we see that the time distortions at the output of the discriminator are the sum of three random variables: the bias distortions, the dynamic time distortions and the stochastic time distortions, which are all independent of each other. Using the formulas for the parameters of the time distortions δ_s, σ_d, σ_p, previously obtained, one can write the formula (6.7) for the upper bound of the error probability in the form

$$\sup_{\psi_0,\{a_n\}} \Pr[\text{error} \mid a_n = i] = \operatorname{erfc}\left\{\frac{(az_0/2\pi)[\Delta\omega/C_1 + (1-2i)\delta_s T] + \sqrt{3}\sigma_d T - \sup\limits_{\{a_n\}} |\overline{X(T_n)} - A_i|}{\sigma_p T}\right\}.$$

This formula, given δ_s, σ_p, σ_d, enables us to calculate the upper bound of the error probability. Moreover, we can approximate the average error probability, which is one order less than the upper bound. Note that the parameter $\sup\limits_{\{a_n\}} |\overline{X(T_n)} - A_i|$ is by (5.13) independent of the input noise, and it is constant for a given discriminator.

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