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On periodic solutions of a certain third-order non-linear differential equation

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1. Consider the third-order differential equation

$$x''' + ax'' + bx' + f(x) = p(t), \tag{1}$$

where a, b are positive constants and functions $f(x)$, $p(t)$ are defined and continuous for all x and t and such that (1) has a solution satisfying any initial conditions. Moreover, suppose that $f(x)$ and $p(t)$ satisfy the following assumptions:

$$f(x)x > 0 \quad \text{for} \quad x \neq 0, \tag{2}$$

$$|f(x)| \leqslant M \quad \text{for all } x, \tag{3}$$

$$F(x) \equiv \int_0^x f(u)\,du \to \infty \quad \text{as} \quad |x| \to \infty, \tag{4}$$

$$|p(t)| \leqslant p, \quad |P(t)| \equiv \left|\int_0^t p(s)\,ds\right| \leqslant p \quad \text{for all } t. \tag{5}$$

THEOREM 1. *If* (2), (3), (4), (5) *hold and function* $p(t)$ *is periodic with the period* ω, *then equation* (1) *has at least one periodic solution with the period* ω.

This theorem strengthens the result of J. O. C. Ezeilo ([1]), who has proved the existence of a solution of (1) with the period $n\omega$, where $n \geqslant 1$ is an integer. In [1] in place of condition (2) it is assumed that $f(x)x > 0$ beyond the segment $\langle x_1, x_2\rangle$ $(-1 < x_1 \leqslant 0 \leqslant x_2 < 1)$, but this is inessential. As can be proved, the replacement of the function $f(x)$ in a compact set by any continuous function has no effect upon the boundedness of the solutions. Moreover, the assumption $f(x)\operatorname{sgn} x \geqslant m > 0$, $|x| \geqslant 1$ used in [1] is replaced by the more general assumption (4).

The method of proof is the same as the one used in the 3 part of [1]. It consists in showing, by the use of the Leray-Schauder technique for Banach spaces, that a certain operator in the Banach space induced by (1) has a fixed point.

2. Proof of Theorem 1. Let us consider alongside with (1) the family (E_μ^*) of differential equations

$$(E_\mu^*) \qquad x''' + a(\mu)x'' + \beta(\mu)x' + \gamma(\mu)x + \mu^2 f(x) = \mu p(t), \qquad \mu \in \langle 0, 1\rangle ,$$

where

$$(6) \qquad a(\mu) = \mu a + (1-\mu)a_1, \qquad \beta(\mu) = \mu b + (1-\mu)b_1, \qquad \gamma(\mu) = (1-\mu)^2 c .$$

Numbers a_1, b_1, c are certain positive constants chosen in such a way that the equation

$$\varrho^3 + a_1\varrho^2 + b_1\varrho + c = 0$$

has real distinct roots, and the following inequalities are satisfied:

$$(7) \qquad c < \min_\mu a(\mu) \min_\mu \beta(\mu) ,$$

$$(8) \qquad c < \min_{\mu \in \langle 0,1\rangle} \big(\beta^2(\mu)/a(\mu)\big) .$$

We shall always use in the sequel a, β, γ instead of $a(\mu)$, $\beta(\mu)$, $\gamma(\mu)$.

Let X^* be a space of continuous functions with the period ω having the norm $\|x\| = \max|x(t)|$. We define in X^* the operator L by ascribing to any function $\varphi(t) \in X^*$ the function $\xi(t)$, which is the solution with the period ω of the differential equation

$$(9) \qquad x''' + a_1x'' + b_1x' + cx = \varphi(t) .$$

It can easily be demonstrated, by finding the general solution of (9) and remembering the conditions imposed upon a_1, b_1, c, that the operator L is well-defined on the whole X^* and that L is linear in X^*.

The equation (E_μ^*) is equivalent to the equation

$$x''' + a_1x'' + b_1x' + cx = \mu\Phi(\mu, x) ,$$

where

$$\Phi(\mu, x) = (a_1 - a)x'' + (b_1 - b)x' + (2-\mu)cx - \mu f(x) + p(t) ,$$

and this is not periodic in t unless x and p are themselves periodic. Thus if, in the sequel, we suppose that $p(t)$ has a period ω, then it follows that any continuously differentiable function $x(t)$ with a period ω is a solution of (E_μ^*) if and only if

$$x = \mu L\Phi(\mu, x) .$$

Let X denote the linear space of twice continuously differentiable functions $x(t)$ with a period ω, having the norm $\|x\| = \max|x''| + \max|x'| + \max|x|$. A function $x \in X$ is a solution of (E_μ^*) if and only if it satisfies the equation

$$(\mathcal{E}_\mu) \qquad x - \mu T(\mu, x) = 0 ,$$

where the operator $T(\mu, x)$ is defined by

$$T(\mu, x) = L\Phi(\mu, x), \quad (\mu, x) \in \langle 0, 1\rangle \times X .$$

(Observe that from the definition of Φ and L it follows that the operator $T(\mu, x)$ is continuous in both arguments and completely continuous.)

Thus in order to prove Theorem 1 it is sufficient to show that the equation

$$(\mathcal{E}_1) \qquad x - T(1, x) = 0$$

has at least one solution.

Suppose that an a priori bound $\|x\| < A$ is known to exist for any x satisfying $(\mathcal{E}_\mu)$, where A is a constant independent of μ. Then from the Leray-Schauder theory it follows that the equation $(\mathcal{E}_1)$ has at least one solution $x \in X$.

We shall demonstrate that solutions of (E_μ^*) are bounded with their first and second derivatives by a constant independent of μ. In view of the definition of the norm in the space X, it follows that solutions of $(\mathcal{E}_\mu)$ are uniformly bounded with respect to μ.

If we consider the differential system

$$(\mathrm{E}_\mu) \qquad \begin{aligned} x' &= y ,\\ y' &= z ,\\ z' &= -\alpha z - \beta y - \mu^2 f(x) - \gamma x + p(t)\mu , \qquad 0 \leqslant \mu \leqslant 1 \end{aligned}$$

corresponding to (E_μ^*), then to prove the boundedness of solutions of (E_μ^*) it is enough to prove the following theorem:

3. THEOREM 2. *If* (2), (3), (4), (5) *hold, then there exists a constant D independent of μ such that for any point (x_0, y_0, z_0) the solution $x(t)$, $y(t)$, $z(t)$ of* (E_μ) *determined by the initial conditions*

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0$$

satisfies the inequality

$$|x(t)| + |y(t)| + |z(t)| < D ,$$

for $t \geqslant t_0 + T$, where T depends only on particular (x_0, y_0, z_0) chosen.

The proof of the theorem consists in constructing in the space $E^3 \times I$ $(I = \{t\colon t \geqslant 0\})$ a domain Δ, with the boundary Σ, included in the domain $|x| + |y| + |z| < D$, $t \in I$ and such that:

(i) all trajectories of (E_μ) cross Σ only inwards,

(ii) each trajectory of (E_μ) beginning outside Δ reaches Σ after a finite time, which depends only upon the initial point of the trajectory.

Σ consists of surfaces defined by conditions $W_1(x, y, z; \mu) = C$, $W_2(x, y, z, t; \mu) = K$, where C, K are positive constants suitably chosen and W_1, W_2 are some functions stated in the sequel.

4. LEMMA 1. *Let the functions a, β, γ be defined by* (6), *where the numbers a_1, b_1, c satisfy* (7) *and let k be a constant independent of μ such that*

$$1/a < k < \beta/c \quad \text{for all } \mu \in \langle 0, 1\rangle \tag{10}$$

(*by* (7) *such a number can always be chosen*); *then there exist positive numbers $\varepsilon, \lambda_1, \lambda_2$ such that for $e = \varepsilon$, $l = \lambda_1$, $L = \lambda_2$ the quadratic forms*

$$\Phi(x, y, z; \mu) = \gamma x^2 + k(\beta y^2 + z^2) + ay^2 + 2yz + 2\gamma(exz + kxy) - \\ - l(\gamma x^2 + y^2 + z^2),$$

$$\Psi(x, y, z; \mu) = e\gamma^2 x^2 + y^2(\beta - k\gamma) + z^2(ka - 1) - \gamma e(yz - axz - \beta xy) - \\ - L(\gamma^2 x^2 + y^2 + z^2), \qquad \mu \in \langle 0, 1\rangle$$

are positive semi-definite and this property holds uniformly in $\langle 0, 1\rangle$.

Proof. We shall prove that if $\varepsilon, \lambda_1, \lambda_2$ are sufficiently small, then for $\mu \in \langle 0, 1)$, $e = \varepsilon$, $l = \lambda_1$, $L = \lambda_2$ the principal minors of the matrices of the forms Φ and Ψ are positive. These minors are

$$M_1(l, e, \mu) = k - l,$$

$$M_2(l, e, \mu) = (k\beta + a - l)(k - l) - 1,$$

$$M_3(l, e, \mu) = \gamma(1 - l)(k\beta + a - l)(k - l) + 2k\gamma^2 e - \gamma^2 e^2(k\beta + a - l) - \\ - \gamma(1 - l) - \gamma^2 k^2(k - l),$$

$$N_1(L, e, \mu) = ka - 1 - L,$$

$$N_2(L, e, \mu) = (ka - 1 - L)(\beta - k\gamma - L) - \tfrac{1}{4}\gamma^2 e^2,$$

$$N_3(L, e, \mu) = \gamma^2[(e - L)(\beta - k\gamma - L)(ka - 1 - L) - 1/4 e^2 a\beta\gamma - \\ - \tfrac{1}{4} e^2 a^2(\beta - k\gamma - L) - \tfrac{1}{4} e^2\beta^2(ka - 1 - L) - \tfrac{1}{4} e^2\gamma^2(e - L)].$$

We have

$$M_i(0, 0, \mu) > 0, \quad N_i(0, 0, \mu) > 0 \quad (i = 1, 2) \quad \text{for} \quad \mu \in \langle 0, 1\rangle;$$

$$M_3(0, 0, \mu) = \gamma(ka - 1) + \gamma(\beta - \gamma k)k^2 > 0,$$

$$\frac{\partial}{\partial e} N_3(0, 0, \mu) = \gamma^2(\beta - k\gamma)(ka - 1) > 0$$

by (10) for $0 \leqslant \mu < 1$. In view of the continuity of functions M_j, N_j $(j = 1, 2, 3)$ the above inequalities imply that for $\varepsilon, \lambda_1, \lambda_2$ small enough the quantities $M_j(\lambda_1, \varepsilon, \mu)$, $N_j(\lambda_2, \varepsilon, \mu)$ are positive for all μ satisfying $0 \leqslant \mu < 1$. It is easy to show that $\varepsilon, \lambda_1, \lambda_2$ can be chosen independent of μ. Thus for $e = \varepsilon$, $l = \lambda_1$, $L = \lambda_2$ Φ and Ψ are in the segment $\langle 0, 1)$ positive definite. Since $M_3(l, e, 1) = N_3(L, e, 1) = 0$, then from the above considerations it follows immediately that $\Phi(x, y, z; 1)$, $\Psi(x, y, z; 1)$ are positive semi-definite, and this ends the proof of Lemma 1.

5. Suppose that k and ε are defined as in section 4. We define functions $W_1(x, y, z; \mu)$, $W_2(x, y, z, t; \mu)$ as follows:

$$2W_1(x, y, z; \mu) = 2\mu^2 F(x) + \gamma x^2 + k(\beta y^2 + z^2) + ay^2 + 2yz + 2\gamma(\varepsilon xz + kxy),$$
$$W_2(x, y, z, t; \mu) = z + ay + \beta x - \mu P(t).$$

By Lemma 1 we have the inequality $2W_1 - 2\mu^2 F(x) - \lambda_1(\gamma x^2 + y^2 + z^2) = \Phi(x, y, z; \mu) \geqslant 0$, which yields

$$g(x) + \lambda_1(y^2 + z^2) \leqslant 2W_1(x, y, z; \mu), \tag{11}$$

where

$$g(x) = \min_{\mu\in\langle 0,1\rangle} [2\mu^2 F(x) + \lambda_1\gamma x^2] = F(x)\frac{2c\lambda_1 x^2}{2F(x) + c\lambda_1 x^2}.$$

Since $g(x) > 0$, for all x and $g(x)\to\infty$ as $|x|\to\infty$ $W_1(x, y, z; \mu)$ is positive definite and

$$\lim_{(x,y,z)\to\infty} W_1(x, y, z; \mu) = \infty \tag{12}$$

uniformly with respect to μ.

We define in $E^3\times I$ the sets

$$\Delta_1(\mu, C) = \{(x, y, z, t):\ t\in I,\ W_1(x, y, z; \mu) \leqslant C\},$$
$$\Delta_2(\mu, K) = \{(x, y, z, t):\ t\in I,\ |W_2(x, y, z, t; \mu)| \leqslant K\},$$
$$\Delta(\mu, K, C) = \Delta_1(\mu, C)\cap\Delta_2(\mu, K), \qquad \Sigma(\mu, K, C) = \operatorname{Fr}\Delta(\mu, K, C),$$
$$E(r, q) = \{(x, y, z, t):\ t\in I,\ |x| \geqslant q,\ y^2 + z^2 \geqslant r^2\}.$$

For arbitrary fixed positive constants d and s_1, we choose a number r_0 such that for $\gamma^2 x^2 + y^2 + z^2 \geqslant r_0^2$

$$-\lambda^2(\gamma^2 x^2 + y^2 + z^2) + d(|\gamma x| + |y| + |z|) \leqslant -s_1. \tag{13}$$

It can be found by straightforward calculation that there exists a constant $h_1(r_0)$ such that for sufficiently large C the surface $W_1(x, y, z; \mu) = C$ considered for $y^2 + z^2 \leqslant r_0^2$ lies in the layer $|x - x_c| \leqslant h_1(r_0)$, where the number $x_c \geqslant \sqrt{2C/c}$ is the solution of the equation $W_1(u, 0, 0; \mu) = C$. From this it follows that a constant h_0 can be found such that for C large the points of intersection of the surface $W_1(x, y, z; \mu) = C$ and the plane

$$z + ay + \beta x = \beta\left(\sqrt{2C/c} - h_0\right) + p, \qquad \mu\in\langle 0, 1\rangle \tag{14}$$

lie outside the domain $y^2+z^2 \leqslant r_0^2$. Indeed, it is enough to choose h_0 so large that the points of (14) considered for $y^2+z^2 \leqslant r_0^2$ will lie outside the domain $|x-x_c| \leqslant h_1(r_0)$.

LEMMA 2. *Let us put $K(C) = \beta(\sqrt{2C/c}-h_0)$, where h_0 is chosen as above. Suppose that the hypotheses of Theorem 2 are satisfied and let q be a positive fixed constant; then there exists a number C_0 such that*

$$\mathrm{Fr}\,\Delta_1(\mu, C) \cap \mathrm{Fr}\,\Delta_2(\mu, K(C)) \subset E(r_0, q) \tag{15}$$

holds for $C \geqslant C_0$ and all $\mu \in \langle 0, 1\rangle$.

Proof. The inequality

$$|z+ay+\beta x-\mu P(t)| \geqslant |z+ay+\beta x-p|$$

implies the inclusion of $\Delta_2(\mu, K(C))$ in the set $|z+ay+\beta x-p| \leqslant K(C)$. Hence by the definition of $K(C)$ all points of the product $\mathrm{Fr}\,\Delta_1(\mu, C) \cap \cap \mathrm{Fr}\,\Delta_2(\mu, K(C))$ are in the domain $y^2+z^2 \geqslant r_0^2$ for $C \geqslant C^*$. To prove the inclusion of this product in the layer $|x| \geqslant q$, we will show that if C is large, then for arbitrary fixed $t \in I$ the system of equations

$$W_1(q, y, z; \mu) = C\,,$$
$$W_2(q, y, z, t; \mu) = K(C)$$

has no real roots.

Calculating z from the second equation and putting it into the first we get the equation

$$\tfrac{1}{2}y^2A_1-y(A_2\sqrt{C}+A_3)+B_1C+B_2\sqrt{C}+B_3 = 0\,, \tag{16}$$

where $A_1 = k\beta+ka^2-a$, $A_2 = \sqrt{2/c}(ka\beta-\beta)$, $B_1 = k\beta^2/c-1$ and coefficients B_2, B_3, A_3 are bounded for all $(t, \mu) \in I\times\langle 0, 1\rangle$.

The expression $A_2^2-2A_1B_1 = 2/c[(ka-1)(ac-\beta^2)+k\beta^2(c-k\beta^2)]$ in virtue of (8) is negative; thus

$$(A_2\sqrt{C}+A_3)^2-2A_1(B_1C+B_2\sqrt{C}+B_3) < 0$$

for large C (say $C \geqslant C^{**}$), whence equation (16) has no real solutions for $C \geqslant C^{**}$. For a constant C satisfying $C \geqslant C_0 = \max(C^*, C^{**})$ we get (15).

6. Proof of Theorem 2. Let us put $d = \max(kM+kp, p, \varepsilon p)$ and define the constants $r_0, q, C_0, K(C)$ as previously in section 5. For $c > c_0$ we define the function $V(x, y, z, t; \mu)$ by

$$V(x, y, z, t; \mu) = \begin{cases} W_1(x, y, z; \mu) \\ \qquad \text{for} \quad (x, y, z, t) \in \mathrm{Fr}\,\Delta_1(\mu, C) \cap \Sigma(\mu, K(C), C)\,, \\ \left(\dfrac{1}{\beta}W_2(x, y, z, t; \mu)+h_0\right)^2 c/2 \\ \qquad \text{for } (x, y, z, t) \in \mathrm{Fr}\,\Delta_2(\mu, K(C)) \cap \Sigma(\mu, K(C), C)\,. \end{cases}$$

Each point (x, y, z, t) of $\Sigma(\mu, K(C), C)$ satisfies the equation

$$V(x, y, z, t; \mu) = C .$$

By (12) V is positive definite for all $\mu \in \langle 0, 1\rangle$ and

$$\lim_{(x,y,z)\to\infty} V(x, y, z, t; \mu) = \infty \tag{17}$$

uniformly in $(t, \mu) \in I \times \langle 0, 1\rangle$. By (11) the surface $\Sigma(\mu, K(C), C)$ is included in the cylindrical domain $D(C)$

$$D(C) = \{(x, y, z, t):\ t \in I,\ \tfrac{1}{2}g(x) + \tfrac{1}{2}\lambda_1(y^2+z^2) \leqslant C\}, \tag{18}$$

whose directrix is bounded for each positive C.

The derivative of V along any solution of (E_μ) is negative. In fact, if $(x, y, z, t) \in \operatorname{Fr}\Delta_2 \cap \Sigma$, then $V = W_2$ and by Lemma 2 $|x| \geqslant q$. Hence

$$V' = \frac{c}{\beta}\left(\frac{1}{\beta}W_2 + h_0\right)W_2' = -\sqrt{\frac{2Cc}{\beta^2}}f(x)\operatorname{sgn}x \leqslant -s_2 < 0 . \tag{19}$$

s_2 is positive because the expression $\sqrt{2Cc/\beta^2}$ is positive for all $\mu \in \langle 0, 1\rangle$, $C \geqslant C_0$ and from the inequality $|x| \geqslant q$ and the boundedness of $\Sigma(\mu, K(C), C)$ it follows that $|f(x)|$ has a positive lower bound. If $(x, y, z, t) \in \operatorname{Fr}\Delta_1 \cap \Sigma$, then $V = W_1$ and the derivative of V along the trajectory of (E_μ) is given by the expression

$$\begin{aligned} V' = W_1' = &-\mu^2\gamma\varepsilon f(x)x - \varepsilon\gamma^2x^2 - y^2(\beta - k\gamma) - z^2(ka - 1) + \\ &+ \varepsilon\gamma(yz - axz - \beta xy) - kz\mu^2 f(x) + p(t)(kz + y + \varepsilon\gamma x). \end{aligned}$$

By Lemma 2 from $(x, y, z, t) \in \operatorname{Fr}\Delta_1 \cap \Sigma$ it follows that $y^2 + z^2 \geqslant r_0^2$ and this implies (13). Next

$$\begin{aligned} W_1' &= -\Psi(x, y, z; \mu) - \mu^2\gamma\varepsilon f(x)x - kz\mu^2 f(x) + p(t)(kz + y + \varepsilon\gamma x) \\ &\leqslant -\Psi(x, y, z; \mu) + d(|\gamma x| + |y| + |z|) - \lambda_2(\gamma^2x^2 + y^2 + z^2) \leqslant -s_1 \end{aligned} \tag{20}$$

in virtue of Lemma 1 and (13). Putting $s = \min(s_1, s_2)$, we get

$$V' \leqslant -s < 0 \tag{21}$$

along the solutions of (E_μ). Notice that the constant s is independent of t_0 and μ. Hence the surface Σ defined as $\Sigma = \Sigma(\mu, K(C_0), C_0)$ satisfies (ii) for each $\mu \in \langle 0, 1\rangle$. Σ has property (i) because (21) remains valid also for $C = C_0$. There exists a constant $D > 0$ such that the set $D(C_0)$ defined by (18) is contained in the domain $t \in I$, $|x| + |y| + |z| < D$, and so Σ also lies in this domain. Thus Theorem 2 is proved completely.

7. For $\mu = 1$ Theorem 2 gives an elementary proof of the result obtained in a different, more complicated way by J. O. C. Ezeilo ([2]). Moreover, his condition $f(x)\operatorname{sgn} x \geqslant m > 0$, for $|x| \geqslant 1$ is relaxed to (4).

References

[1] J. O. C. Ezeilo, *On the existence of periodic solutions of a certain third-order differential equation*, Proc. Cambr. Phil. Soc. 56 (1960), pp. 381-389.

[2] — *On the boundedness of solutions of a certain differential equation of the third order*, Proc. Lond. Math. Soc. (3) 9 (1959), pp. 74-114.

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