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ON ANISOTROPIC BESOV AND BESSEL POTENTIAL SPACES

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1. Introduction

The purpose of these lectures is to represent an elementary and short introduction to some aspects of anisotropic Besov spaces $B^p_{\gamma,q}$ and Bessel potential spaces L^p_γ on the whole Euclidean space $\boldsymbol{R}^n$. A unified treatment of the isotropic as well as of the anisotropic case is made possible by replacing the standard dilation matrix $\operatorname{diag}(t, \ldots, t)$ by a general one $A_t = t^P := e^{P\log t}$, where P is a real $n \times n$-matrix with eigenvalues λ_j, $\operatorname{Re}\lambda_j > 0$, and, consequently, by replacing the Euclidean distance by an A_t-homogeneous distance function. This idea is basic in the work of Calderón and Torchinsky [6] on parabolic maximal functions.

In the case of anisotropic function spaces with respect to diagonal dilation groups we refer the reader to the book of Nikolskii [18] for an extensive discussion and relevant literature. In this connection let us mention that, when working with diagonal dilation matrices, one is lead to use the special concrete coordinate system in the proofs (e.g. when differentiating) whereas in the case of general dilations the methods must be independent of a particular coordinate system.

The techniques used here are standard Fourier multiplier methods; this implies that we restrict ourselves to the mostly occurring case $1 \leqslant p, q \leqslant \infty$ and $\gamma \geqslant 0$ (the latter restriction not being necessary, cf. [20]); but there is little doubt that one can carry over the "maximal function" method in the book of Triebel [29] to obtain the missing $\gamma \in \boldsymbol{R}$, $0 < p, q \leqslant \infty$ results. With the books of Bergh and Löfström [1], Brenner, Thomée and Wahlbin [5], Nikolskii [18], Peetre [19] and Triebel [29] in mind, the results displayed here will cause a "dejà vu" impression. Nevertheless, the simplicity of the derivation and the generality of the results might a little surprise.

Start with a real $n \times n$-matrix P whose eigenvalues λ_j have positive real parts. Denote by ν the trace of P and define

$$\Lambda_m = \min_j \operatorname{Re} \lambda_j, \qquad \Lambda_M = \max_j \operatorname{Re} \lambda_j.$$

Following Besov, Il'in and Lizorkin [3] associate to the dilation matrix $A_t = t^P$ a positive, *A_t-homogeneous distance function* $r(x)$, i.e., a continuous function r on $\mathbf{R}^n$ with

$$(1.1) \qquad r(x) > 0 \quad \text{for } x \neq 0; \quad r(A_t x) = t r(x) \quad \text{for } t > 0,\ x \in \mathbf{R}^n.$$

We mention that there always exists at least one A_t-homogeneous $C^\infty(\mathbf{R}^n_0)$-distance function, $\mathbf{R}^n_0 = \mathbf{R}^n \setminus \{0\}$: e.g. $r^*(x) = 1/t$, where t is the solution of

$$BA_t x A_t x = 1 \quad \text{for } x \neq 0 \quad \text{with } B = \int_0^\infty e^{-tP'} e^{-tP} dt,$$

P' being the adjoint of P (see Madych [16], Stein and Wainger [24]). In concrete diagonal situations one can easily list up many distance functions associated to one A_t; e.g.

(i) $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_t = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$;

$r_p(x) = (|x_1|^p + |x_2|^p)^{1/p}$, $p > 0$.

(ii) $P = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$, $A_t = \begin{bmatrix} t^{1/2} & 0 \\ 0 & t^{1/4} \end{bmatrix}$;

$(|x_1|^p + |x_2|^{2p})^{2/p}$, $p > 0$, $x_1^2 - c|x_1| x_2^2 + x_2^4$ for $c < 2$,

$2\sqrt{x_1^4 + x_2^8} - x_1^2$, etc. are admissible choices for A_t-homogeneous distance functions.

(iii) $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A_t = \begin{bmatrix} t & t \log t \\ 0 & t \end{bmatrix}$;

in this case, an explicit representation of some $r(x)$ seems to be complicated to derive.

We list up some properties of such distance functions:

$$(1.2) \qquad r(x+y) \leqslant C\big(r(x) + r(y)\big), \qquad C \geqslant 1;$$

$$(1.3) \qquad r(sx) \leqslant C r(x), \qquad |s| \leqslant 1,\ s \in \mathbf{R};$$

$$(1.4) \qquad r(x) \asymp r^*(x), \qquad x \in \mathbf{R}^n,$$

i.e., there exist positive constants c_1, c_2 such that

$$c_1 r(x) \leqslant r^*(x) \leqslant c_2 r(x),$$

where r^* is the distance function constructed below (1.1). In particular (1.4) implies that all distance functions, homogeneous with respect to one fixed dilation A_t, are comparable. Concerning derivatives we have: to each $\varepsilon > 0$ ($\varepsilon < \Lambda_m$) and $\alpha \in N_0^n$ there exist constants $C_{\varepsilon,\alpha}$, $C'_{\varepsilon,\alpha} > 0$ with (r is assumed to be sufficiently smooth)

$$(1.5)\qquad \begin{aligned} |D^\alpha r(x)| &\leqslant C_{\varepsilon,\alpha} r(x)^{1-|\alpha|(\Lambda_M+\varepsilon)}, \quad |x| < 1,\\ |D^\alpha r(x)| &\leqslant C'_{\varepsilon,\alpha} r(x)^{1-|\alpha|(\Lambda_m-\varepsilon)}, \quad |x| > 1. \end{aligned}$$

Also the following comparison with the Euclidean distance holds ($0 < \varepsilon < \Lambda_m$)

$$(1.6)\qquad \begin{aligned} C_\varepsilon |x|^{1/(\Lambda_m-\varepsilon)} &\leqslant r(x) \leqslant C'_\varepsilon |x|^{1/(\Lambda_M+\varepsilon)}, \quad |x| \to 0,\\ C_\varepsilon |x|^{1/(\Lambda_M+\varepsilon)} &\leqslant r(x) \leqslant C'_\varepsilon |x|^{1/(\Lambda_m-\varepsilon)}, \quad |x| \to \infty. \end{aligned}$$

Now define the Fourier transformation on the space S of infinitely differentiable, rapidly decreasing functions on R^n by

$$\mathscr{F}[f](\xi) \equiv f^\wedge(\xi) = \int_{R^n} f(x) e^{-i\xi x} dx;$$

denote by $\mathscr{F}^{-1}$ its inverse and extend these definitions to the space S' of tempered distributions. Concerning dilations there holds

$$(1.7)\qquad [t^\nu f(A_t x)]^\wedge(\xi) = f^\wedge(A'_{1/t}\xi),$$

where $A'_t = t^{P'}$ is the adjoint of A_t. Let ϱ be an A'_t-homogeneous distance function and ϱ^* be constructed analogously to r^*. It is clear that ϱ, ϱ^* satisfy properties (1.2)–(1.6). For proofs see [2], [3], [24], [7].

The Besov spaces we want to discuss are now defined as follows: Let $0 \leqslant \Phi \in C^\infty(R)$ be a non-negative bump function with

$$\operatorname{supp}\Phi \subset [\tfrac{1}{2}, 2], \qquad \sum_{-\infty}^{\infty} \Phi(2^{-k}t) = 1 \quad \text{for } t > 0.$$

Set

$$\varphi_k = \Phi\circ(2^{-k}\varrho) = \Phi\circ 2^{-k}\varrho, \qquad k \in Z,$$

where ϱ is an A'_t-homogeneous distance function with $\varrho \in C^{n^*}(R_0^n)$, $n^* = [n/2] + 1$, and define

$$(1.8)\qquad \|f\|_{\dot{B}^p_{\gamma,q}} = \Big(\sum_{k=-\infty}^{\infty} [2^{k\gamma}\|\mathscr{F}^{-1}[\varphi_k] * f\|_p]^q\Big)^{1/q}, \qquad \gamma \geqslant 0$$

(with the usual modification for $q = \infty$; we do not indicate a dependence on ϱ and φ, since it will turn out (see Theorem 3) that for different ϱ's and φ's all semi-norms $\|\cdot\|_{\dot{B}^p_{\gamma,q}}$ are equivalent); the anisotropic Besov spaces to be discussed are now given by

$$B^p_{\gamma,q} = \{f \in L^p\colon \|f\|_{B^p_{\gamma,q}} = \|f\|_p + \|f\|_{\dot{B}^p_{\gamma,q}} < \infty\} \tag{1.9}$$

(for the case of diagonal dilations see e.g. [29], p. 270). $B^p_{\gamma,q}$ coincides with the classical isotropic Besov space (cf. [5], Ch. 2) when $A_t = \operatorname{diag}(t, \dots, t)$, $\varrho(\xi) = |\xi|$. Also we will consider the following anisotropic Bessel potential spaces

$$L^p_{\gamma,\varrho} = \{f\colon f = \mathscr{F}^{-1}[(1+\varrho)^{-\gamma}] * g,\ g \in L^p\}, \qquad \gamma \geqslant 0,\ 1 \leqslant p \leqslant \infty, \tag{1.10}$$

which we norm by

$$\|f\|_{p,\gamma} = \|g\|_p. \tag{1.11}$$

For $A_t = \operatorname{diag}(t^{1/2}, \dots, t^{1/2})$, $\varrho(\xi) = |\xi|^2$, the latter spaces coincide with the classical Bessel potential spaces of order 2γ (cf. [23], p. 134) and with the Triebel–Lizorkin spaces $F^\gamma_{p,2}$, $1 < p < \infty$ (cf. [29], p. 146). In the case of diagonal dilations anisotropic Bessel potential kernels and spaces have been discussed by Lizorkin [14], [15], Sadosky and Cotlar [21], Torchinsky [28] and others (clearly, anisotropic $F^\gamma_{p,q}$-spaces can be introduced by replacing $|\xi|$ by $\varrho(\xi)$, cf. [29], p. 270). Since, contrary to the classical situation, one cannot expect explicit representations or asymptotic estimates of $\mathscr{F}^{-1}[(1+\varrho)^{-\gamma}]$ without much work, we will try to work only with the Fourier transform $(1+\varrho(\xi))^{-\gamma}$. To this end, in Section 2 we list up some Fourier multiplier criteria and use them for a first discussion of the Bessel potential spaces (1.10); Section 3 deals with the Besov spaces (1.9) and in Section 4 we revisit the Bessel potential spaces to clarify questions on multiplication algebras and pointwise multipliers.

2. Some Fourier multiplier criteria

In the following let $\varrho, \varrho_1, \varrho^*, \ldots \in C^{n^*}(\boldsymbol{R}^n_0)$, $n^* = [n/2]+1$, always be A'_t-homogeneous distance functions. We start with a variant of the Bernstein Lemma.

THEOREM A (Madych [16]). *Let the sufficiently smooth and vanishing at infinity m be such that*

$$\sum_{k=-\infty}^{\infty} \sup_{\substack{1\leqslant j\leqslant n\\ 0\leqslant \alpha_j\leqslant n^*}} \left(\int_{1/2\leqslant \varrho(\xi)\leqslant 2} \left| \left(\frac{\partial}{\partial \xi_j}\right)^{\alpha_j} m(A'_{2^k}\xi) \right|^2 d\xi \right)^{1/2} \leqslant B < \infty.$$

Then $m \in [L^1(\boldsymbol{R}^n)]^\wedge$, i.e., m is the Fourier transform of an integrable function and

$$\|\mathscr{F}^{-1}[m]\|_1 \leqslant CB.$$

As an application one has (use (1.5))

$$(2.1)\qquad (1+\varrho_1)^{\delta}(1+\varrho)^{-\gamma}\in[L^1]^{\wedge}, \quad 0\leqslant\delta<\gamma,$$

ϱ_1, ϱ being homogeneous with respect to the same A_t'.

In the case that m is *quasi-radial*, i.e., $m = m_0\circ\varrho$, m_0: $\boldsymbol{R}_+ \to \boldsymbol{C}$ (e.g. the Fourier transform of the anisotropic Bessel potential kernel: $(1+\varrho)^{-\gamma}$ or $\varphi_k = \Phi\circ 2^{-k}\varrho$) the following version of the quasi-convexity criterion turns out to be quite useful.

THEOREM B (Dappa and Trebels [8]). *Let m_0: $\boldsymbol{R}_+ \to \boldsymbol{C}$ be sufficiently smooth and vanishing at infinity. Then (ν being the trace of P)*

$$\|\mathscr{F}^{-1}[m_0\circ\varrho]\|_p \leqslant C\int_0^\infty t^{n^*+\nu/p'}|dm_0^{(n^*)}(t)|, \quad 1\leqslant p\leqslant\infty,$$

provided that the right hand side is finite; here $1/p+1/p' = 1$.

In [8], the proof is only given for diagonal dilations; but it immediately carries over to the general situation considered here.

We mention two applications of Theorem B.

$$(2.2)\qquad (1+\varrho)^{-\mu}\in[L^q]^{\wedge}, \quad \mu>\nu\left(1-\frac{1}{q}\right), \quad 1\leqslant q\leqslant\infty,$$

which for $q = 1$ in particular shows that the definition (1.10) makes sense.

$$(2.3)\qquad \left(\frac{(1+\varrho^{\delta})^{1/\delta}}{1+\varrho}\right)^{\gamma}\in M^{\wedge} \quad \text{for } \delta>0,\ \gamma\in\boldsymbol{R};$$

i.e. this function is the Fourier transform of a bounded measure. To realize that (2.3) is true we only need to observe that

$$m_0(t) = ((1+t^{\delta})^{1/\delta}/(1+t))^{\gamma}-1$$

satisfies the hypotheses of Theorem B and that 1 is the Fourier transform of the bounded Dirac measure supported at the origin. In particular, if we choose $\delta = \gamma$ and then $\delta = -\gamma$ we obtain the anisotropic analogon of Stein's Lemma ([23], p. 133):

$$(2.4)\qquad \varrho^{\gamma} = (1+\varrho)^{\gamma}[d\omega_1]^{\wedge}, \quad \int_{\boldsymbol{R}^n}|d\omega_1|<\infty,$$

$$(2.5)\qquad (1+\varrho)^{\gamma} = (1+\varrho^{\gamma})[d\omega_2]^{\wedge}, \quad \int_{\boldsymbol{R}^n}|d\omega_2|<\infty.$$

Up to a one time use of the Wiener–Levy theorem, Theorems A and B turn out to be sufficient for our discussion of anisotropic Besov spaces, where we will only use the convolution theorems ([23], p. 27, [25], p. 31)

$$\|f * d\mu\|_p \leqslant \|\mu\|_M \|f\|_p, \quad 1 \leqslant p \leqslant \infty, \tag{2.6}$$

$$\|f * g\|_q \leqslant \|g\|_r \|f\|_p, \quad 0 \leqslant \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1, \; p \geqslant 1. \tag{2.7}$$

For the treatment of the Bessel potential spaces we also need a multiplier criterion of Hörmander-type and one on fractional integration.

THEOREM C (Madych [16]). *Suppose m is sufficiently smooth and*

$$\|m\|_\infty + \sup_{\substack{1 \leqslant j \leqslant n \\ 0 \leqslant \alpha_j \leqslant n^*}} \left(\sup_{t>0} \int_{1/2 \leqslant \varrho(\xi) \leqslant 2} \left| \left(\frac{\partial}{\partial \xi_j} \right)^{\alpha_j} m(A_t' \xi) \right|^2 d\xi \right)^{1/2} \leqslant B.$$

Then

$$\|\mathscr{F}^{-1}[m] * f\|_p \leqslant CB\|f\|_p, \quad f \in S, \; 1 < p < \infty. \tag{2.8}$$

If an inequality of type (2.8) holds we call m a *multiplier of type* (p, p): $m \in M_p^p$. As an application of Theorem C we have

$$\left(\frac{\varrho}{\varrho^*} \right)^\gamma \in M_p^p, \quad 1 < p < \infty, \; \gamma \in \mathbf{R}, \tag{2.9}$$

since $(\varrho/\varrho^*)^\gamma$ is homogeneous of degree zero it is not hard to verify the assumptions of Theorem C to hold.

THEOREM D (Calderón and Torchinsky [6]). *For $1 < p < \infty$, $0 < \mu < \nu/p$ and $1/q = 1/p - \mu/\nu$ there holds*

$$\|\mathscr{F}^{-1}[\varrho^{-\mu} f\hat{\ }]\|_q \leqslant C\|f\|_p, \quad f \in S.$$

The assertion is proved for ϱ^* in [6]; the general case now follows from this by (2.9) since

$$\|\mathscr{F}^{-1}[\varrho^{-\mu} f\hat{\ }]\|_q = \left\| \mathscr{F}^{-1}\left[\left(\frac{\varrho^*}{\varrho} \right)^\mu \right] * \mathscr{F}^{-1}[\varrho^{*-\mu} f\hat{\ }] \right\|_q \leqslant C\|f\|_p.$$

As an application we have that $\varrho^{-\mu}$ in Theorem D can be replaced by $(1+\varrho)^{-\mu}$, since by (2.4) we have

$$\begin{aligned} \|\mathscr{F}^{-1}[(1+\varrho)^{-\mu} f\hat{\ }]\|_q &= \left\| \mathscr{F}^{-1}\left[\left(\frac{\varrho}{1+\varrho} \right)^\mu \right] * \mathscr{F}^{-1}[\varrho^{-\mu} f\hat{\ }] \right\|_q \\ &\leqslant C\|\mathscr{F}^{-1}[\varrho^{-\mu} f\hat{\ }]\|_q \leqslant C'\|f\|_p. \end{aligned} \tag{2.10}$$

With the aid of the above applications we can briefly give a first discussion of the above Bessel potential spaces.

THEOREM 1. *Let $\gamma > 0$ and ν be the trace of P. Then*

(a) $L^p_{\gamma,\varrho}$, $1 \leqslant p \leqslant \infty$, *is a non trivial Banach space;*

(b) $L^p_{\gamma,\varrho} \subset L^p_{\delta,\varrho}$, $0 \leqslant \delta < \gamma$, $1 \leqslant p \leqslant \infty$, *in the sense of continuous embedding*;

(c) $L^p_{\gamma,\varrho} = L^p_{\gamma/N,\varrho^N}$, $1 \leqslant p \leqslant \infty$; $N > 0$, *with equivalent norms; thus, on account of* (1.6), ϱ *may be assumed to be sufficiently smooth at the origin*;

(d) $L^p_{\gamma,\varrho} = L^p_{\gamma,\varrho_1}$, $1 < p < \infty$; *i.e., the anisotropic Bessel potential spaces depend only on* A_t *if* $1 < p < \infty$; *in this case we omit* ϱ *and write* L^p_γ;

(e) (i) $L^p_{\gamma,\varrho} \subset L^q_{\delta,\varrho_1}$, $1 \leqslant p \leqslant q \leqslant \infty$, $\gamma-\delta > \nu\left(\frac{1}{p}-\frac{1}{q}\right)$;

(ii) $L^p_\gamma \subset L^q_\delta$, $1 < p < q < \infty$, $\gamma-\delta \geqslant \nu\left(\frac{1}{p}-\frac{1}{q}\right)$.

Proof. (a) By the convolution theorem and (2.2) for $q = 1$ we see that

$$f = \mathscr{F}^{-1}[(1+\varrho)^{-\gamma}] * g, \quad g \in L^p,$$

is well defined and belongs to L^p. Since L^p is a Banach space, the same is true for $L^p_{\gamma,\varrho}$ by the definition of its norm (1.11).

(b) is again a consequence of the convolution theorem and (2.2) since

$$\mathscr{F}^{-1}[(1+\varrho)^{-\gamma}] * g = \mathscr{F}^{-1}[(1+\varrho)^{-\delta}] * \mathscr{F}^{-1}[(1+\varrho)^{\delta-\gamma}] * g$$

implies

$$\|f\|_{p,\delta} = \|\mathscr{F}^{-1}[(1+\varrho)^{\delta-\gamma}] * g\|_p \leqslant C\|g\|_p = C\|f\|_{p,\gamma}.$$

(c) follows directly from (2.3).

(d) is proved by (2.9).

(e) (i) is established by (2.1) when $q = p$ and by a combination of (2.1) and (2.2) if $q > p$; (ii) is contained in a combination of (2.10), part (b) and part (c) of the theorem.

One may ask after a relation between the above Bessel potential spaces and the classical Sobolev spaces

$$W^p_k = \left\{f \in L^p\colon \|f\|_{W^p_k} = \|f\|_p + \sum_{j=1}^n \left\|\left(\frac{\partial}{\partial x_j}\right)^k f\right\|_p < \infty\right\}.$$

THEOREM 2. (a) *For all* p, $1 \leqslant p \leqslant \infty$, *there holds*

(i) $$L^p_{\gamma,\varrho} \subset W^p_k, \quad k < \gamma/\Lambda_M,$$

(ii) $$W^p_l \subset L^p_{\gamma,\varrho}, \quad l > \gamma/\Lambda_m.$$

(b) *If* P *is diagonal,* $1 < p < \infty$, *and* $\alpha_j = \gamma/\lambda_j \in N$, *then one can identify* L^p_γ *with the anisotropic Sobolev space*

$$\left\{f\colon \|f\|_p + \sum_{j=1}^n \left\|\left(\frac{\partial}{\partial x_j}\right)^{\alpha_j} f\right\|_p < \infty\right\}.$$

Proof. (b) is shown in [8]. Concerning (a), an application of Theorem A (use (1.5) and (1.6)) yields $(i\xi_j)^k(1+\varrho)^{-\gamma}\in[L^1]^\wedge$ for $k<\gamma/\Lambda_M$ and

$$(1+\varrho)^\gamma(1+i\xi_1^l+\ldots+i\xi_n^l)^{-1}\in[L^1]^\wedge \quad \text{for } l>\gamma/\Lambda_m.$$

But then it follows by the convolution theorem that, e.g.,

$$\left\|\left(\frac{\partial}{\partial x_j}\right)^k f\right\|_p = \|\mathscr{F}^{-1}[(i\xi_j)^k(1+\varrho)^{-\gamma}]*g\|_p \leqslant C\|g\|_p = C\|f\|_{p,\gamma}.$$

Analogously (ii) is proved and thus also (a) is established.

3. Anisotropic Besov spaces

Before studying the embedding behavior of these spaces, let us first characterize them. We follow here notes of A. Seeger on unpublished work of Rivière [20], [17] which itself is based on ideas of Shapiro [22] (see also Boman and Shapiro [4]). For the sake of completeness and for the convenience of the reader also Rivière's proofs are given.

THEOREM E (Rivière [20]). *Let $f\in B^p_{\gamma,q}$, $\gamma>0$, $1\leqslant p,q\leqslant\infty$. Let $(\eta_y)\subset M^\wedge$, i.e., a family of Fourier transforms of bounded measures, with the map $y\to\eta_y*f$ being continuous in y on L^p and*

$$\sup_y\|\mathscr{F}^{-1}[\eta_y]\|_M\leqslant B,\qquad \sup_y\|\mathscr{F}^{-1}[\varrho^{-\mu}\eta_y]\|_M\leqslant B \quad \textit{for } \mu>\gamma.$$

Then we have with the notation $\delta_t' f^\wedge(\xi)=f^\wedge(A_t'\xi)$ that

$$\left(\int_0^\infty[t^{-\gamma}\sup_y\|\mathscr{F}^{-1}[\delta_t'\eta_y]*f\|_p]^q\frac{dt}{t}\right)^{1/q}\leqslant CB\|f\|_{B^p_{\gamma,q}},\qquad q<\infty,$$

and

$$\sup_{t>0}[t^{-\gamma}\sup_y\|\mathscr{F}^{-1}[\delta_t'\eta_y]*f\|_p]\leqslant CB\|f\|_{B^p_{\gamma,\infty}}.$$

Proof. Set $\mathscr{F}^{-1}[\varphi_k]*f=f_k$. By assumption there holds

$$\|\mathscr{F}^{-1}[\delta_t'\eta_y]*f_k\|_p\leqslant B\|f_k\|_p$$

and, observing that $\varphi_k=(\varphi_{k-1}+\varphi_k+\varphi_{k+1})\varphi_k$,

$$\begin{aligned}
&\|\mathscr{F}^{-1}[\delta_t'\eta_y]*f_k\|_p\\
&\quad=\|\mathscr{F}^{-1}[\delta_t'(\eta_y\varrho^{-\mu})]*\mathscr{F}^{-1}[(t\varrho)^\mu(\varphi_{k-1}+\varphi_k+\varphi_{k+1})]*f_k\|_p\\
&\quad\leqslant B\|\mathscr{F}^{-1}[(2^{-k}\varrho)^\mu(\varphi_{k-1}+\varphi_k+\varphi_{k+1})]*f_k\|_p(t2^k)^\mu\\
&\quad\leqslant CB(2^kt)^\mu\|f_k\|_p,
\end{aligned}$$

since, by Theorem B,

$$\|\mathscr{F}^{-1}[(2^{-k}\varrho)^{\mu}(\varphi_{k-1}+\varphi_k+\varphi_{k+1})]\|_1 = O(1).$$

Now

$$\sup_y \|\mathscr{F}^{-1}[\delta_t' \eta_y] * f\|_p \leqslant \sum_{k=-\infty}^{\infty} \sup_y \|\mathscr{F}^{-1}[\delta_t' \eta_y] * f_k\|_p$$

and therefore, since the sup's are measurable by hypothesis (observe that for $p = \infty$ the Besov space $B^{\infty}_{\gamma,q}$ contains only continuous functions, cf. (3.4)),

$$\left(\int_0^{\infty} [t^{-\gamma} \sup_y \|\mathscr{F}^{-1}[\delta_t' \eta_y] * f\|_p]^q \frac{dt}{t}\right)^{1/q}$$

$$\leqslant CB\left(\sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} [t^{-\gamma} \sum_{k=-\infty}^{\infty} \min\{1, (2^k t)^{\mu}\} \|f_k\|_p]^q \frac{dt}{t}\right)^{1/q}$$

$$\leqslant CB\left(\sum_{i=-\infty}^{\infty} [\sum_{k=-\infty}^{\infty} \min\{2^{-\gamma(i+k)}, 2^{(\mu-\gamma)(i+k)}\} 2^{k\gamma} \|f_k\|_p]^q\right)^{1/q}$$

$$\leqslant CB \|f\|_{\dot{B}^p_{\gamma,q}}$$

by the convolution theorem for sequences ([13], p. 123), since $(\min\{2^{-\gamma k}, 2^{(\mu-\gamma)k}\}) \in l^1(\mathbf{Z})$.

LEMMA F (Rivière [20]). *Let $\eta \in C(\mathbf{R}^n_0)$ satisfy a Tauberian condition (i.e., given $\xi \in \mathbf{R}^n_0$, there exists a $t > 0$ such that $\eta(A_t'\xi) \neq 0$), and let $K \subset \mathbf{R}^n_0$ be compact. Then there exist $\chi_i \in C^{\infty}(\mathbf{R}^n)$ with compact support, positive numbers $a_i > 0$, $|c_i| = 1$, $i = 1, \ldots, m(K) < \infty$, such that*

$$F(\xi) = \sum_{i=1}^{m} c_i \eta(A'_{a_i}\xi)\chi_i(\xi) \neq 0 \quad \text{on } K.$$

Proof. Given $\xi \in K$ choose $t(\xi)$, $c(\xi)$ such that

$$\operatorname{Re}[c(\xi)\eta(A_{t(\xi)}\xi)] > 0 \quad \text{in } \xi + U_{\xi}.$$

Consider the covering $\{\xi + \frac{1}{2}U_{\xi}\colon \xi \in K\}$. Since K is compact we can choose a finite subcovering $\{\xi_i + \frac{1}{2}U_i\}$. Set $c_i = c(\xi_i)$, $a_i = t(\xi_i)$; $\chi_i = 1$ on $\xi_i + \frac{1}{2}U_i$, $\operatorname{supp}\chi_i \subset \xi_i + U_i$ ($\chi_i \geqslant 0$ as in the hypotheses) and the lemma follows.

Theorem E has now the following converse.

THEOREM G (Rivière [20]). *Let $\eta \in M^{\wedge}$ satisfy a Tauberian condition (w.r.t. A_t). Then*

$$\|f\|_{\dot{B}^p_{\gamma,q}} \leqslant C\left(\int_0^{\infty} [t^{-\gamma}\|\mathscr{F}^{-1}[\delta_t' \eta] * f\|_p]^q \frac{dt}{t}\right)^{1/q}, \quad q < \infty$$

and the usual modification for $q = \infty$ also holds.

Proof. Let Φ be as in the definition of $\|\cdot\|_{\dot{B}^p_{\gamma,q}}$ and set

$$\operatorname{supp}\Phi\circ\varrho = K.$$

Choose F as in Lemma F, then

$$\|f\|_{\dot{B}^p_{\gamma,q}} = \Big(\sum_{k=-\infty}^{\infty}[2^{k\gamma}\|\mathscr{F}^{-1}[\varphi_k]*f\|_p]^q\Big)^{1/q}$$
$$= C\left(\sum_{k=-\infty}^{\infty}\left[2^{k\gamma}\left\|\mathscr{F}^{-1}\left[\varphi_k\int_0^{\infty}\frac{(\Phi\circ t\varrho)^2}{F(A_t'\cdot)}F(A_t'\cdot)\frac{dt}{t}f^{\wedge}\right]\right\|_p\right]^q\right)^{1/q}$$
$$\leqslant C\left(\sum_{k=-\infty}^{\infty}\left[2^{k\gamma}\|\mathscr{F}^{-1}[\varphi_k]\|_1\int_{2^{-k-2}}^{2^{-k+2}}\left\|\mathscr{F}^{-1}\left[\frac{(\Phi\circ t\varrho)^2}{F(A_t'\cdot)}\right]\right\|_1\right.\right.$$
$$\left.\left.\times\|\mathscr{F}^{-1}[F(A_t'\cdot)f^{\wedge}]\|_p\frac{dt}{t}\right]^q\right)^{1/q}.$$

But

$$\|\mathscr{F}^{-1}[\varphi_k]\|_1 = \|\mathscr{F}^{-1}[\Phi\circ\varrho]\|_1 = O(1)$$

by Theorem B and

$$\left\|\mathscr{F}^{-1}\left[\frac{(\Phi\circ t\varrho)^2}{F(A_t'\cdot)}\right]\right\|_1 = \|\mathscr{F}^{-1}[(\Phi\circ\varrho)^2/F]\|_1 = O(1)$$

by the Wiener–Levy theorem. Hence

$$\|f\|_{\dot{B}^p_{\gamma,q}} \leqslant C\left(\sum_{k=-\infty}^{\infty}\left[2^{k\gamma}\int_{2^{-k-2}}^{2^{-k+2}}\sum_{i=1}^{m}\|\mathscr{F}^{-1}[\delta_t'\chi_i]\|_1\,\|\mathscr{F}^{-1}[\delta_{a_it}'\eta]*f\|_p\frac{dt}{t}\right]^q\right)^{1/q}$$
$$\leqslant C\sum_{i=1}^{m}\left(\sum_{k=-\infty}^{\infty}2^{k\gamma q}\int_{2^{-k-2}}^{2^{-k+2}}\|\mathscr{F}^{-1}[\delta_{a_it}'\eta]*f\|_p^q\frac{dt}{t}\left(\int_{2^{-k-2}}^{2^{-k+2}}\frac{dt}{t}\right)^{1/q'}\right)^{1/q}$$

by the triangle and the Hölder inequality. Thus, finally,

$$\|f\|_{\dot{B}^p_{\gamma,q}} \leqslant C\sum_{i=1}^{m}\left(\int_0^{\infty}[t^{-\gamma}\|\mathscr{F}^{-1}[\delta_{a_it}'\eta]*f\|_p]^q\frac{dt}{t}\right)^{1/q}$$
$$\leqslant C\left(\int_0^{\infty}[t^{-\gamma}\|\mathscr{F}^{-1}[\delta_t'\eta]*f\|_p]^q\frac{dt}{t}\right)^{1/q}.$$

We are now ready to give the announced characterizations of $\|\cdot\|_{\dot{B}^p_{\gamma,q}}$. With the notation

$$\Delta_y f(x) = f(x+y)-f(x), \qquad \Delta_y^m = \Delta_y\Delta_y^{m-1},$$
$$\bar{\Delta}_y f(x) = f(x+\tfrac{1}{2}y)-f(x-\tfrac{1}{2}y), \qquad \bar{\Delta}_y^m = \bar{\Delta}_y\bar{\Delta}_y^{m-1},$$

we have

THEOREM 3. *Let $\bar{m} \geq m > \gamma/A_m$, $\bar{m}$ be even; let $r \in C^1(R^n_0)$ be an A_t-homogeneous distance function. For $1 \leq p, q \leq \infty$ and $\gamma > 0$ we have the following characterizations*

$$(3.1) \qquad \|f\|_{\dot{B}^P_{\gamma,q}} \approx \|f\|_{(1)} = \left(\int_0^\infty [t^{-\gamma}\|t^{-\nu}\int_{r(y)\leq t} \bar{\Delta}^{\bar{m}}_y f\,dy\|_p]^q \frac{dt}{t}\right)^{1/q}$$

$$\approx \|f\|_{(2)} = \left(\int_{R^n} [r(y)^{-\gamma}\|\Delta^m_y f\|_p]^q r(y)^{-\nu} dy\right)^{1/q}$$

$$\approx \|f\|_{(3)} = \left(\int_0^\infty [t^{-\gamma} \sup_{r(y)\leq t} \|\Delta^m_y f\|_p]^q \frac{dt}{t}\right)^{1/q}.$$

In particular, these characterizations show that the definition of $B^P_{\gamma,q}$ depends only on P and does not depend on the choice of particular φ and ϱ.

Proof. Consider

$$\eta = \int_{r(y)\leq 1} (e^{i\xi y/2} - e^{-i\xi y/2})^{\bar{m}} dy.$$

Then

$$\eta = \sum_{i=0}^{\bar{m}} \binom{\bar{m}}{i}(-1)^i \int_{r(y)\leq 1} e^{i\xi(\bar{m}-2i)y/2} dy \in M^{\wedge},$$

since each single term of the sum is the Fourier transform of a bounded measure. Furthermore, since $\bar{m}$ is even, the Tauberian condition is satisfied and

$$\delta'_t \eta = t^{-\nu} \int_{r(y)\leq t} (e^{i\xi y/2} - e^{-i\xi y/2})^{\bar{m}} dy.$$

Thus

$$\mathcal{F}^{-1}[\delta'_t \eta] * f = t^{-\nu} \int_{r(y)\leq t} \bar{\Delta}^{\bar{m}}_y f\,dy$$

and Theorem G gives

$$\|f\|_{\dot{B}^P_{\gamma,q}} \leq C\left(\int_0^\infty [t^{-\gamma}\|t^{-\nu}\int_{r(y)\leq t} \bar{\Delta}^{\bar{m}}_y f\,dy\|_p]^q \frac{dt}{t}\right)^{1/q} = C\|f\|_{(1)}.$$

Trivially, by the Minkowski inequalities and the translation invariance of the norm,

$$\|f\|_{(1)} \leq C\left(\int_0^\infty t^{-\gamma q}[t^{-\nu}\int_{r(y)\leq t} \|\Delta^m_y f\|_p dy]^q \frac{dt}{t}\right)^{1/q}$$

$$\leq C\left(\int_0^\infty t^{-\gamma q-\nu}\int_{r(y)\leq t} \|\Delta^m_y f\|^q_p dy \left(t^{-\nu}\int_{r(y)\leq t} dy\right)^{q/q'} \frac{dt}{t}\right)^{1/q}$$

$$= C\left(\int_{R^n} \|\Delta^m_y f\|^q_p \int_{r(y)}^\infty t^{-\gamma q-\nu-1} dt\right)^{1/q} = C\|f\|_{(2)}.$$

Introducing now polar coordinates: $y = A_t y'$, $y' \in \Sigma_r = \{h\colon r(h) = 1\}$ and $d\omega$ being a smooth measure on Σ_ϱ, we obtain

$$\|f\|_{(2)} = C\Big(\int_0^\infty t^{-\nu-\gamma q} \int_{\Sigma_r} \|\Delta^m_{A_t y'} f\|_p^q \, d\omega(y') \, t^{\nu-1} \, dt\Big)^{1/q}$$

$$\leqslant C\left(\int_0^\infty [t^{-\gamma} \sup_{r(y)\leqslant t} \|\Delta^m_y f\|_p]^q \frac{dt}{t}\right)^{1/q} = \|f\|_{(3)}.$$

Since $e^{ia\xi} \in M^{\wedge}$ for all $a \in \mathbf{R}^n$, we have

$$\eta_y(\xi) = (e^{i\xi y} - 1)^m \in M^{\wedge}.$$

Furthermore, Theorem A shows that $\hat{g_y} = \varrho^{-\mu} \eta_y \in [L^1]^{\wedge}$ for each $y \in \mathbf{R}^n$, $m > \mu/\Lambda_m > \gamma/\Lambda_m$, and in particular that $\sup_{r(y)\leqslant 1} \|g_y\|_1 \leqslant B$. But then Theorem E may be applied to give

$$\|f\|_{(3)} = \left(\int_0^\infty [t^{-\gamma} \sup_{r(z)\leqslant 1} \|\Delta^m_{A_t z} f\|_p]^q \frac{dt}{t}\right)^{1/q} \leqslant C \|f\|_{\dot{B}^p_{\gamma,q}}.$$

If one introduces the semi-group of operators $P(t)$, defined by

$$P(t) f = \mathscr{F}^{-1}[e^{-t\varrho(\xi)}] * f, \quad f \in L^p$$

(by Theorem B there holds $\|\mathscr{F}^{-1}[e^{-\varrho}]\|_1 \leqslant C$), one obtains the further equivalence

$$\|f\|_{\dot{B}^p_{\gamma,q}} \approx \left(\int_0^\infty \left[t^{m-\gamma} \left\|\frac{\partial^m P(t) f}{\partial t^m}\right\|_p\right]^q \frac{dt}{t}\right)^{1/q}, \quad m > \gamma. \tag{3.2}$$

For the proof consider $\eta = \varrho^m e^{-\varrho}$ which is Tauberian; an application of Theorem B also gives

$$\|\mathscr{F}^{-1}[\eta]\|_1 \leqslant B, \quad \|\mathscr{F}^{-1}[\varrho^{-m}\eta]\|_1 \leqslant B$$

so that the assertion (3.2) follows from Theorems E and G. If A_t is isotropic and $\varrho(\xi) = |\xi|$ or $\varrho(\xi) = |\xi|^2$, then (3.2) just represents the characterizations via Cauchy–Poisson and Gauss–Weierstrass semigroups, resp. (cf. [29], p. 184). Let us mention that, if P is diagonal, one can deduce essentially by the same methods (cf. [8]):

Set $0 < \gamma/\lambda_j = \Gamma_j$, choose $\alpha \in \mathbf{N}_0^n$ such that $0 < \Gamma_j - \alpha_j = \theta_j \leqslant 1$; define

$$\Delta_j(s) f(x) = f(x_1, \ldots, x_j + s, \ldots, x_n) - f(x_1, \ldots, x_n),$$

$$\Delta_j^2(s) = \Delta_j(s) \Delta_j(s).$$

Then

$$\|f\|_{\dot{B}^p_{\gamma,q}} \approx \sum_{j=1}^n \left(\int_0^\infty \left[t^{-\theta_j} \sup_{|s|\leqslant t} \left\|\Delta_j^2(s) \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} f\right\|_p\right]^q \frac{dt}{t}\right)^{1/q}.$$

After these clarifications let us briefly touch on the embedding properties of the Besov spaces and their relation to the Bessel potential spaces. Proofs of results we do not give can directly be carried over from Chapter 2 of Brenner, Thomée and Wahlbin [5].

$B^p_{\gamma,q}$ is a Banach space under the norm $\|\cdot\|_{B^p_{\gamma,q}}$ given in (1.9). If one defines

$$\psi_k = \varphi_k \quad \text{for } k \in N \quad \text{and} \quad \psi_0 = 1 - \sum_{k=1}^{\infty} \varphi_k,$$

one has as another equivalent norm

$$\|f\|_{B^p_{\gamma,q}} \approx \Big(\sum_{k=0}^{\infty} [2^{k\gamma} \|\mathscr{F}^{-1}[\psi_k] * f\|_p]^q\Big)^{1/q}. \tag{3.3}$$

Concerning embedding results we have

(3.4) $B^p_{\gamma,q} \subset B^p_{\gamma_1,q_1}$ if $1 \leqslant p \leqslant \infty$ and (i) $\gamma > \gamma_1$, $1 \leqslant q, q_1 \leqslant \infty$ or (ii) $\gamma = \gamma_1$ and $1 \leqslant q < q_1 \leqslant \infty$.

$$B^p_{\gamma,q} \subset B^{p_1}_{\gamma_1,q}, \quad \gamma - \gamma_1 = \nu\left(\frac{1}{p} - \frac{1}{p_1}\right), \quad p \leqslant p_1 \leqslant \infty,\ 1 \leqslant q \leqslant \infty, \tag{3.5}$$

since, by Young's inequality (2.7) and the identity

$$\psi_k = (\psi_{k-1} + \psi_k + \psi_{k+1})\psi_k \quad (\text{where } \psi_{-1} = 0),$$

it follows that

$$\|\mathscr{F}^{-1}[\psi_k] * f\|_{p_1} \leqslant \sum_{l=k-1}^{k+1} \|\mathscr{F}^{-1}[\psi_l]\|_r \|\mathscr{F}^{-1}[\psi_k] * f\|_p.$$

But Theorem B gives

$$\|\mathscr{F}^{-1}[\psi_l]\|_r \leqslant C\, 2^{l\nu/r'} \approx 2^{k\nu/r'} \quad \text{for } l-1 \leqslant k \leqslant l+1.$$

Since $\gamma_1 + \nu/r' = \gamma$ one arrives at (3.5):

$$\|f\|_{B^{p_1}_{\gamma_1,q}} \leqslant C \|f\|_{B^p_{\gamma,q}}.$$

$$B^p_{\gamma,1} \subset L^p_{\gamma,\varrho} \subset B^p_{\gamma,\infty}, \quad 1 \leqslant p \leqslant \infty. \tag{3.6}$$

We show only the right-hand side embedding, since the other follows along the same lines. Thus start with $f = \mathscr{F}^{-1}[(1+\varrho)^{-\gamma}] * g \in L^p_{\gamma,\varrho}$; then, for $k \in N$,

$$\begin{aligned} 2^{k\gamma} \|\mathscr{F}^{-1}[\psi_k] * f\|_p &= 2^{k\gamma} \left\| \mathscr{F}^{-1}\left[\left(\frac{\varrho}{1+\varrho}\right)^{\gamma}\right] * \mathscr{F}^{-1}[\psi_k \varrho^{-\gamma}] * g \right\|_p \\ &\leqslant C\, 2^{k\gamma} \|\mathscr{F}^{-1}[\psi_k \varrho^{-\gamma}] * g\|_p \leqslant C' \|g\|_p = C' \|f\|_{p,\gamma} \end{aligned}$$

uniformly in k. Here we used (2.4) for the first inequality and Theorem B for the second.

A combination of (3.6) with the characterization (3.1) yields

$$\sup_y r(y)^{-\gamma} \|\Delta_y^m f\|_p \leqslant C \|f\|_{p,\gamma}, \quad m > \gamma/\Lambda_m, \tag{3.7}$$

and a further combination with (3.5) for $p_1 = \infty$ leads to

$$\sup_y r(y)^{\nu/p-\gamma} \|\Delta_y^m f\|_\infty \leqslant C \|f\|_{p,\gamma}. \tag{3.8}$$

It should be mentioned that one can show as in Bergh and Löfström ([1], p. 152) the following strengthening of (3.6):

$$B_{\gamma,p}^p \subset L_\gamma^p \subset B_{\gamma,2}^p, \quad 1 < p \leqslant 2,$$
$$B_{\gamma,2}^p \subset L_\gamma^p \subset B_{\gamma,p}^p, \quad 2 \leqslant p < \infty.$$

(Here one may use anisotropic Littlewood–Paley theory [16] and [8]

$$J_\delta B_{\gamma,q}^p = B_{\gamma+\delta,q}^p, \quad J_\delta f = \mathscr{F}^{-1}[(1+\varrho)^{-\delta}] * f,$$

which is valid for $1 \leqslant p, q \leqslant \infty$, γ, $\gamma+\delta \geqslant 0$.)

As in [5] a combination of Theorem 2 (a) with (3.6) gives that the C^∞-functions with compact support or whose Fourier transforms have compact support are dense in $B_{\gamma,q}^p$, $\gamma > 0$, $1 \leqslant p, q < \infty$. Also as in [5] one can verify the following Sobolev-type embedding result

$$B_{\nu/p,1}^p \subset L^\infty, \quad 1 \leqslant p \leqslant \infty. \tag{3.9}$$

We conclude this section with a partial result for the problem, when the Besov spaces form multiplication algebras (for the isotropic case see e.g. [29], p. 145).

COROLLARY 4. *If $B_{\gamma,q}^p \subset L^\infty$, then $f, g \in B_{\gamma,q}^p$ implies $fg \in B_{\gamma,q}^p$ and*

$$\|fg\|_{B_{\gamma,q}^p} \leqslant C \|f\|_{B_{\gamma,q}^p} \|g\|_{B_{\gamma,q}^p}.$$

Proof. By Leibniz' formula

$$\Delta_y^m f(x) g(x) = \sum_{i=0}^m \binom{m}{i} \Delta_y^{m-i} f(x) \Delta_y^i g(x+(m-i)y)$$

it follows from the characterization (3.1) (with even m so large that $\max_{1 \leqslant i \leqslant m} \{m-i, i\} > 2\gamma/\Lambda_m$) that

$$\|fg\|_{B_{\gamma,q}^p} \leqslant C \left\{ \|f\|_\infty \|g\|_p + \left(\int_{\mathbb{R}^n} [r(y)^{-\gamma} \| \Delta_y^m (fg)\|_p]^q \frac{dy}{r(y)^\nu} \right)^{1/q} \right\}$$
$$\leqslant C \left\{ \|f\|_{B_{\gamma,q}^p} \|g\|_{B_{\gamma,q}^p} + \sum_{i=0}^{m/2} \|g\|_\infty \left(\int_{\mathbb{R}^n} [r(y)^{-\gamma} \|\Delta_y^{m-i} f\|_p]^q \frac{dy}{r(y)^\nu} \right)^{1/q} \right.$$

$$+ \sum_{i=m/2+1}^{m} \|f\|_\infty \left(\int_{\mathbf{R}^n} [r(y)^{-\gamma} \|\Delta_y^i g(\cdot + (m-i)y)\|_p]^q \frac{dy}{r(y)^\nu} \right)^{1/q} \Bigg\}$$

$$\leqslant C \|f\|_{B^p_{\gamma,q}} \|g\|_{B^p_{\gamma,q}},$$

where we again used (3.1) and the hypotheses $\|\cdot\|_\infty \leqslant C\|\cdot\|_{B^p_{\gamma,q}}$.

4. Bessel potentials revisited in case $1 < p < \infty$

Here we want to characterize the set of pointwise multipliers on L^p_γ:

$$\mathcal{M}L^p_\gamma = \{g \in L^\infty\colon \|g\|_{\mathcal{M},p,\gamma} = \inf\{C\colon \|gf\|_{p,\gamma} \leqslant C\|f\|_{p,\gamma} \text{ for all } f \in L^p_\gamma\} < \infty\}$$

and to discuss the question when L^p_γ forms a multiplication algebra:

$$L^p_\gamma L^p_\gamma \subset L^p_\gamma.$$

Our starting point is the following characterization of L^p_γ via hypersingular integrals.

THEOREM H (Dappa and Trebels [9]). *Let m be an even integer larger than γ/Λ_m. On L^p_γ, $1 < p < \infty$, the following norms are equivalent*

$$\|f\|_{p,\gamma} \approx \|f\|_p + \sup_{\varepsilon>0} \Big\| \int_{r(y)\geqslant\varepsilon} r(y)^{-\gamma-\nu} \bar{\Delta}_y^m f\, dy \Big\|_p$$

$$\approx \|f\|_p + \sup_{\varepsilon>0} \Big\| \int_{\varepsilon\leqslant r(y)\leqslant 1} r(y)^{-\gamma-\nu} \bar{\Delta}_y^m f\, dy \Big\|_p.$$

A first sufficient condition for $g \in \mathcal{M}L^p_\gamma$ (for the corresponding isotropic result see [29], p. 143) is contained in

LEMMA 5. *For $\chi \in B^\infty_{\gamma',\infty}$, $\gamma' > \gamma > 0$, there holds*

$$\|\chi f\|_{p,\gamma} \leqslant C \|\chi\|_{B^\infty_{\gamma',\infty}} \|f\|_{p,\gamma}.$$

Proof. By the above characterization it is sufficient to show that

$$\|\chi f\|_p + I_\varepsilon \leqslant C\|\chi\|_{B^\infty_{\gamma',\infty}} \|f\|_{p,\gamma}, \qquad I_\varepsilon = \Big\| \int_{\varepsilon\leqslant r(y)\leqslant 1} r(y)^{-\gamma-\nu} \bar{\Delta}_y^m (\chi f)\, dy \Big\|_p$$

uniformly in $\varepsilon > 0$. First

$$\|\chi f\|_p \leqslant \|\chi\|_\infty \|f\|_p \leqslant C\|\chi\|_{B^\infty_{\gamma',\infty}} \|f\|_{p,\gamma}. \tag{4.1}$$

Now apply Leibniz' formula

$$\bar{\Delta}_y^m (\chi f)(x) = \sum_{i=0}^{m} \binom{m}{i} \bar{\Delta}_y^{m-i} f(x-iy)\, \bar{\Delta}_y^i \chi(x+(m-i)y),$$

where we choose the even integer m larger than $2\gamma/\Lambda_m$ to obtain

$$\begin{aligned} I_\varepsilon \leqslant C \{ \|\chi & \int_{\varepsilon \leqslant r(y) \leqslant 1} r(y)^{-\gamma-\nu} \bar{\Delta}_y^m f\, dy\|_p \\ &+ \int_{r(y) \leqslant 1} r(y)^{-\gamma-\nu} \|\Delta_y^m f\|_p \|\chi(\cdot + my) - \chi(\cdot)\|_\infty \, dy \\ &+ \sum_{i=1}^{m/2} \int_{r(y) \leqslant 1} r(y)^{-\gamma-\nu} \|\Delta_y^{m-i} f\|_p \|\Delta_y^i \chi\|_\infty \, dy \\ &+ \sum_{i=1+m/2}^{m-1} \int_{r(y) \leqslant 1} r(y)^{-\gamma-\nu} \|\Delta_y^{m-i} f\|_p \|\bar{\Delta}_y^i \chi\|_\infty \, dy \\ &+ \int_{r(y) \leqslant 1} r(y)^{-\gamma-\nu} \|\Delta_y^m \chi\|_\infty \|f(\cdot - my) - f(\cdot)\|_p \, dy \\ &+ \|f \int_{\varepsilon \leqslant r(y) \leqslant 1} r(y)^{-\gamma-\nu} \bar{\Delta}_y^m \chi \, dy\|_p \} = \sum_{i=1}^{6} I_\varepsilon^i. \end{aligned}$$

By Theorem H it is clear that

$$I_\varepsilon^1 \leqslant C \|\chi\|_\infty \|f\|_{p,\gamma} \leqslant C' \|\chi\|_{B_{\gamma',\infty}^\infty} \|f\|_{p,\gamma};$$

by hypotheses

$$\begin{aligned} I_\varepsilon^6 &\leqslant C \|f\|_p \int_{r(y) \leqslant 1} r(y)^{-\gamma-\nu} r(y)^{\gamma'} \|\chi\|_{B_{\gamma',\infty}^\infty} \, dy \\ &\leqslant C \|\chi\|_{B_{\gamma',\infty}^\infty} \|f\|_{p,\gamma}. \end{aligned}$$

By (3.7) and the hypotheses on χ, the same estimate holds for I_ε^2 and I_ε^3 (one may use polar coordinates). Concerning I_ε^4 and I_ε^5 we at least have

$$\begin{aligned} \|\Delta_y^{m-i} f\|_p &\leqslant C r(y)^{\gamma(m-i)/m} \|f\|_{p,\gamma}, \\ \|\bar{\Delta}_y^i \chi\|_\infty &\leqslant C r(y)^{\gamma'} \|\chi\|_{B_{\gamma',\infty}^\infty}, \end{aligned}$$

thus the desired estimate also for these terms and Lemma 5 is established. This proof even shows more, namely (for the isotropic case see [29], p. 146).

COROLLARY 6. $\|fg\|_{p,\gamma} \leqslant C \|f\|_{p,\gamma} \|g\|_{p,\gamma}$, $\gamma > \nu/p$, *ν being the trace of P.*

Proof. Since $L_\gamma^p \subset L^\infty$ by Theorem 1, e (ii), (4.1) carries at once over with $\|\chi\|_{B_{\gamma',\infty}^\infty}$ replaced by $\|g\|_{p,\gamma}$. Form now terms analogous to $I_\varepsilon^1, \ldots, I_\varepsilon^3$ with χ replaced by g and use (3.8) to obtain the desired estimates. The remaining terms analogous to $I_\varepsilon^4, \ldots, I_\varepsilon^6$ are symmetric to the first three terms, thus interchange the roles of f and g and all is proved.

In particular, Corollary 6 implies

$$L_\gamma^p \subset \mathscr{M} L_\gamma^p, \qquad \gamma > \nu/p. \tag{4.2}$$

We mention that the proof of Corollary 6 is quite different from Strichartz' [26] deduction of the isotropic case. With the aid of Theorem H one can also prove (along the lines of Lemma 5) the following

UNIFORM LOCALIZATION THEOREM [11]. *The following norms are equivalent on* L^p_γ

$$\|f\|_{p,\gamma} \approx \Big(\sum_{\alpha\in Z^n} \|f\eta_\alpha\|^p_{p,\gamma}\Big)^{1/p}, \qquad \gamma > 0,$$

where $\eta_\alpha(x) = \eta(x+\alpha)$, $\eta \in C^\infty(R^n)$, $0 \leqslant \eta \leqslant 1$, $\eta = 1$ *on* $Q = \{x\colon |x_k| \leqslant 1\}$ *and* $\operatorname{supp}\eta \subset 2Q$.

A combination of that theorem with (4.2) leads us to the anisotropic analog of Strichartz' [26] classical result on pointwise multipliers.

COROLLARY 7. *Let* $\gamma > \nu/p$. *Then* $g \in \mathcal{M}L^p_\gamma$ *if and only if*

$$\sup_{\alpha\in Z^n} \|g\eta_\alpha\|_{p,\gamma} \doteq K < \infty \tag{4.3}$$

(i.e., g belongs uniform locally to L^p_γ*). K gives an equivalent norm on* $\mathcal{M}L^p_\gamma$.

Proof. Let $g \in \mathcal{M}L^p_\gamma$; clearly $\eta \in L^p_\gamma$ and $\|\eta_\alpha\|_{p,\gamma} = \|\eta\|_{p,\gamma} = C$. Thus, by hypothesis,

$$\|g\eta_\alpha\|_{p,\gamma} \leqslant \|g\|_{\mathcal{M},p,\gamma} \|\eta_\alpha\|_{p,\gamma} \leqslant C\|g\|_{\mathcal{M},p,\gamma}$$

uniformly in α; i.e., $K \leqslant C\|g\|_{\mathcal{M},p,\gamma}$.

Conversely, assume (4.3) to be true. Observe that η^2 satisfies the same conditions as η; then, by the Uniform Localization Theorem and (4.2),

$$\|gf\|_{p,\gamma} \leqslant C\Big(\sum_{\alpha\in Z^n} \|gf\eta^2_\alpha\|^p_{p,\gamma}\Big)^{1/p}$$
$$\leqslant CK\Big(\sum_{\alpha\in Z^n} \|f\eta_\alpha\|^p_{p,\gamma}\Big)^{1/p} \leqslant C'K\|f\|_{p,\gamma},$$

and the corollary is proved.

Let us conclude with a further characterization of the L^p_γ-spaces which reads [10]:

$$\|f\|_{p,\gamma} \approx \|f\|_p + \left\|\left(\int_0^\infty \Big[t^{-\gamma} \int_{r(y)\leqslant 1} |\bar{\Delta}^m_{A_t y} f|\,dy\Big]^2 \frac{dt}{t}\right)^{1/2}\right\|_p, \qquad \gamma > 0, \tag{4.4}$$

where m is even and larger than γ/Λ_m. An immediate consequence of (4.4) (via the Leibniz formula) is that $L^p_\gamma \subset L^\infty$ implies $L^p_\gamma L^p_\gamma \subset L^p_\gamma$.

All results of this paper can also be obtained for anisotropic Triebel-Lizorkin spaces by appropriate modifications of the methods in Triebel's book [29]; see also [12] and [26]. In [29], p. 270, it is indicated how to define these in the case of diagonal dilations. As already mentioned at the beginning, a substitution of the Euclidean distance $|\xi|$ by an A_t'-homogeneous distance function $\varrho(\xi)$ seems to be the natural definition to start with in the general situation. One may characterize these spaces by analogs of (4.4) and [29], p. 108 (16), which allow easy access to results on pointwise multipliers and multiplication algebras via the Leibniz formula; for characterizations in the case $p > 0$, $0 < q \leqslant \infty$ see the contribution of A. Seeger in these Proceedings.

Concluding let us point out that the approach to the above result (Theorem 1 to Corollary 7) is quite simple: only elementary Fourier multiplier techniques are used in the simultaneous discussion of isotropic and anisotropic function spaces.

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