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DENSITY THEOREMS FOR MEASURABLE TRANSFORMATIONS

BY

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1. Introduction. Let m denote Lebesgue measure on the Borel σ-algebra of the unit interval $[0, 1]$. A function τ: $[0, 1] \to [0, 1]$ which is Borel-measurable and nonsingular (i.e. $m(A) = 0 \Rightarrow m\tau^{-1}(A) = 0$) is called a *transformation.* We identify transformations which differ only on a set of measure zero. A transformation τ is called *invertible* if τ^{-1} exists and is also a transformation. τ is called *measure-preserving* if $m\tau^{-1}(A) = m(A)$ for all Borel sets A. The group of all invertible transformations is denoted by $\mathscr{G}$. By $\mathscr{G}_m$ we denote the group of all invertible measure-preserving transformations.

Every invertible transformation τ induces a positive invertible isometry $T_\tau^{(p)}$ of $L^p(m)$, $1 \leqslant p < \infty$, defined by

$$T_\tau^{(p)}(f)(t) = \omega_\tau^{1/p}(t) f(\tau^{-1}(t)),$$

where $f \in L^p(m)$, $\omega_\tau = dm\tau^{-1}/dm$. If $\tau \in \mathscr{G}_m$, then $\omega_\tau = 1$.

By a classical result (see, e.g., [2], footnote 3), for every p $(1 \leqslant p < \infty)$ we can identify $\mathscr{G}$ with the group $\mathscr{G}^{(p)}$ of all positive invertible isometries of $L^p(m)$ (i.e. with the set of all Banach lattice automorphisms of $L^p(m)$). Therefore, we can define a topology in $\mathscr{G}$ as the strong operator topology inherited from $\mathscr{L}(L^p(m))$. For all p $(1 \leqslant p < \infty)$ these topologies coincide (see [1], Theorem 8). Moreover, the L^p-strong and L^p-weak operator topologies in $\mathscr{G}_m$ coincide, since the strong and weak topologies in the unitary group in $\mathscr{L}(L^2(m))$ are the same and all L^p-weak operator topologies coincide on the compact set of doubly stochastic operators. It is not hard to see that the family of sets of the form

$$\{\tau \in \mathscr{G}\colon\ m(\tau(A_i)\,\Delta\,\sigma(A_i)) < \varepsilon \text{ for } i = 1, \ldots, n \text{ and } \|\omega_\tau - \omega_\sigma\|_1 < \varepsilon\},$$

where $\varepsilon > 0$, $\sigma \in \mathscr{G}$, and $A_1, \ldots, A_n$ is a partition of the interval $[0, 1]$ into subintervals, is a neighborhood base for the strong operator topology in $\mathscr{G}$.

In this paper we prove that the groups $\mathscr{G}_m$ and $\mathscr{G}$ are topologically finitely generated (Theorems 1 and 2). We also prove that for $1 < p < \infty$ the automorphisms of the Banach lattice $L^p(m)$ span a dense subspace of

$\mathscr{L}(L^p(m))$ (Theorem 3). Iwanik [4] has shown that this is so for $p = 2$ but does not hold for $p = 1$.

The function spaces $L^p(m)$ considered in this paper can be viewed as either real or complex.

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2. Invertible measure-preserving transformations. We shall use the following property of permutations:

LEMMA 1. *Let n be a natural number. Then the group of all permutations of* $\{1, \ldots, 2n\}$ *is generated by the following two elements*:

$$\alpha = \begin{pmatrix} 1, 2, \ldots, 2n \\ 2, 3, \ldots, 1 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1, 2, \ldots, n-1, n, n+1, \ldots, 2n \\ 2, 3, \ldots, \quad n, \quad 1, n+1, \ldots, 2n \end{pmatrix}.$$

Proof. Since every permutation can be decomposed into transpositions, it suffices to show that α and β generate every transposition. Moreover, because of the nature of α and β it is enough to prove that some transposition, e.g.,

$$\gamma = \begin{pmatrix} 1, 2, \ldots, \quad n, n+1, \ldots, 2n \\ 1, 2, \ldots, 2n, n+1, \ldots, \quad n \end{pmatrix},$$

can be expressed as a composition of α and β. In fact, it is not hard to see that $\gamma = \alpha^{n-1}\beta\alpha^n\beta$.

For $a \in [0, 1]$ we write $\alpha_a(t) = t + a \pmod 1$. Moreover, we define

$$\beta_a(t) = \begin{cases} t+a \pmod{1/2} & \text{for } 0 \leqslant t < 1/2, \\ t & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases}$$

Obviously, $\alpha_a, \beta_a \in \mathscr{G}_m$.

THEOREM 1. *If a and b are irrational numbers, then the group generated by* α_a *and* β_b *is dense in* $\mathscr{G}_m$.

Proof. It is easy to see that for every real number c the transformations α_c and β_c belong to the closure $\mathscr{H}_m$ in $\mathscr{G}_m$ of the group generated by α_a and β_b.

Now, given $n \in N$, we partition $[0, 1]$ into $2n$ subintervals of equal length. It is sufficient to show that $\mathscr{H}_m$ contains every piecewise linear transformation ξ which permutes these subintervals. By Lemma 1 we can express ξ as a certain composition of transformations α_c and β_c for $c = 1/2n$.

3. Invertible transformations. Let $I_1, \ldots, I_n$ and $J_1, \ldots, J_n$ be partitions of the interval $[0, 1]$ into subintervals. The notation

$$\varphi\colon I_1 \to J_1, \ldots, I_n \to J_n$$

means that φ is the piecewise linear transformation that maps I_i linearly (with positive slope) onto J_i for all $i \leqslant n$.

Now, for $a \in [0, 1]$ and $b > 0$ we define

$$\varphi_{a,b}\colon [0, a/(b+1)] \to [0, ab/(b+1)], \quad (a/(b+1), a] \to (ab/(b+1), a], \ (a, 1] \to (a, 1].$$

Note that the first interval is stretched and the second one is shrunk by the factor of b.

Let $\mathscr{H}$ denote the group generated by $\mathscr{G}_m$ and ψ, where

$$\psi = \varphi_{1/4,2} \circ \alpha_{1/2} \circ \varphi_{1/4,3} \circ \alpha_{1/2}\colon [0, 1/12) \to [0, 1/6), \ [1/12, 1/4) \to [1/6, 1/4), \ [1/4, 1/2) \to [1/4, 1/2), \ [1/2, 9/16) \to [1/2, 11/16), \ [9/16, 3/4) \to [11/16, 3/4), \ [3/4, 1] \to [3/4, 1].$$

LEMMA 2. *If* $a \in [0, 1]$, *then* $\varphi_{a,2}, \varphi_{a,3} \in \mathscr{H}$.

Proof. We may assume that $a \leqslant 1/12$ (since $\mathscr{G}_m \subset \mathscr{H}$, several conjugates of φ can be composed together, if necessary). Let $\xi \in \mathscr{G}_m$ be defined by

$$\xi\colon [0, 1/6) \to [1/12+2a, 2a+1/4), \ [1/6, 2a+1/4) \to [0, 2a+1/12), \ [2a+1/4, 1/2) \to [2a+1/4, 1/2), \ [1/2, 11/16) \to [9/16, 3/4), \ [11/16, 3/4) \to [1/2, 9/16), \ [3/4, 1] \to [3/4, 1].$$

The transformation $\varphi = \psi\xi\psi$ maps linear intervals

$$I_1 = (1/12 - a, 1/12) \quad \text{and} \quad I_2 = (1/4, 1/4+2a)$$

onto $(1/4, 2a+1/4)$ and $(1/6, a+1/6)$, respectively. It is easy to check that for all intervals I such that $I \cap (I_1 \cup I_2) = \emptyset$ we have $m(\varphi(I)) = m(I)$. This implies the existence of two transformations $\xi_1, \xi_2 \in \mathscr{G}_m$ such that $\varphi_{a,2} = \xi_1 \varphi \xi_2$. Therefore, we obtain $\varphi_{a,2} \in \mathscr{H}$. By analogous arguments we have $\varphi_{a,3} \in \mathscr{H}$.

LEMMA 3. *If* $\varphi_{a,b}, \varphi_{a,c} \in \mathscr{H}$ *for all* $a \in [0, 1]$ *and some* $b, c > 0$, *then* $\varphi_{a,bc} \in \mathscr{H}$ *for all* $a \in [0, 1]$.

Proof. Let $\delta > 0$ be such that $\delta(b+1)(c+1) \leqslant 1/2$. We put

$$\eta = \varphi_{b+1,b} \circ \alpha_{1/2} \circ \varphi_{b(c+1),c} \circ \alpha_{1/2}\colon [0, \delta] \to [0, \delta b], \ (\delta, \delta(b+1)] \to (\delta b, \delta(b+1)], \ (\delta(b+1), 1/2] \to (\delta(b+1), 1/2], \ (1/2, \delta b+1/2] \to (1/2, \delta bc+1/2], \ (\delta b+1/2, b(c+1)+1/2] \to (\delta bc+1/2, b(c+1)+1/2], \ (\delta b(c+1)+1/2, 1] \to (\delta b(c+1)+1/2, 1].$$

Then we have $\eta \in \mathscr{H}$.

Let now $\xi \in \mathcal{G}_m$ be defined by

$$\begin{aligned}\xi\colon\ &[0, \delta b) \to [1/2, \delta b+1/2),\\ &[\delta b, \delta(b+1)) \to [0, \delta),\ [\delta(b+1),\ 1/2) \to [\delta(b+1), 1/2),\\ &[1/2, \delta bc+1/2) \to [\delta b+1/2, \delta b(c+1)+1/2),\\ &[\delta bc+1/2, \delta b(c+1)) \to [\delta, \delta(b+1)),\\ &[\delta b(c+1)+1/2, 1] \to [\delta b(c+1)+1/2, 1].\end{aligned}$$

The transformation $\varphi = \eta \circ \xi \circ \eta$ maps the intervals

$$I_1 = (0, \delta] \quad \text{and} \quad I_2 = (\delta b+1/2,\ \delta b(c+1)+1/2]$$

onto $(1/2, \delta bc+1/2]$ and $(\delta b, \delta(b+1)]$, respectively, and for intervals I such that $I \cap (I_1 \cup I_2) = \emptyset$ we have $m(\eta \circ \xi \circ \eta(I)) = m(I)$. Hence $\varphi_{a,bc} = \sigma\varphi\tau$ for some $\sigma, \tau \in \mathcal{G}_m$, and we obtain $\varphi_{a,bc} \in \mathcal{H}$.

COROLLARY. *The closure of $\mathcal{H}$ contains all $\varphi_{a,b}$ for $0 \leqslant a \leqslant 1$ and $b > 0$.*

Proof. The transformation $\varphi_{a,b}$ belongs to $\mathcal{H}$ if and only if $\varphi_{a,1/b}$ belongs to $\mathcal{H}$, since $\mathcal{G}_m \subset \mathcal{H}$. Therefore, using Lemmas 2 and 3 we infer that $\varphi_{a,b} \in \mathcal{H}$ for $b = 2^k/3^m$ with $k, m \in \boldsymbol{N}$. Since the set $\{2^k/3^m\colon k, m \in \boldsymbol{N}\}$ is dense in $\boldsymbol{R}_+$, the proof is complete.

The following proposition is implicitly contained in [3], so we omit its proof.

PROPOSITION. *Let D be a dense subset of* $[0, 1]$. *Then the family of all invertible transformations τ of the form*

$$\tau\colon\ I_1 \to J_1, \ldots, I_n \to J_n,$$

where (I_k) and (J_k) are partitions of $[0, 1]$ *into subintervals with endpoints in $D \cup \{0, 1\}$, is dense in $\mathcal{G}$.*

THEOREM 2. *If a and b are irrational numbers, then the group generated by α_a, β_b, and ψ is dense in $\mathcal{G}$.*

Proof. By Theorem 1 and the Proposition it is sufficient to show that for any partitions $0 = a_0 < a_1 < \ldots < a_{n+1} = 1$ and $0 = b_0 < b_1 < \ldots \ldots < b_{n+1} = 1$ with a_i, b_i of the form $2^k/3^m$ for $1 \leqslant i \leqslant n$ there exists a transformation in $\mathcal{H}$ which maps $[a_i, a_{i+1})$ linearly onto $[b_i, b_{i+1})$ for $i = 1, \ldots, n$.

By the Corollary we may assume that $a_n \leqslant 1/4$ and $b_n \leqslant 1/4$.

Now, $\xi_1 = \varphi_{(a_1+b_1),(b_1/a_1)}$ maps $[0, a_1)$ onto $[0, b_1)$. The function

$$\begin{aligned}\varphi\colon\ &[0, b_1) \to [0, b_1),\ [b_1, \xi_1(a_2)) \to [b_1, b_2),\\ &[\xi_1(a_2), \xi_1(a_2)+b_2-b_1) \to [b_2, \xi_1(a_2)+b_2-b_1),\\ &[\xi_1(a_2)+b_2-b_1, 1] \to [\xi_1(a_2)+b_2-b_1, 1]\end{aligned}$$

clearly satisfies the equality $\varphi = \sigma\varphi_{x,y}\tau$ for some $\sigma, \tau \in \mathcal{G}_m$ and $x, y \in \boldsymbol{R}_+$, and so $\varphi \in \mathcal{H}$. Therefore, $\xi_2 = \varphi\xi_1 \in \mathcal{H}$ and it is easy to see that ξ_2 maps $[a_i, a_{i+1})$ onto $[b_i, b_{i+1})$ for $i = 0, 1$. Continuing this process by induction, we can construct a transformation $\xi_n \in \mathcal{H}$ such that ξ_n maps $[a_i, a_{i+1})$ onto $[b_i, b_{i+1})$ for $i = 0, 1, \ldots, n$.

4. Linear operators on L^p.

LEMMA 4. *Let* $1 < p < \infty$. *For every measurable characteristic function* χ *the multiplication operator* $f \to \chi f$ *is in the strong operator closure of* $\operatorname{conv}\mathcal{G}$ *in* $\mathcal{L}(L^p(m))$.

Proof. We may and do assume that χ is the characteristic function of the interval $[0, a)$, $0 < a < 1$. Let $A_1, \ldots, A_N$ be the partition of $[a, 1]$ into subintervals of equal length. We define $\tau_i \in \mathcal{G}$ to be the identity on $[0, a)$ and linear on each A_k with $A_N \to [a, 1]\backslash A_i$ and $[a, 1]\backslash A_N \to A_i$. Now we define $T_i = T_{\tau_i}^{(p)}$. In other words,

$$T_i f(t) = \omega_i(t)^{1/p} f(\tau_i^{-1}(t)), \quad \text{where } \omega_i(t) = dm\tau_i^{-1}/dm(t).$$

Clearly,

$$\omega_i(t) = \begin{cases} 1 & \text{for } t \in [0, a), \\ 1/(N-1) & \text{for } t \in [a, 1]\backslash A_i, \\ N-1 & \text{for } t \in A_i. \end{cases}$$

We put

$$R_N = \frac{1}{N}\sum_{i=1}^{N} T_i \in \operatorname{conv}\mathcal{G}.$$

For $|f| \leqslant 1$ we have $R_N f(t) = f(t)$ if $t \in [0, a)$ and

$$|R_N f(t)| \leqslant \frac{1}{N}\sum_{i=1}^{N} |T_i f(t)| \leqslant \frac{1}{N}\sum_{i=1}^{N} (\omega_i(t))^{1/p}$$
$$= \left((N-1)^{1-1/p} + (N-1)^{1/p}\right)\frac{1}{N} \to 0 \quad \text{for } t \in [a, 1)$$

as $N \to \infty$. Consequently, R_N converges to the required operator in the strong operator topology.

THEOREM 3. *Let* $1 < p < \infty$. *Then the linear span* $\operatorname{lin}\mathcal{G}$ *of* $\mathcal{G}$ *is strong operator dense in* $\mathcal{L}(L^p(m))$.

Proof. Let

$$S_i^{(n,k)} f(t) = \begin{cases} f(\psi_i(t)) & \text{for } t \in ((k-1)/2^n, k/2^n), \\ 0 & \text{otherwise,} \end{cases}$$

where $n \in N$, $1 \leqslant k \leqslant 2^n$, and the transformation

$$\psi_i\colon \big((k-1)/2^n, k/2^n\big) \to \big((i-1)/2^n, i/2^n\big)$$

is linear for $i = 1, 2, \ldots, 2^n$. By Lemma 4, $S_i^{(n,k)}$ belongs to the strong operator closure of $\operatorname{lin} \mathscr{G}$. Now we fix a finite measure $\mu \ll m$ and two numbers a, b $(a < b)$ in $[0, 1]$. Write

$$S^{(n)} = \sum_k \sum_{i=1}^{2^n} \mu\Big(\big((i-1)/2^n, i/2^n\big)\Big) S_i^{(n,k)},$$

where the first summation is over all k such that

$$\big((k-1)/2^n, k/2^n\big) \subset (a, b).$$

Obviously, $S^{(n)} f(t) = 0$ for $t \notin [a, b]$. If f is a step function (constant on subintervals), then for $t \in (a, b)$ we have $S^{(n)} f(t) \to \int f \, d\mu$. Therefore, by Lemma 4 the one-dimensional operators $f \to \chi \int f \, d\mu$ belong to the strong operator closure of $\operatorname{lin} \mathscr{G}$ and, consequently all finite-dimensional operators are in the closure of $\operatorname{lin} \mathscr{G}$. But the set of all finite-dimensional operators is strong operator dense in $\mathscr{L}\big(L^p(m)\big)$.

It should be noted that, by Theorems 2 and 3, for every p $(1 < p < \infty)$ there exist three positive invertible isometries which generate a strong operator dense subalgebra of $\mathscr{L}\big(L^p(m)\big)$.

Added in proof. An extension of Theorem 1 is contained in [5].

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