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GAMMA-MINIMAX ESTIMATORS FOR THE PARAMETERS OF A MULTINOMIAL DISTRIBUTION

Abstract. In this paper Γ-minimax estimators for the parameters of a multinomial distribution are determined when the loss is a weighted sum of squared errors. Classes Γ are considered which consist of all priors having first moments between some given bounds. The Γ-minimax estimators are linear estimators and they are Bayes with respect to least favourable Dirichlet priors. If Γ consists of all priors, then the usual minimax estimators are obtained which are due to Steinhaus [3] and Trybuła [6].

1. Notation and introduction. The parameters $\theta_1, \ldots, \theta_k$ of a multinomial distribution $M(n; \theta_1, \ldots, \theta_k)$ are to be estimated, where the parameter vector $(\theta_1, \ldots, \theta_k)$ is an element of the parameter space

$$\Theta = \{(\theta_1, \ldots, \theta_k) \in [0, 1]^k \mid \theta_1 + \ldots + \theta_k = 1\}.$$

Let Δ be the set of all (non-randomized) estimators δ: $X \to \Theta$, where

$$X = \{(x_1, \ldots, x_k) \in \{0, \ldots, n\}^k \mid x_1 + \ldots + x_k = n\}$$

denotes the sample space. A Borel probability measure π on the parameter space Θ is called a *prior* and the set of all priors is denoted by Π. The *Bayes risk of an estimator* δ *with respect to a prior* π is defined by

$$r(\pi, \delta) = \int_{\Theta} R(\theta, \delta)\pi(d\theta),$$

where $R(\cdot, \delta)$ denotes the risk function of $\delta = (\delta_1, \ldots, \delta_k)$ given by

$$R(\theta, \delta) = \sum_{x \in X} \sum_{i=1}^{k} s_i(\theta_i - \delta_i(x))^2 f(\theta, x)$$

with

$$f(\theta, x) = (n!) \sum_{i=1}^{k} \frac{\theta_i^{x_i}}{x_i!}, \qquad x = (x_1, \ldots, x_k) \in X,$$

for $\theta = (\theta_1, \ldots, \theta_k) \in \Theta$ when the loss is assumed to be a weighted sum of squared errors with positive weights $s_1, \ldots, s_k$.

If there is precise prior information on the distribution of the unknown parameters $\theta_1, \ldots, \theta_k$ which can be described by a single prior π, then usually the Bayes principle is applied. If on the other hand no prior information is available, then the minimax principle can be used. In this paper an intermediate approach between the Bayes and the minimax principle is chosen. The use of the Γ-minimax principle is appropriate if vague prior information is available which can be described by a subset Γ of the set Π of all priors. A Γ-minimax estimator δ^* minimizes the maximum Bayes risk with respect to the elements of Γ, i.e.,

$$\sup_{\pi\in\Gamma} r(\pi, \delta^*) = \inf_{\delta\in\Delta} \sup_{\pi\in\Gamma} r(\pi, \delta).$$

Solutions of Γ-minimax problems for various classes Γ of priors have been obtained, e.g., in [1], [2], and [5]. In this paper subsets of priors of the form

$$\Gamma = \{\pi\in\Pi \mid \nu_i \leqslant \int_{\Theta} \theta_i \pi(d\theta) \leqslant \mu_i,\ 1 \leqslant i \leqslant k\}$$

are considered for some fixed bounds with

$$0 \leqslant \nu_i \leqslant \mu_i \leqslant 1, \quad 1 \leqslant i \leqslant k,$$

$$\nu_1 + \ldots + \nu_k \leqslant 1 \leqslant \mu_1 + \ldots + \mu_k,$$

i.e., Γ consists of all priors whose first moments are between given bounds.

2. Gamma-minimax estimators. In the sequel, weights $s_1, \ldots, s_k$ as well as lower and upper bounds $\nu_1, \ldots, \nu_k$ and $\mu_1, \ldots, \mu_k$ are fixed according to the definitions of the loss function and the subset of priors, respectively. For $c\in \boldsymbol{R}$ and $1 \leqslant i \leqslant k$ put

$$\alpha_i(c) = \operatorname{med}\left(\nu_i\sqrt{n},\ \frac{1}{2\sqrt{n}}(n - c/s_i),\ \mu_i\sqrt{n}\right),$$

where $\operatorname{med}(u, v, w)$ denotes the middle of the three numbers u, v, and w. Obviously, there exists a $c_0\in\boldsymbol{R}$ with

$$\sum_{i=1}^{k} \alpha_i(c_0) = \sqrt{n}.$$

THEOREM. *The estimator $\delta^* = (\delta_1^*, \ldots, \delta_k^*)$ given by*

$$\delta_i^*(x) = \frac{x_i + \alpha_i(c_0)}{n + \sqrt{n}}, \quad 1 \leqslant i \leqslant k,\ x = (x_1, \ldots, x_k)\in \boldsymbol{X},$$

is Γ-minimax.

Proof. (i) Put $\alpha_i = \alpha_i(c_0)$, $1 \leqslant i \leqslant k$, and

$$\Theta_0 = \{(\theta_1, \ldots, \theta_k)\in\Theta \mid \theta_i = 0 \text{ if } \alpha_i = 0,\ 1 \leqslant i \leqslant k\}.$$

Let π_0 be the Dirichlet prior with $\pi_0(\Theta_0) = 1$ and density

$$f(\theta) = \Gamma(\sqrt{n}) \sum_{\substack{i=1 \\ \alpha_i > 0}}^{k} \frac{\theta_i^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \qquad \theta = (\theta_1, \ldots, \theta_k) \in \Theta_0,$$

with respect to Lebesgue measure on Θ_0. It is well known (see, e.g., [4], Chapter 5) that the estimator δ^* is Bayes with respect to π_0.

(ii) A straightforward calculation shows that the risk function of the estimator δ^* is given by

$$R(\theta, \delta^*) = \frac{1}{(n+\sqrt{n})^2} \sum_{i=1}^{k} s_i((n-2\alpha_i\sqrt{n})\theta_i + \alpha_i^2), \qquad \theta = (\theta_1, \ldots, \theta_k) \in \Theta.$$

Let

$$v_i(\pi) = \int_{\Theta} \theta_i \pi(d\theta), \qquad 1 \leqslant i \leqslant k,$$

denote the first moments of a prior π. Then the Bayes risk of the estimator δ^* with respect to a prior π is given by

$$r(\pi, \delta^*) = \frac{1}{(n+\sqrt{n})^2} \sum_{i=1}^{k} (s_i(n-2\alpha_i\sqrt{n}) - c_0)\gamma_i(\pi) + \frac{1}{(n+\sqrt{n})^2}(c_0 + \sum_{i=1}^{k} s_i\alpha_i^2).$$

Since $\gamma_i(\pi_0) = \alpha_i/\sqrt{n}$, $1 \leqslant i \leqslant k$, it follows that

$$\sup_{\pi \in \Gamma} r(\pi, \delta^*) = r(\pi_0, \delta^*),$$

which completes the proof of the Theorem.

3. Concluding remarks. (a) Suppose that $\mu_1 = \ldots = \mu_k = 1$, i.e., Γ consists of all priors whose first moments exceed some given lower bounds. It can be assumed without loss of generality that

$$s_1(1-2v_1) \geqslant \ldots \geqslant s_k(1-2v_k).$$

The subsequent results follow after some straightforward calculations which are similar to those of Trybuła (cf. [4], Chapter 5). Put

$$L = \max\{1 \leqslant l \leqslant k \mid s_l(1-2v_l) \sum_{j=1}^{l} \frac{1}{s_j} \geqslant l-2+2 \sum_{j=l+1}^{k} v_j\}.$$

Then the estimator $\delta^* = (\delta_1^*, \ldots, \delta_k^*)$ given by

$$\delta_i^*(x) = \begin{cases} \dfrac{x_i + \frac{1}{2}\sqrt{n}\left(1-(L-2+2\sum_{j=L+1}^{k} v_j)/(s_i \sum_{j=1}^{L} s_j^{-1})\right)}{n+\sqrt{n}}, & 1 \leqslant i \leqslant L, \\[2ex] \dfrac{x_i + v_i\sqrt{n}}{n+\sqrt{n}}, & L+1 \leqslant i \leqslant k, \end{cases}$$

for $x = (x_1, \ldots, x_k) \in X$ is Γ-minimax. If $\nu_1 = \ldots = \nu_k = 0$, i.e., $\Gamma = \Pi$, then δ^* is the usual minimax estimator (cf. [4], Chapter 5, and [6], Chapter 4). If additionally $s_1 = \ldots = s_k$, then $L = k$ and

$$\delta_i^*(x) = \frac{x_i + k^{-1}\sqrt{n}}{n + \sqrt{n}}, \quad 1 \leqslant i \leqslant k, \ x = (x_1, \ldots, x_k) \in X,$$

(cf. [3], Chapter 7).

(b) Suppose that $\nu_i = \mu_i$, $1 \leqslant i \leqslant k$, i.e., Γ consists of all priors with fixed first moments. Then the estimator $\delta^* = (\delta_1^*, \ldots, \delta_k^*)$ given by

$$\delta_i^*(x) = \frac{x_i + \nu_i\sqrt{n}}{n + \sqrt{n}}, \quad 1 \leqslant i \leqslant k, \ x = (x_1, \ldots, x_k) \in X,$$

is Γ-minimax. Observe that this estimator does not depend on the weights $s_1, \ldots, s_k$ of the loss function.

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