ANNALES
POLONICI MATHEMATICI
XLIII (1983)

Infinitesimal automorphisms of some G-structures

by ROBERT WOLAK (Kraków)

Abstract. The space of 1-jets of infinitesimal automorphisms of G-structures allows to define a class of G-structures which have properties similar to those of infinitesimally transitive structures. A generalized version of Amemiya's theorem for germs of vector fields is proved and used to establish integrability of some distributions. Several other propositions are proved for distributions and almost product structures including a generalization of Gauchman's theorem on maximally mobile Riemannian almost product structures.

In the paper we consider the space of 1-jets of infinitesimal automorphisms of G-structures. This space allows to define a class of G-structures which have properties similar to those of infinitesimally transitive structures. A generalized version of Amemiya's theorem for germs of vector fields is proved and used to establish integrability of some distributions. In the last part of the paper we generalize Gauchman's theorem on maximally mobile Riemannian almost product structures. Several other theorems on distributions and almost product structures are proved using similar method.

In the paper there are considered smooth manifolds and vector fields only. A G-structure B is said to be *infinitesimally transitive* if the sheaf $\tilde{\mathcal{U}}(B)$, obtained by lifting the sheaf $\mathcal{U}$ of germs of infinitesimal automorphisms of B from the manifold M to B, acts transitively on B. Let us consider that action. If f_t is the flow of an infinitesimal automorphism X, the vector field $\tilde{X}$ has the flow $\tilde{f}_t$. We can always consider B as a submanifold of the manifold $J^1(M)$ of 1-jets on the manifold M.

Then

$$\tilde{f}_t(y_1, \ldots, y_n;\ y_j^i) = \left(y_1 + a_1(t, y), \ldots, y_n + a_n(t, y);\ \sum_{k=1}^{n} a_k^i(t, y)\, y_j^k\right),$$

where

$$j_y^1 f_t = \left(a_1(t, y), \ldots, a_n(t, y);\ a_j^i(t, y)\right)$$

and

$$\tilde{X}(y) = \frac{d}{dt} f_t(y) = \left(\frac{\partial}{\partial t} a_1(t, y), \ldots, \frac{\partial}{\partial t} a_n(t, y); \sum_k \frac{\partial}{\partial t} a_k^i(t, y) y_j^k \right)$$
$$= \left(X_1(y), \ldots, X_n(y); \sum_k \frac{\partial}{\partial t} \frac{\partial f_t^i}{\partial x_k} \cdot y_j^k \right)$$
$$= \left(X_1(y), \ldots, X_n(y); \sum_k \frac{\partial X_i}{\partial x_k}(y) \cdot y_j^k \right).$$

According to Proposition 3 of [7], a sheaf $\mathscr{L}$ of Lie algebras of germs of vector fields is transitive if and only if the distribution $\mathscr{M}(\mathscr{L})$ is of maximal dimension. Applying this proposition to our case, we get that the distribution $\mathscr{M}(\tilde{\mathscr{U}})$ has to be of maximal dimension, so the space

$$\left\{ \left(X_1(y), \ldots, X_n(y); \frac{\partial X_i}{\partial x_k}(y) \right); (X)_y \in \mathscr{U}_y \right\}$$

at each point y of the manifold M has to have the dimension equal to the dimension of B. Thus the following proposition has been proved:

Proposition 1. *Let B be a G-structure on a connected manifold M, $m = \dim B$ and G be a connected Lie group. The G-structure B is infinitesimally transitive if and only if the distribution $\mathscr{M}(B)$ is of dimension m, where*

$$\mathscr{M}(B)_y = \left\{ \left(X_1(y), \ldots, X_n(y); \frac{\partial X_i}{\partial x_k}(y) \right) : X = \sum_{i=1}^{n} X_i \partial_i \in \mathscr{U}(\operatorname{dom} X),\ y \in \operatorname{dom} X \right\}.$$

For infinitesimally transitive structures it is possible to give a theorem similar to that formulated by Bernard [2].

Theorem 1. *Let B be a G-structure and B' a G'-structure on M.*

(i) *If the sheaf $\widetilde{\mathscr{U} \cap \mathscr{U}'}(B)$ is transitive on B, then there exists $l \in \mathrm{GL}(V)$ such that $B' \supset B \cdot l$.*

(ii) *If the sheaf $\widetilde{\mathscr{U} \cap \mathscr{U}'}(B)$ is transitive on B and the sheaf $\widetilde{\mathscr{U} \cap \mathscr{U}'}(B')$ is transitive on B', then there exists $l \in \mathrm{GL}(V)$ such that $B' = B \cdot l$.*

Now we are going to derive several useful propositions concerning some special cases.

Proposition 1. *Let B be a G-structure on a connected manifold M, m and $\mathscr{M}'$ a regular distribution of constant dimension k' on a connected manifold M. If there exists a G-structure B related to $\mathscr{M}$ such that $\mathscr{M}(B)$ is of maximal dimension and $\mathscr{U}' \supset \mathscr{U}$, then $\mathscr{M} = \mathscr{M}'$.*

Proof. The orbits of $\mathcal{M}(\tilde{\mathcal{U}})$ are the connected components of B. The orbits are G^0-structures (G^0 – the connected component of e in G). As infinitesimal automorphisms preserve connected components, there exists $l \in \mathrm{GL}(V)$ such that $B'(G'^0) \supset B(G^0)l$ (B' – any G'-structure related to the distribution $\mathcal{M}'$). Therefore, $G'^0 \supset l^{-1}G^0 l$. According to the construction of G-structures related to distributions $G = G(V_1)$, $G' = G(V_2)$, where V_1 (resp. V_2) is a vector subspace of V of dimension k (resp. k') and $G(V_i)$ is the group of all automorphisms of V which preserve V_i. Then $G^0(V_2) \supset G^0(l^{-1}V_1)$ and $g(V_2) \supset g(l^{-1}V_1)$ which implies that $V_2 = l^{-1}V_1$. Therefore,

$$\mathcal{M}' = B'(V_2) = Bl(V_2) = Bl(l^{-1}V_1) = B(V_1) = \mathcal{M}.$$

PROPOSITION 3. *Let $\mathcal{M}$, $\mathcal{M}'$ be two almost product structures on a connected manifold M and $\mathcal{M}^i = \bigoplus_{j \in S_i} \mathcal{M}_j$, where $S_i = \{j\colon \dim \mathcal{M}_j = i\}$. Then $\mathcal{M}^i = \mathcal{M}'^i$ if there exist G-structures B and B' – the former related to $\mathcal{M}$, the latter to $\mathcal{M}'$ – such that the distributions $\mathcal{M}(B)$ and $\mathcal{M}(B')$ are of maximal dimension, and $\mathcal{U}' \supset \mathcal{U}$.*

Proof. The similar argumentation as in Proposition 2 proves that $g' = l^{-1}gl$ for some $l \in \mathrm{GL}(V)$. The vector space V, both as a g'-module and a $l^{-1}gl$ module, $(l^{-1}V)^p \supset V_j'$ when $\dim V_j = p$, so $(l^{-1}V)^p = (V')^p$. Analogously, isomorphic. Let $p = \min \dim V_i'$. Since $(l^{-1}V)^p + V_i'$ is a direct factor of the $l^{-1}gl$ module, $(l^{-1}V)^p \supset V_j'$ when $\dim V_j = p$, so $(l^{-1}V)^p = (V')^p$. Analogously, we get $(l^{-1}V)^i = (V')^i$ for $1 \leqslant i \leqslant n$, hence $(\mathcal{M}')^i = B'(V')^i = Bl(l^{-1}V)^i = \mathcal{M}^i$.

One of the most important problems in the theory of infinitesimal automorphisms is the assessment of the dimension of the stalks of the sheaf $\mathcal{U}$ for a given G-structure. The following proposition deals with the problem of the codimension of stalks.

PROPOSITION 4. *Let B be a G-structure and $\mathcal{U}$ be the sheaf of germs of the infinitesimal automorphisms of B. If the stalks of the sheaf $\mathcal{U}$ have finite codimension, then $G = \mathrm{GL}(V)^+$ or $G = \mathrm{GL}(V)$.*

The proof of the proposition is based on the following algebraic lemma:

LEMMA 1. *Let A be the Lie algebra of all germs at a point m of vector fields on a manifold M. If C is a proper subalgebra of A of finite codimension, then $C \subset C(m)$, where $C(m) = \{X \in A\colon X(m) = 0\}$.*

Proof. We are going to follow Amemiya's method. Let us assume that $C \not\subset C(m)$. Then there exists a vector field Z, $(Z)_m \in C$ which does not disappear at the point m, i.e., $Z_m \neq 0$. We can find a germ $(f)_m$ (f a smooth function on M) for which $(Z(f))_m$ is invertible in $C_m^\infty(M)$. Later on we denote $(Z)_m$ by X_0 and $(f)_m$ by f_0.

Put $C' = \{X \in C\colon [X, Y] \in C \text{ for every } Y \in A\}$; C' is an ideal of C. For $X \in C$ ad X; $Y \to [X, Y]$ induces a linear transformation T_X of the finite dimensional space A/C. The ideal C', being the kernel of the mapping T, is of finite codimension in A. The set E of $g \in C_m^\infty(M)$ for which $gX_0 \in C'$ and $gf_0X_0 \in C'$ is a subspace of $C_m^\infty(M)$ of finite codimension, since C' is of finite codimension in A.

Now we can define the following set of germs of functions:

$$F = \{g \in C_m^\infty(M)\colon\ gA \subset C\}.$$

First of all, we are going to prove that the codimension of F is finite, by showing that $E \subset F$.

Let $X \in A$ and $g \in E$; then

$$C \ni [gf_0X_0, X] = f_0[gX_0, X] - X(f_0)gX_0$$

and

$$C \ni [gX_0, f_0X] = f_0[gX_0, X] + gX_0(f_0)X;$$

hence

$$C \ni X(f_0)gX_0 + gX_0(f_0)X. \tag{1}$$

Substituting X by $(1/X_0(f_0))X(f_0)X_0$ in (1), we obtain

$$C \ni X(f_0)gX_0.$$

Taking into account (1), we get

$$C \ni gX_0(f_0)X.$$

Substituting X by $(1/X_0(f_0))X$, we finally have

$$C \ni gX.$$

That ends the proof of the relation $E \subset F$.

Let $g \in F$. Then for any $X \in A$

$$C \ni [gX, X_0] = g[X, X_0] - X_0(g)X.$$

Since $g[X, X_0] \in C$, $X_0(g)X$ is also an element of the algebra C. Thus $g \in F$ implies $X_0(g) \in F$.

There exists a non-zero polynomial P such that $P(f_0) \in F$, since F is a subspace of finite codimension. Then $X_0(P(f_0)) = P'(f_0)X_0(f_0) \in F$, which means that

$$P'(f_0)X_0(f_0)(1/X_0(f_0))X \in C \quad \text{for any } X \in A.$$

Therefore $P'(f_0) \in F$. We analogously get $P''(f_0) \in F$, so after a finite number of such steps we prove that

$$1 \in F,$$

which implies that $A \subset C$.

Proof of the proposition. The stalk $\mathcal{U}_m$ is of finite codimension, so either $\mathcal{U}_m = \mathcal{X}_m$ or $\mathcal{U}_m \subset C_m$. It is possible to find an open set U such that

$$\forall\, m \in U, \quad \mathcal{U}_m = \mathcal{X}_m.$$

Otherwise, the set $L = \{m \in M\colon \mathcal{U}_m \subset C_m\}$ is dense in M. Let X be an infinitesimal automorphism of B. Then $X|_{L\cap \mathrm{dom} X} \equiv 0$ and since $\overline{L \cap \mathrm{dom}\, X} \supset \mathrm{dom}\, X$, we have $X \equiv 0$. A contradiction.

The G^0-structure $B^0|\,U$ has the sheaf of germs of the infinitesimal automorphisms $\mathcal{U}|\,U = \mathcal{X}|\,U$. According to Theorem 1, $G^0 = \mathrm{GL}(V)^+$.

We can improve Lemma 1 by emphasizing the most important points of the proof. Let A be any Lie algebra of germs of vector fields at a point m. Then $R = \{f \in C_m^\infty(M)\colon fA \subset A\}$ is a ring.

LEMMA 2. *Let A be a Lie algebra of germs of vector fields at a point m, and C a subalgebra of A finite codimension. If there exist $X_0 \in C$ and $f_0 \in R$ such that $1/X_0(f_0) \in R$ and $(1/X_0(f_0))X(f_0)X_0 \in A$ for any $X \in A$, then $C = A$.*

Proof. Let us define

$$C' = \{X \in C\colon [X, Y] \in C, \text{ for any } Y \in A\},$$
$$E = \{g \in R\colon gX_0 \in C', gf_0X_0 \in C'\},$$
$$F = \{g \in R\colon gA \subset C\}.$$

With the sets C', E and F defined as above, the proof of Lemma 2 is the same as that of Lemma 1.

Lemma 2 applied to the case of two distributions yields the following result.

THEOREM 2. *Let $\mathcal{M}$, $\mathcal{M}'$ be two distributions. If the following conditions are fulfilled:*

(i) *the sheaf $\mathcal{U}$ is transitive;*

(ii) *$\mathcal{M}'$ is a completely integrable distribution* (1);

(iii) *there exist a point m_0 and an open neighbourhood W of that point such that for any point m $\dim(\mathcal{U}'/\mathcal{U})_m < +\infty$;*

(iv) *$\mathcal{M}^*$, the smallest involutive distribution containing $\mathcal{M}$ and $\mathcal{M}'$, is at most of dimension $n-1$ ($n = \dim M$);*

then $\mathcal{M}$ is a completely integrable distribution. If, additionally, $\mathcal{U}' \supset \mathcal{U}$, then $\mathcal{M} = \mathcal{M}'$.

Proof. At the very beginning we will determine the algebras $\mathcal{U}'_m$ in relation to the adopted maps of $\mathcal{M}'$ and $\mathcal{M}^*$. To achieve this we need to prove a lemma.

(1) This condition is equivalent to the hypothesis that $\mathcal{M}'$ is of constant dimension and the distribution $\mathcal{M}(B)$ is of maximal dimension for a G-structure related to $\mathcal{M}'$ (see Proposition 8).

LEMMA 3. *Let $\mathcal{M}$, $\mathcal{M}'$ be two completely integrable distributions and $\mathcal{M} \subset \mathcal{M}'$. Then for any point m of the manifold M there exists a map (U, φ) at this point such that $\mathcal{M}|U = \rangle\partial_1, \ldots, \partial_k\langle$ and $\mathcal{M}'|U = \rangle\partial_1, \ldots, \partial_{k'}\langle$, where* $\dim \mathcal{M} = k$ *and* $\dim \mathcal{M}' = k'$.

Proof. Because of the theorem proved in [6] it is sufficient to find locally a completely integrable distribution $\mathcal{M}''$ complementary to $\mathcal{M}$ in $\mathcal{M}''$, i.e., $\mathcal{M}''+\mathcal{M}|U = \mathcal{M}'|U$.

Let (U, φ) be an adopted map for the distribution $\mathcal{M}'$ at a point m_0, i.e., $\mathcal{M}'|U = \rangle\partial_1, \ldots, \partial_{k'}\langle$. Since $\mathcal{M}$ is of constant dimension, it is possible to find $X_1, \ldots, X_k$, C^∞ independent vector fields on U (we can always choose a smaller U) such that $\mathcal{M}|U = \rangle X_1, \ldots, X_k\langle$.

Then $X_i = \sum_{j=1}^{k'} f_i^j \partial_j$.

At each point m of U, rank $(f_i^j(m)) = k$. We may assume that $(f_i^j(m_0))$, $i, j = 1, \ldots, k$ has the rank k. Taking a small enough U, we see that the rank of $(f_i^j(m))$, $i, j = 1, \ldots, k$, is equal to k for $m \in U$.

Let $A(m) = (a_l^i(m))$ be the inverse matrix of $(f_i^j(m))$, $i, j = 1, \ldots, k$. Then

$$Y_l = \sum_{i=1}^{k} \sum_{j=1}^{k'} a_l^i f_i^j \partial_j = \partial_l + \sum_{j=k+1}^{k'} c_l^j \partial_j.$$

The vector fields Y_l for $l = 1, \ldots, k$ are C^∞ independent. The distribution $\mathcal{M}'' = \rangle\partial_{k+1}, \ldots, \partial_{k'}\langle$ is the distribution we have been looking for.

According to Lemma 3, in a certain neighbourhood W of m_0

$$\mathcal{M}^*|W = \rangle\partial_1, \ldots, \partial_l\langle, \quad l \leqslant n-1, \quad \mathcal{M}'|W = \rangle\partial_1, \ldots, \partial_{k'}\langle.$$

Then $\mathcal{U}'(W) = \{X\colon X = \sum f_i \partial_i,\ \partial_j(f_i) \equiv 0 \text{ for } i > k',\ j \leqslant k'\}$. In this case $R_m = \{f \in C_m^\infty(M)\colon \partial_j(f) \equiv 0 \text{ for } j \leqslant k'\}$. We will prove that $(\partial_n)_m \in \mathcal{U}_m$ for a slightly modified coordinate system at a certain point $m \in W$. The vector space $\{(f\partial_n)_{m_0}\}$, f – a smooth function of x_n, is of infinite dimension. Hence, for a certain f, $(f\partial_n)_{m_0} \in \mathcal{U}_{m_0}$, which means that $f\partial_n|V$ – V is an open neighbourhood of m_0, $V \subset W$ – is an infinitesimal automorphism of $\mathcal{M}$. There exists $m \in V$ such that $f(m) \neq 0$ and $(f\partial_n)_m \in \mathcal{U}_m$. Therefore $\mathcal{U}_m$ and $\mathcal{U}'_m$ conform to the hypothesis of Lemma 2, taking $X_0 = \partial'_n$, $f_0 = x'_n$. Hence $\mathcal{U}_m = \mathcal{U}'_m$. Let us choose infinitesimal automorphisms $X_1, \ldots, X_r$ at the point m such that

$$\dim \rangle j_m^1 X_i \langle = \dim B \quad (B \text{ – a } G\text{-structure related to } \mathcal{M}').$$

This relation is true for a certain open neighbourhood U of m. According to Proposition 2, $\mathcal{M}'|U = \mathcal{M}|U$. Since the sheaf $\mathcal{U}$ is transitive, $\mathcal{M}$ is a completely integrable distribution (see [5], [7]). In the case of $\mathcal{U}' \supset \mathcal{U}$, of course, $\mathcal{M} = \mathcal{M}'$.

A proposition very similar to Proposition 4 can be proved for almost product structures, here we do not assume these distributions to be of constant dimension.

PROPOSITION 5. *Let $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_k)$ be an almost product structure, for which the sheaf $\mathcal{U}$ is regular. If the codimensions of the stalks of the sheaf $\mathcal{U}$ are at most denumerable, then $k = 1$ and $\mathcal{M}_1 = \mathcal{M}_0$. ($\mathcal{M}_0 = T_m M$ for each point m of the manifold M.)*

Proof. The distributions $\mathcal{M}_i$ are of constant dimension as the sheaf $\mathcal{U}$ is transitive (see [7]).

Then

$$\dim (\mathcal{M}_i/\mathcal{C}^i)_m \leqslant \dim (\mathcal{M}_i/\mathcal{M}_i \cap \mathcal{U})_m \leqslant \dim (\mathcal{X}/\mathcal{U})_m \leqslant \aleph_0,$$

where $\mathcal{C}^i$ is the distribution generated by $\mathcal{M}_i \cap \mathcal{U}$. The distribution $\mathcal{C}^i$ is completely integrable. Of course $\mathcal{U} \subset \mathcal{U}_{\mathcal{C}^i}$ and $\mathcal{C}^i$ is involutive.

Now, we will prove that $\mathcal{M}_i = \mathcal{C}^i$. Let $m \in M$. As distributions $\mathcal{M}_i$, $\mathcal{C}^i$ are of constant dimension, it is possible to construct a distribution S such that $\mathcal{M}_i | U = \mathcal{C}_i | U + S$ for a certain open neighbourhood U of the point m. Then $\dim S_m \leqslant \dim (\mathcal{M}_i/\mathcal{C}^i)_m$, hence $S \equiv 0$. (See [7].)

Therefore $\mathcal{M}_i$ is a completely integrable distribution. This is equivalent to the following condition:

$$\forall\, m \in M \;\exists (U, \varphi) \in Atl(M, m),\; \varphi(x) = (X_1, \ldots, X_n)\colon\; \rangle \partial_1, \ldots, \partial_p \langle \;=\; \mathcal{M}_i | U.$$

If we assume that $\mathcal{M}_i$ is not trivial, it means that $0 < p < n$. Let us consider smooth functions on U such that $\partial_1(f) \not\equiv 0$. The dimension of the space of the germs of $f\partial_n$ at a given point is greater than c. Then

$$[\partial_1, f\partial_n] = f[\partial_1, \partial_n] + \partial_1(f)\partial_n = \partial_1(f)\partial_n \not\equiv 0.$$

Therefore $f\partial_n$ is not an infinitesimal automorphism of the distribution $\mathcal{M}_i$, which leads to the contradiction of the hypothesis on codimension.

Note. It is obvious that an infinitesimally transitive G-structure B is a connected manifold. Hence it is impossible to talk about infinitesimally transitive structures for $G(V_1)$ or $G(V_1, \ldots, V_k)$. These two Lie groups are not connected. At most we can tell whether the distribution $\mathcal{M}(B)$ is of maximal dimension or not. We propose to call G-structures for which the dimension of $\mathcal{M}(B)$ is maximal, "almost infinitesimally transitive".

For almost infinitesimally transitive structures the following propositions are true:

PROPOSITION 6. *An integrable G-structure is almost infinitesimally transitive.*

PROPOSITION 7. *The structure tensor of an almost infinitesimally G-structure B is constant on connected components of B.*

EXAMPLE 1. Let $\mathcal{M}$ be a distribution on a manifold M. In this case only integrable structures are almost infinitesimally transitive.

PROPOSITION 8. *$\mathcal{M}$ is a completely integrable distribution if and only if each G-structure related to $\mathcal{M}$ is almost infinitesimally transitive.*

Proof. According to Proposition 7 the structure tensor of a G-structure is constant on connected components of B, then the structure tensor is zero; hence the distribution is completely integrable (see [3]).

Note. Exactly the same proposition is true for almost product structures.

EXAMPLE 2. (Riemannian almost product structures.)

DEFINITION. A Riemannian almost product structure of the type $(n_1, \ldots, n_r)$ is an n-dimensional Riemannian manifold M with the positively defined metric and r regular distributions $\mathcal{M}_i$ of constant dimension on M satisfying the following conditions:

(i) $\dim \mathcal{M}_i = n_i$,

(ii) $TM = \mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_r$,

(iii) the subspaces $\mathcal{M}_i(m)$ are mutually orthogonal.

Gauchman in his paper [4] introduces the following special structure:

DEFINITION. A Riemannian almost product structure of the type $(n_1, \ldots, n_r)$ is called a *maximally mobile Riemannian almost product structure* or simply *M-structure of the type* $(n_1, \ldots, n_r)$ if for any orthonormal adapted frames $R = (x; e_1, \ldots, e_n)$ (i.e., $e_i \in \mathcal{M}_{j_i}$ for some j_i) and $R_1 = (y; c_1, \ldots, c_n)$ having the same orientation, there exists a local automorphism of the M-structure mapping R onto R_1.

The proofs of the next two propositions are straightforward.

PROPOSITION 9. *The structure tensor of an M-structure is constant on connected components.*

PROPOSITION 10. *An almost infinitesimally transitive Riemannian almost product structure is an M-structure.*

H. Gauchman proved the following proposition:

PROPOSITION 11. *If $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_r)$ is an M-structure of the type $(n_1, \ldots, n_r)$ and for some i $(1 \leqslant i \leqslant r)$ $n_i \neq 2$, then the distribution $\mathcal{M}_i$ is involutive.*

Proposition 7 allows us to formulate a second version of the previous proposition. The proof is based on a lemma which is proved below.

LEMMA 4. *Let $\mathcal{M}$ be a Riemannian almost product structure without distributions of dimension one. If the structure tensor of $\mathcal{M}$ is constant on connected components, then the distributions are completely integrable.*

Proof. Let us choose any distribution $\mathcal{M}_i$ from $\mathcal{M}$. By B we will denote a G-structure related to $\mathcal{M}$ and by B_i a G_i-structure related to $\mathcal{M}_i$. The structure tensor c_i of B_i $(c(p) \in \mathrm{Hom}(V \wedge V, V)/\partial \mathrm{Hom}(V, g)$

$\simeq \mathrm{Hom}(W \wedge W, V/W)$ (see [3])) is obtained from the structure tensor c of B, since $B \subset B_i$. Therefore c_i is constant on connected components of B which means that

$$c(pa) = c(p) = u \quad \text{for } p \in B,\ a \in G^0; \tag{1}$$

computing, we get

$$c(pa) = r(a^{-1})c(p) = r(a^{-1})u = u,$$

$$r(a)u(v_1, v_2) = au(a^{-1}v_1, a^{-1}v_2) = u(v_1, v_2). \tag{2}$$

In our case, $\mathrm{Hom}(W \wedge W, V/W)$ is isomorphic to $\mathrm{Hom}(W \wedge W, W_1)$ with the induced representation, where W_1 is the subspace which is complementary to W and left invariant by G. Therefore, choosing a belonging to $SO(n_1) \times \ldots \times SO(n_r)$ such that $a_i = I$, (2) can be written in the form

$$au(v_1, v_2) = u(v_1, v_2)$$

for any $a \in S = SO(n_1) \times \ldots \times SO(n_{i-1}) \times SO(n_{i+1}) \times \ldots \times SO(n_r)$.

This means that $u(v_1, v_2)$ generates 1-dimensional invariant subspace of W_1. Since W_1 as an S-module is semi-simple and representations of $SO(n_j)$ on W_j are simple, $u = 0$. Hence $\mathcal{M}_i$ is involutive.

THEOREM 3. *The underlying almost product structure $\mathcal{M}$ of an M-structure with $n_i \neq 1$ is involutive.*

Proof. Because of Lemma 4 and Proposition 9 the distributions of $\mathcal{M}$ are involutive.

THEOREM 4. *The underlying almost product structure of an almost infinitesimally transitive Riemannian almost product structure without distributions of dimension one is involutive.*

References

[1] I. Amemiya, *Lie algebra of vector fields and complex structure*, J. Math. Soc. Japan 27 (1975), p. 545–549.

[2] B. Bernard, *Sur la géométrie différentielle des G structures*, Ann. Inst. Fourier 10 (1960), p. 151–270.

[3] A. Fujimoto, *The theory of G structures*, Tokyo 1972.

[4] H. H. Gauchman, *On maximally mobile Riemannian almost product structure*, Tensor (N.S.) 31 (1977), p. 13–26.

[5] P. Lecompte, *Infinitesimal automorphisms and integrability of distributions*, Bull. Soc. Roy. Sci. Liège 46 (1977), p. 44–50.

[6] J. Lehmann-Lejeune, *Intégrabilité des G structures définies par une 1-forme 0-déformable à valeurs dans le fibre tangent*, Ann. Inst. Fourier 16 (1966), p. 339–387.

[7] R. Wolak, *Infinitesimal automorphism of distributions* (to appear).

Reçu par la Rédaction le 15.10.1979