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A remark concerning the dependence on the initial data of the solution of the Cauchy problem for a linear equation with analytic coefficients

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Let us consider the Cauchy problem

$$D_n^m u = \sum_{\substack{|\alpha|=m \\ \alpha_n<m}} a_\alpha(x) D^\alpha u + f(x) \qquad (x \in \Omega), \tag{1}$$

$$D_n^k u(x', 0) = \varphi_k(x) \qquad (k = 0, 1, \ldots, m-1;\ x' \in \Omega') \tag{2}$$

(here Ω denotes a domain of the n-dimensional real vector space R_n having a non-empty intersection Ω' with the plane $x_n = 0$; for $x = (x_1, \ldots, x_n) \in R_n$ we write $x = (x', x_n)$). According to the Cauchy–Kowalewski theorem, the initial value problem (1), (2) has a unique analytic solution defined in some n-dimensional neighbourhood of the set Ω' if the functions a_α, f, φ_k are analytic. As Hörmander has noticed in [2], this solution need not depend continuously on the initial data, although the topology in the space of the data is very strong and that in the space of the solutions is a very weak one. In the present note we give a simple illustration of this remark by means of a slight modification of the well-known example due to Hadamard [1].

Namely, let us consider for fixed natural n the Cauchy problem in the (x, y)-plane

$$\Delta u = 0, \quad u(x, 0) = g_n(x), \quad D_y u(x, 0) = 0,$$

with $g_n(x) = e^{-\sqrt{n}} \sin(2\pi n x)$. As is easy to verify, its unique solution has the form

$$u_n(x, y) = e^{-\sqrt{n}} \sin(2\pi n x) \cosh(2\pi n y),$$

and so it is defined in the whole plane. For every fixed $k = 0, 1, 2, \ldots$ we have

$$\lim_{n\to\infty} \sup_{x \in R_1} |g_n^{(k)}(x)| = 0.$$

We show that for an arbitrarily fixed $\sigma > 0$ the sequence $\{u_n\}$ does not tend to zero in the space $D'(\Omega_\sigma)$, where Ω_σ denotes the strip $0 < y < \sigma$. Let $\alpha \in C_0^\infty(R_1)$ be a positive-valued function vanishing outside the interval $(0, \frac{1}{2})$, let $\beta \in C_0^\infty(-\sigma, \sigma)$ be a positive-valued function which equals 1 for $\sigma/4 \leqslant y \leqslant \sigma/2$, and let us put

$$\varphi_n(x, y) = e^{-\sqrt{n}}\alpha(nx)\beta(y)$$

for $n = 1, 2, \ldots$ Evidently $B = \{\varphi_n\}$ is a bounded subset of $D(\Omega_\sigma)$ (see [3]) and it is easy to prove that the sequence $\langle\varphi, u_n\rangle$ does not tend to zero uniformly on $\varphi \in B$. Namely, applying the substitution $nx = t$, we have

$$\langle\varphi_n, u_n\rangle = \frac{1}{n}e^{-2\sqrt{n}}\int_0^{\frac{1}{2}}\alpha(t)\sin(2\pi t)\,dt\int_{-\sigma}^{\sigma}\beta(y)\cosh(2\pi ny)\,dy;$$

thus, denoting the first integral by c, we obtain

$$\langle\varphi_n, u_n\rangle \geqslant \frac{c}{n}e^{-2\sqrt{n}}\int_{\sigma/4}^{\sigma/2}\cosh(2\pi ny)\,dy. \tag{3}$$

As the integral on the right of (3) exceeds the number

$$\frac{\sigma}{4}\sup_{\sigma/4<y<\sigma/2}\cosh(2\pi ny) = \frac{\sigma}{4}\cosh(\pi n\sigma),$$

we have

$$\langle\varphi_n, u_n\rangle \geqslant \frac{c\sigma}{4n}e^{-2\sqrt{n}}\cosh(\pi n\sigma),$$

and the expression on the right tends to infinity as $n \to \infty$. This completes the proof.

References

[1] J. Hadamard, *Lectures on Cauchy's problem in linear partial differential equations*, New York 1952.

[2] L. Hörmander, *Linear partial differential operators*, Berlin 1963 (particularly point 5.0).

[3] L. Schwartz, *Théorie des distributions I*, Paris 1957.

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