

ON CONTRACTIBLE DENDROIDS

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Investigating smooth fans [5], [9] and, more general, smooth dendroids [6], we have observed that these curves are contractible (see [5], Theorem 11, p. 30 and Corollary 19, p. 31, and [6], Corollary 12, p. 311). Further, Mohler [10] has characterized smooth dendroids as those which are contractible by a retracting homotopy in a way such that some point remains fixed throughout the homotopy ([10], Theorem 1.14, p. 370, and Theorem 1.16, p. 371). Answering a question raised by Mohler in [10] we show that Mohler's condition concerning the fixation of some point throughout the homotopy is superfluous. Answering another question asked by Mohler in the same paper we prove that each contractible dendroid is uniformly arcwise connected, and we give an example of a fan which is not contractible but is uniformly arcwise connected. Finally several questions concerning contractibility of dendroids are raised.

1. A metric space is said to be a *continuum* if it is compact and connected. A continuum is said to be *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected. A continuum is said to be a *dendroid* if it is hereditarily unicoherent and arcwise connected. A *dendrite* means a locally connected dendroid (i.e., a locally connected continuum without any simple closed curve).

For the remainder of the paper we let I denote the unit interval $[0, 1]$ of real numbers. If $h: X \times I \rightarrow X$ is a homotopy on a topological space X , we let h_t for $t \in I$ denote the map $h_t: X \rightarrow X$ defined by $h_t(x) = h(x, t)$ for each $x \in X$.

A space X is said to be *contractible* if there exists a homotopy $h: X \times I \rightarrow X$ such that h_0 is the identity on X and h_1 is a constant map (i.e., h_1 maps X onto a point).

THEOREM 1. *If a dendroid X is contractible, then each subdendroid of X is contractible.*

Proof. Let X be a contractible dendroid and let a homotopy $h: X \times I \rightarrow X$ be such that h_0 is the identity and h_1 is a constant map. Further,

let Y be a subdendroid of X . For each point $y \in Y$ define a subset A_y of I by

$$A_y = \{t \in I : h(y, t) \in Y\}.$$

In particular, for $t = 0$ we have $h(y, 0) = y \in Y$, thus $0 \in A_y$ for each y in Y .

Since the mapping h is continuous and Y is a closed subset of X , it is easy to verify that A_y is closed. It implies that the set $I \setminus A_y$ is open in I , thus it has at most countably many components each of which is an open interval. Denote them by $C_{y,n}$ ($n = 1, 2, \dots$) and denote ends of $C_{y,n}$ by $t'_{y,n}$ and $t''_{y,n}$ with $t'_{y,n} < t''_{y,n}$. Putting

$$D_{y,n} = \{h(y, t) : t'_{y,n} \leq t \leq t''_{y,n}\}$$

we see that $D_{y,n}$ is a continuous image of the closed interval $[t'_{y,n}, t''_{y,n}]$, thus a locally connected continuum which lies in the dendroid X . Hence $D_{y,n}$ is a dendrite. If $C_{y,n}$ does not contain 1, i.e. if $t''_{y,n} \neq 1$, then by the definition of A_y exactly two points of $D_{y,n}$ lie in Y , namely $h(y, t'_{y,n})$ and $h(y, t''_{y,n})$. But the intersection $D_{y,n} \cap Y$ is connected by virtue of the hereditary unicoherence of X . It implies that if $t''_{y,n} \neq 1$, then $h(y, t'_{y,n}) = h(y, t''_{y,n})$. Using this equality it is not too hard to verify that the mapping $H: Y \times I \rightarrow Y$ defined by

$$H(y, t) = \begin{cases} h(y, t) & \text{if } t \in A_y, \\ h(y, t'_{y,n}) & \text{if } t \in C_{y,n} \subset I \setminus A_y \end{cases}$$

is continuous.

We shall prove that H is a homotopy with H_0 — the identity and H_1 — a constant map. In fact, for $t = 0$ we have $H(y, 0) = h(y, 0)$ since $0 \in A_y$, so that $H_0(y) = y$ for each $y \in Y$, i.e. $H_0: Y \rightarrow Y$ is the identity.

For $t = 1$ put $h_1(X) = c$ and consider two cases. Firstly, let $c \in Y$, i.e., $h(y, 1) \in Y$. Then $1 \in A_y$ for each $y \in Y$. Thus we have $H(y, 1) = h(y, 1)$ by the definition of H , whence $H(y, 1) = c$ for each y in Y , i.e., $H_1: Y \rightarrow Y$ is a constant map. Secondly, let $c \in X \setminus Y$, i.e., $h(y, 1) \in X \setminus Y$. Then $1 \in I \setminus A_y$ for each $y \in Y$. Thus, for any fixed y in Y there is a component $C_{y,n}$ of $I \setminus A_y$ which contains 1. Let $t'_{y,n} < 1 = t''_{y,n}$ be the end point of $C_{y,n}$, i.e., $C_{y,n} = \{t \in I : t'_{y,n} < t \leq 1\}$. Therefore $H(y, 1) = h(y, t'_{y,n})$ by the definition of H . Further, if $c \in X \setminus Y$, the hereditary unicoherence of X implies that there is one and only one arc qc in X such that $Y \cap qc = \{q\}$ (see [4], T20, p. 195, and T27, p. 197). The set $D_{y,n} = \{h(y, t) : t'_{y,n} \leq t \leq 1\}$ is a dendrite which contains $h(y, 1) = c$. Since $C_{y,n}$ is a component of the set $I \setminus A_y$, hence its left end point $t'_{y,n}$ is the only point of the closed interval $[t'_{y,n}, 1]$ which belongs to A_y . Thus $h(y, t'_{y,n})$ is the only point of $D_{y,n}$ which lies in Y . So $D_{y,n}$ contains the arc from $h(y, t'_{y,n})$ to c and

we have $Y \cap h(y, t'_{y,n})c = \{h(y, t'_{y,n})\}$ which implies $h(y, t'_{y,n}) = q$ by the uniqueness of the point q defined above. Therefore we have shown that $H(y, 1) = q$, i.e., that $H_1: Y \rightarrow Y$ is a constant map.

2. A dendroid X is said to be *smooth* (cf. [6], p. 298) if there exists a point $p \in X$, called an *initial point* of X , such that given any sequence of points a_n in X with $\lim_{n \rightarrow \infty} a_n = a$, it follows that the sequence of arcs $a_n p$ is convergent, and $\lim_{n \rightarrow \infty} a_n p = ap$.

Recall that if a continuous mapping f of a topological space X onto $f(X)$ has the properties $f(X) \subset X$ and $f(x) = x$ for each $x \in f(X)$, then f is said to be a *retraction* ([1], p. 154). A homotopy $h: X \times I \rightarrow X$ on a topological space X is said to be a *retracting homotopy* ([5], p. 31 and [10], p. 370) if, for every $t \in I$, the map $h_t: X \rightarrow X$ is a retraction.

Mohler has proved ([10], Theorem 1.14, p. 370) the following

THEOREM A. *If a dendroid X admits a homotopy $h: X \times I \rightarrow X$ satisfying*

- (i) *h is a retracting homotopy,*
- (ii) *h contracts X to a point $p \in X$,*
- (iii) *$h(p, t) = p$ for every $t \in I$,*

then X is smooth.

Further, he has asked ([10], p. 372) whether condition (iii) is necessary in establishing the above theorem; that is, suppose that X is a dendroid which admits a homotopy $h: X \times I \rightarrow X$ satisfying conditions (i) and (ii) of Theorem A, does it still follow that X is smooth? We shall prove now that condition (iii) is not necessary, i.e. that conditions (i) and (ii) are sufficient for a dendroid X to be smooth.

A point x of a topological space X will be called a *fixed point* of a homotopy $h: X \times I \rightarrow X$ provided $h_t(x) = x$ for each $t \in I$.

Let Y be a subcontinuum of a continuum X , and let $p \in Y$. Denote by $T_p(Y)$ the set of all points $y \in Y$ such that if K is a subcontinuum of Y containing y in its interior with respect to Y , then $p \in K$ (see [6], p. 304). For the remainder of this section let X be a dendroid, d be a metric on X , $Q(x, r)$ — open metric ball in X with centre x and radius r , i.e., $Q(x, r) = \{y \in X: d(x, y) < r\}$, and let $h: X \times I \rightarrow X$ be a retracting homotopy on X . We shall prove the following

LEMMA 1. *Suppose x is a fixed point of h and $p \in h_0(X)$ such that $x \in T_p(h_0(X))$. Then*

- (a) *p is a fixed point of h ,*
- (b) *$x \in T_p(h_1(X))$.*

Proof. Assume $p \neq x$, for otherwise the lemma is trivial.

- (a) Let $\varepsilon = \frac{1}{3}d(x, p)$. The homotopy h being continuous and x being

a fixed point of h , there is, for each $t \in I$, a pair of open sets $U_t \subset X$ and $V_t \subset I$ such that $x \in U_t$, $t \in V_t$ and $h(U_t \times V_t) \subset Q(x, \varepsilon)$. By the compactness of I there exist t_1, t_2, \dots, t_n in I such that the sets $V_{t_1}, V_{t_2}, \dots, V_{t_n}$ cover I . Let $U = \bigcap_{i=1}^n U_{t_i}$. Then U is an open set in X containing x with the property that $h(U \times I) \subset Q(x, \varepsilon)$.

Let $t \in I$. We shall show that $h_t(p) = p$. To see this, let W be an open set about p (assume for convenience that $W \subset Q(p, \varepsilon)$). Since $x \in T_p(h_0(X))$, there is a point $y \in h_0(X) \cap U$ such that $xy \cap W \neq \emptyset$. Now the arc xy is contained in the union of two arcs $xh_t(y)$ and $yh_t(y)$. However, $yh_t(y) \subset h(\{y\} \times I) \subset h(U \times I) \subset Q(x, \varepsilon)$ and so $yh_t(y) \cap W = \emptyset$. Therefore it must be $xh_t(y) \cap W \neq \emptyset$. Pick $z \in xh_t(y) \cap W$. Then $z \in h_t(X)$ since $h_t(X)$ is a continuum containing $h_t(x) = x$ and $h_t(y)$. Thus $h_t(z) = z$ since h_t is a retraction, and we have shown that each open set W about p contains a point z such that $h_t(z) = z$. From this it follows that $h_t(p) = p$, and this concludes the proof of (a).

(b) Let K be any subcontinuum of $h_1(X)$ containing x in its interior with respect to $h_1(X)$. We wish to show that $p \in K$. Let V and W be disjoint open sets about x and p respectively. We can assume that $V \cap h_1(X) \subset K$. Construct, as in (a), an open set U about x such that $U \subset V$ and $h(U \times I) \subset V$. Since $x \in T_p(h_0(X))$, there exists a point $y \in h_0(X) \cap U$ such that $xy \cap W \neq \emptyset$. As before, we have $xy \subset xh_1(y) \cup yh_1(y)$ and $yh_1(y) \cap W = \emptyset$. Hence it must be $xh_1(y) \cap W \neq \emptyset$. But $x \in K$ and $h_1(y) \in V$, hence $h_1(y) \in V \cap h_1(X) \subset K$, and so $xh_1(y) \subset K$. Thus $K \cap W \neq \emptyset$ and we have shown that each open set W about p meets K . Since K is closed, it follows that $p \in K$. Hence $x \in T_p(h_1(X))$. This completes the proof of the lemma.

LEMMA 2. *Suppose D is a dendrite and p, q, x and y are distinct points in D with $xy \subset pq$. Then if p_n and q_n are sequences of points in D converging to p and q respectively, there exists a positive integer k such that if $n > k$, then $xy \subset p_n q_n$.*

Proof. Assume $x \in py$ for convenience. Let C_p denote the component of $D \setminus \{x\}$ containing p and C_q denote the component of $D \setminus \{y\}$ containing q . Then C_p and C_q are open disjoint subsets of D . Choose k so large that $p_n \in C_p$ and $q_n \in C_q$ for $n > k$. Then clearly $xy \subset p_n q_n$ for $n > k$.

THEOREM 2. *Suppose $h: X \times I \rightarrow X$ is a retracting homotopy on a dendroid X . Then $h_0(X)$ is smooth if and only if $h_1(X)$ is smooth.*

Proof. In view of the symmetry between $h_0(X)$ and $h_1(X)$ (by replacing in homotopy t with $1-t$), it suffices to assume that $h_0(X)$ is not smooth and to prove that $h_1(X)$ is not smooth. So suppose $h_0(X)$ is not smooth. Then by Theorem 6 of [6], p. 302, there exist distinct points p, q, x and y of $h_0(X)$ such that $x \in T_p(h_0(X)) \cap pq$ and $y \in T_q(h_0(X)) \cap pq$.

Define a set B by

$$B = \{0\} \cup \{t \in (0, 1] : p \text{ and } q \text{ are fixed points of } h|X \times [0, t]\}$$

and put $t_0 = \sup B$. Note $t_0 \in B$.

Suppose $t_0 < 1$. We will show a contradiction. Let $D = h(pq \times [t_0, 1])$. Then D is a dendrite in X . For $t \in [t_0, 1]$, let $p_t = h_t(p)$ and $q_t = h_t(q)$. Note that $p_t q_t \subset D$. We claim that there is a positive number $\delta \in [0, 1 - t_0]$ such that for each t satisfying $t_0 \leq t \leq t_0 + \delta$ we have $xy \subset p_t q_t$. For otherwise there must exist a sequence t_n in $[t_0, 1]$ converging to t_0 such that xy is not contained in $p_{t_n} q_{t_n}$ for $n = 1, 2, \dots$. But by the continuity of h , p_{t_n} and q_{t_n} are sequences in the dendrite D converging to p and q respectively. Hence by Lemma 2 the arc $p_{t_n} q_{t_n}$ contains xy for some n . This contradiction establishes the existence of δ .

Now suppose $t_0 \leq t \leq t_0 + \delta$. Then $xy \subset p_t q_t \subset h_t(pq)$. Hence $h_t(x) = x$ and $h_t(y) = y$ since h_t is a retraction. Thus we infer that x and y are fixed points of the homotopy $h|X \times [0, t_0 + \delta]$. Since $x \in T_p(h_0(X))$ and $y \in T_q(h_0(X))$, p and q are, by Lemma 1 (a), fixed points of the homotopy $h|X \times [0, t_0 + \delta]$, and so $t_0 + \delta \in B$ which contradicts to the definition of t_0 . From this contradiction we conclude that $t_0 = 1$.

Now we know that p and q are fixed points of the homotopy h . Hence x and y are fixed points of h . So by Lemma 1 (b) we have that $x \in T_p(h_1(X))$ and $y \in T_q(h_1(X))$. Thus by Theorem 6 of [6], p. 302, we get that $h_1(X)$ is not smooth. This completes the proof.

COROLLARY. *A dendroid X is smooth if and only if there is a retracting homotopy $h: X \times I \rightarrow X$ such that h_0 is the identity and h_1 is a constant mapping.*

Proof. If X is smooth, then the homotopy h is given in [10], Theorem 1.16, p. 371. If a homotopy h satisfying the conditions is given, then X must be smooth by Theorem 2.

This shows that condition (iii) of Mohler's characterization of smoothness of dendroids ([10], Theorem 1.14, p. 370) is not necessary.

3. A point p in a continuum X is called a *ramification point* of X in the classical sense if it is the common end point of three (or more) arcs in X whose only common point is p . A *fan* means a dendroid with exactly one ramification point (see [5] and [9]).

Investigating contractibility of fans, Mohler has asked in [10], p. 375, whether there exists a fan which fails to be contractible. Answering this question in affirmative we show a class of non-contractible fans; namely we prove that every non-uniformly arcwise connected dendroid is not contractible.

Recall that a subset X of a metric space is said to be *uniformly arcwise connected* (see [4], p. 193, [5], p. 12, and [6], p. 316) if it is arcwise

connected and if for every number $\varepsilon > 0$ there is a positive integer k such that every arc A in X contains points a_0, a_1, \dots, a_k with the properties

$$A = \bigcup_{i=0}^{k-1} a_i a_{i+1}$$

and

$$\delta(a_i a_{i+1}) < \varepsilon \quad \text{for every } i = 0, 1, \dots, k-1$$

(here $\delta(S)$ stands for the diameter of the set S).

THEOREM 3. *Every contractible dendroid is uniformly arcwise connected.*

Proof. Let X be a contractible dendroid and let $h: X \times I \rightarrow X$ be a homotopy on X with h_0 being the identity and h_1 being a constant map. Put $h_1(X) = \{c\}$ and define for each $x \in X$

$$D_x = \{h(x, t): 0 \leq t \leq 1\}.$$

Thus for each $x \in X$, the set D_x is a continuous image of the closed interval and so it is a locally connected continuum contained in the dendroid X ; hence each D_x is a dendrite. By the definition, points x and c are in D_x , thus D_x contains the arc cx . Obviously, we have

$$X = \bigcup \{D_x: x \in X\}.$$

Further, let n be a positive integer. For every $i = 0, 1, \dots, n-1$ put

$$D_x^i = \left\{ h(x, t): \frac{i}{n} \leq t \leq \frac{i+1}{n} \right\}.$$

As previously, D_x^i is a dendrite and

$$D_x = \bigcup_{i=0}^{n-1} D_x^i.$$

We have to prove that X is uniformly arcwise connected. Since it is arcwise connected by definition, it remains to show the uniformity. It is sufficient to prove that, given any $\varepsilon > 0$, every arc $A \subset X$ is contained in the union of a fixed number, say $2k$, of arcs, the diameters of which are less than ε . So let ε be an arbitrary positive number. The mapping h being uniformly continuous, for this ε there exists a positive integer k so large that for every $x \in X$, for every $n \geq k$, and for every $i = 0, 1, \dots, n-1$ we have $\delta(D_x^i) < \varepsilon$. Let $A = ab$ be an arbitrary arc in X . Thus A is contained in the union $ac \cup bc$. As it was said above, we have $ac \subset D_a$ and $bc \subset D_b$, whence, by taking $n = k$, we infer that $A \subset \bigcup_{i=0}^{k-1} (D_a^i \cup D_b^i)$. Therefore we can write

$$A = \bigcup_{i=0}^{k-1} ((A \cap D_a^i) \cup (A \cap D_b^i)).$$

By virtue of the hereditary unicoherence of X , each intersection $A \cap D_a^i$ and $A \cap D_b^i$ is an arc (or a point or the empty set), so the last equality gives the decomposition of the arc A into at most $2k$ subarcs of the form $A \cap D_a^i$ or $A \cap D_b^i$ whose diameters are less than ε by the definition of k . This completes the proof.

It follows from the above theorem that the non-uniformly arcwise connected fan described in [4], (49) and (52), p. 201, is not contractible.

One can ask whether the inverse theorem is true, i. e. suppose X is a uniformly arcwise connected dendroid, does it follow that X is contractible? The answer is negative even if X is a fan. To see this, consider the following example.

Put (in the Cartesian rectangular coordinates in the plane) $a = (0, 1)$, $b = (0, 0)$, $a_n = (-1/n, 1)$, $b_n = (-1/n, 0)$, $c_n = (1/n, 0)$.

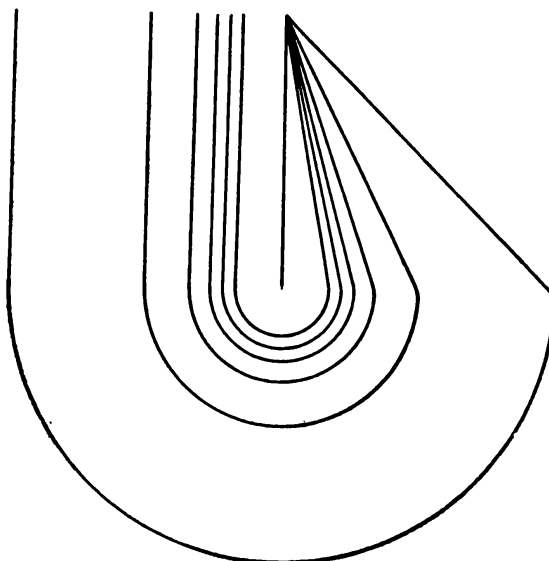
Join a with b and with c_n ($n = 1, 2, \dots$) by the straight line segments ab and ac_n respectively, join c_n with b_n by the semicircle $c_n b_n = \{(x, y) : x^2 + y^2 = 1/n^2 \text{ and } y \leq 0\}$, join b_n with a_n by the straight line segment $b_n a_n$, and put

$$A_0 = ab, \quad A_n = ac_n \cup c_n b_n \cup b_n a_n \text{ for } n = 1, 2, \dots$$

Let

$$X = \bigcup_{n=0}^{\infty} A_n$$

(see the figure). Thus X is a fan with the top a and with end points b, a_1, a_2, \dots . It is easy to see that X is uniformly arcwise connected (see [4], C2, p. 193). We shall prove that X is not contractible. To show this, recall that a space X is contractible if and only if $C(X)$, the cone over X , can be retracted onto its base X (see, e.g., [8], 1.3, p. 317).



So take a point p in the Euclidean 3-space outside the plane (e.g., let $p = (0, 0, 1)$) and join it with each point $x \in X$ by the straight line segment px . The union of these segments, for all x in X , forms the cone $C(X)$. Suppose on the contrary that X is contractible. Then there exists a mapping $f: C(X) \rightarrow X$ such that $f(x) = x$ for each $x \in X$.

Now let T denote the triangle (i.e., a 2-cell) with vertices a, b and p . Thus $T \subset C(X)$. Let a continuum $K \subset C(X)$ be such that $K \subset T$, $p \in K$ and $K \cap A_0 \neq \emptyset$. We shall show that $K \cap f^{-1}(b) \neq \emptyset$. In fact, let $x \in K \cap A_0$ and let K_n be the copy of K lying in the triangle with vertices a_n, b_n and p in the same manner as K lies in T . Thus $p \in K_n$ and $K_n \cap a_n b_n \neq \emptyset$ by construction. Let $x_n \in K_n \cap a_n b_n$ be the copy of x . Therefore we have $K = \lim_{n \rightarrow \infty} K_n$ and $x = \lim_{n \rightarrow \infty} x_n$.

Since every subcontinuum of X is a dendroid (see [3], (49), p. 240), thus an arcwise connected continuum, $f(K_n)$ contains the arc $f(p)x_n$ from $f(p)$ to $f(x_n) = x_n$, hence

$$\text{Ls}_{n \rightarrow \infty} f(p)x_n \subset \text{Ls}_{n \rightarrow \infty} f(K_n).$$

On the first hand, $f(p)x_n = f(p)a \cup ax_n$ for all (but possibly one) naturals n . Since $\text{Ls}_{n \rightarrow \infty} ax_n = ab$,

$$\text{Ls}_{n \rightarrow \infty} f(p)x_n = f(p)a \cup ab.$$

On the other hand, $\text{Ls}_{n \rightarrow \infty} f(K_n) = f(\text{Ls}_{n \rightarrow \infty} K_n)$ (see [7], Lemma 8.4, p. 23), thus

$$\text{Ls}_{n \rightarrow \infty} f(K_n) = f(K).$$

So we have $f(p)a \cup ab \subset f(K)$, whence, in particular, $b \in f(K)$, i.e., $K \cap f^{-1}(b) \neq \emptyset$.

Taking the straight segment ap as K , we see that $ap \cap f^{-1}(b) \neq \emptyset$. Since $b \in f^{-1}(b)$ by the definition of f , the proved property of each continuum K shows that the triangle T contains a continuum L which joins a point of the side ap of T to the point b and which is contained in $f^{-1}(b)$:

$$L \subset T \cap f^{-1}(b).$$

Consider now a copy L_n of L lying in the triangle with vertices a, c_n and p in the same manner as L lies in T (with respect to vertices a, b and p). So we have

$$\bigcap_{n=1}^{\infty} L_n = ap \cap L,$$

$c_n \in L_n$ for each n , and $L = \text{Ls}_{n \rightarrow \infty} L_n$ by construction. Let $q \in ap \cap L$. Thus $q \in L_n$ for each n , and $f(q) = b$. Since $f(c_n) = c_n$ by the definition of f , we conclude using the same arguments as previously that, for each n , the continuum $f(L_n)$ contains the arc bc_n from b to c_n . Since a is in bc_n , we see that each L_n contains a point y_n which goes into a under f , i.e. $y_n \in L_n$ and $f(y_n) = a$. Taking the limit y of a convergent subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ we see that $y \in L$ by construction, thus $f(y) = b$ by the definition of L , but $f(y) = a$ by the continuity of f . The contradiction concludes the proof of the non-contractibility of X .

4. Borsuk [2] has constructed a *non-planar fan* (i.e., a fan which cannot be imbedded in the plane). This fan contains a homeomorphic image of the non-contractible fan X constructed above, thus it is not contractible by Theorem 1. The question arises whether every non-planar fan is not contractible. (P 786)

In [6] we have introduced the concept of n -countably generated dendroids (see [6], p. 303) and we have proved that a dendroid is smooth if and only if each of its 2-countably generated subdendroids is smooth ([6], Theorem 7, p. 304). Similarly to this, it is our conjecture that a dendroid is contractible if and only if every its 1-countable subdendroid is contractible. One way is clear by Theorem 1. So the question is whether the contractibility of every 1-countable subdendroid of a dendroid X implies the contractibility of X . (P 787)

Using the same method as for the example X in the end of Section 3 one can prove that no one of the dendroids D_n described in [6], p. 305 is contractible. Hence, repeating the proof of Theorem 8 of [6], p. 305, with a slight modification, we obtain the following

THEOREM 4. *The property of not being contractible is not finite in the class of dendroids.*

A question is whether the property of not being contractible is not finite in the class of fans. (P 788)

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