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ON CHARACTERIZATION OF NON-LINEAR BEST CHEBYSHEV APPROXIMATIONS

BY

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1. Introduction. Let $C_b(X)$ be the space of real-valued continuous and bounded functions defined on the topological space X with the norm

$$\|f\| = \sup\{|f(x)|: x \in X\}.$$

For any $f \in C_b(X)$ and $a \geqslant 0$ define the following closed set:

$$M_f(a) = \{x \in X: |f(x)| \geqslant a\}.$$

If $a = \|f\|$, then we denote this set by M_f.

Let G be a non-empty subset of $C_b(X)$. We say that an element $g \in G$ is a *best approximation* to $f \in C_b(X)$ in G if $\|f-g\| \leqslant \|f-h\|$ for all $h \in G$. Kammler, assuming additionally that X is an interval of the real axis and G is a linear family, has formulated in [3] necessary and sufficient conditions for g to be a best approximation to f.

In this paper we obtain a Kolmogorov type characterization of a best approximation if G is a non-linear family and X is a topological space.

2. Main results.

Definition 1. A subset G of $C_b(X)$ has the *weak betweenness property* if for any two distinct elements g and h in G and for every non-empty closed subset D of X such that

$$\inf\{|h(x)-g(x)|: x \in D\} > 0$$

there exists a sequence $\{g_i\}$ of elements of G such that

(i) $$\lim_{i\to\infty} \|g-g_i\| = 0,$$

(ii) $$\inf\{[h(x)-g_i(x)][g_i(x)-g(x)]: x \in D\} > 0 \qquad \text{for all integers } i.$$

We note that if D is a compact set, then inequality (ii) is equivalent to the fact that $g_i(x)$ lies strictly between $g(x)$ and $h(x)$ for all $x \in D$ (i.e.

either $g(x) < g_i(x) < h(x)$ or $h(x) < g_i(x) < g(x)$). Therefore, if $X = [a, b]$, then Definition 1 is equivalent to Definition 1 from [7].

Example 1. Let P be a convex subset of $C_b(X)$ and let λ_i be a sequence of real numbers from the interval $(0, 1)$ convergent to zero. It is easy to see that the sequence defined by

$$g_i = \lambda_i h + (1-\lambda_i) g, \quad i = 1, 2, \ldots,$$

satisfies (i) and (ii) in Definition 1. Indeed, we have

$$\lim_{i\to\infty} \|g - g_i\| = \|g - h\| \lim_{i\to\infty} \lambda_i = 0.$$

Additionally, if $\inf\{|h(x) - g(x)|: x \in D\} > 0$, then we obtain

$$\inf\{[h(x) - g_i(x)][g_i(x) - g(x)]: x \in D\}$$
$$= \lambda_i(1-\lambda_i) \inf\{[h(x) - g(x)]^2: x \in D\} > 0.$$

Hence P has the weak betweenness property.

Example 2. Let P and Q denote convex subsets of $C_b(X)$ and let λ_i be such as in Example 1. Assume that $\inf\{q(x): x \in X\} > 0$ for an arbitrary $q \in Q$. Let us set

$$R = \{r = p/q: p \in P \text{ and } q \in Q\}.$$

Now, let $h = p/q$ and $g = s/t$ be two distinct elements from R. Define the sequence g_i by

$$g_i = \frac{\lambda_i p + (1-\lambda_i) s}{\lambda_i q + (1-\lambda_i) t}.$$

Since $\|t\| > 0$, we have

$$\lim_{i\to\infty} \|g - g_i\| \leqslant \frac{\|sq - tp\|}{\|t\|^2} \lim_{i\to\infty} \lambda_i = 0.$$

Additionally, if

$$\inf\left\{\frac{p(x)t(x) - q(x)s(x)}{q(x)t(x)}: x \in D\right\} > 0,$$

then in view of $\lambda_i q + (1-\lambda_i) t \in Q$ we obtain

$$\inf\{[h(x) - g_i(x)][g_i(x) - g(x)]: x \in D\}$$
$$= \lambda_i(1-\lambda_i) \inf\left\{\frac{[p(x)t(x) - q(x)s(x)]^2}{q(x)t(x)[\lambda_i q(x) + (1-\lambda_i)t(x)]^2}: x \in D\right\} > 0.$$

Hence R has a weak betweenness property.

For $X = [a, b]$, other examples of subsets having a weak betweenness property have been given in [7].

THEOREM 1 (characterization theorem of Kolmogorov type). *If a subset G of $C_b(X)$ has the weak betweenness property, then g is a best approximation to $f \in C_b(X)$ if and only if there exist no element $h \in G$ and no positive number $\varepsilon_0 < \|f-g\|$ such that*

$$\inf\{[f(x)-g(x)][h(x)-g(x)]:\ x \in M_{f-g}(\|f-g\|-\varepsilon)\} > 0 \tag{1}$$

for all $0 < \varepsilon < \varepsilon_0$.

Proof. Necessity. Let us suppose on the contrary that there exist an ε_0, $0 < \varepsilon_0 < \|f-g\|$, and $h \in G$ such that (1) holds. Additionally, let us set $U = M_{f-g}(\|f-g\|-\varepsilon_0)$.

Since

$$\inf\{|h(x)-g(x)|:\ x \in U\} \geqslant \frac{1}{\|f-g\|}\inf\{[f(x)-g(x)][h(x)-g(x)]: x \in U\} > 0,$$

it follows from Definition 1 that there exists a sequence $\{g_i\}$ of elements of G satisfying conditions (i) and (ii) with the set D replaced by U.

Let us assume that an integer n has been chosen so that for all $i \geqslant n$ we have

$$\|g-g_i\| < \min\left(\delta, \frac{\varepsilon_0}{2}\right), \quad \text{where } \delta = \inf\{|f(x)-g(x)|:\ x \in U\}.$$

Since $\delta > 0$ by (1), condition (i) implies that it is possible to select such an n. Hence for each fixed $i \geqslant n$ and for all $x \in U$ we have

$$\operatorname{sign}[f(x)-g(x)] = \operatorname{sign}[f(x)-g_i(x)] = \operatorname{sign}[g_i(x)-g(x)]$$

and

$$\begin{aligned} |f(x)-g_i(x)| &= \big([f(x)-g(x)]-[g_i(x)-g(x)]\big)\operatorname{sign}[f(x)-g_i(x)] \\ &= |f(x)-g(x)|-|g_i(x)-g(x)| \leqslant \|f-g\|-\eta_i, \end{aligned}$$

where, by (ii), the number $\eta_i = \inf\{|g_i(x)-g(x)|:\ x \in U\}$ is positive.

Now, for all $i \geqslant n$ and $x \in X \setminus U$ we have

$$\begin{aligned} |f(x)-g_i(x)| &\leqslant |f(x)-g(x)|+|g(x)-g_i(x)| \\ &< \|f-g\|-\varepsilon_0+\frac{1}{2}\varepsilon_0 = \|f-g\|-\frac{1}{2}\varepsilon_0. \end{aligned}$$

Combining this inequality with the earlier inequality for $x \in U$, we obtain

$$\|f-g_i\| \leqslant \|f-g\|-\min\left(\frac{1}{2}\varepsilon_0, \eta_i\right) < \|f-g\| \quad \text{for each fixed } i \geqslant n.$$

Thus we get a contradiction to the fact that g is a best approximation in G to f.

Sufficiency. Let us suppose on the contrary that an element $h \in G$ is a better approximation to f than g, i.e. $\|f-h\| < \|f-g\|$. Write $\delta = \|f-g\| - \|f-h\| > 0$. Let ε_0 be a positive number such that $\varepsilon_0 < \delta/2$. For every ε, $0 < \varepsilon < \varepsilon_0$, and for all $x \in M_{f-g}(\|f-g\| - \varepsilon)$ we have

$$|f(x)-g(x)| - \|f-h\| \geqslant \|f-g\| - \varepsilon_0 - \|f-h\| > \frac{\delta}{2}.$$

Hence for these ε and x we obtain

$$\begin{aligned}
&[f(x)-g(x)][h(x)-g(x)] \\
&\quad = |f(x)-g(x)|\big(|f(x)-g(x)| - [f(x)-h(x)]\operatorname{sign}[f(x)-g(x)]\big) \\
&\quad > |f(x)-g(x)|\left(\|f-h\| + \frac{\delta}{2} - [f(x)-h(x)]\operatorname{sign}[f(x)-g(x)]\right) \\
&\quad \geqslant \frac{\delta}{2}\,|f(x)-g(x)| \geqslant \frac{\delta}{2}\left(\|f-h\| + \frac{\delta}{2}\right).
\end{aligned}$$

This implies that (1) is satisfied. Therefore, the proof is completed.

Note that the proof of the sufficiency of Theorem 1 does not require any assumption about the structure of G.

Now, we assume that X is a compact metric space. In this case we denote the space $C_b(X)$ by $C(X)$. We often characterize the best Chebyshev approximation for functions from $C(X)$ by the following criterion:

KOLMOGOROV CRITERION. *An element $g \in G$ is a best approximation to $f \in C(X)$ in G if and only if there is no element $h \in G$ such that*

$$[f(x)-g(x)][h(x)-g(x)] > 0 \quad \textit{for all } x \in M_{f-g}.$$

Obviously, this criterion is true only for the sets G satisfying some additional restrictions (see, e.g., [2] and [4]).

THEOREM 2. *A necessary and sufficient condition for the Kolmogorov criterion to hold for all $f \in C(X)$ is that G has the weak betweenness property.*

Proof. Necessity. Let h, g be arbitrary fixed distinct elements of G and let D be any non-empty closed subset of X such that

$$\delta_1 = \min\{|h(x)-g(x)|:\ x \in D\} > 0.$$

Define a closed subset Z of X by $Z = \{x:\ g(x) = h(x)\}$. Obviously, we have $D \cap Z = \emptyset$. Let λ_i $(i = 1, 2, \ldots)$ be any sequence of positive numbers convergent to zero. Denote by f_1 the function from $C(X)$ defined by

$$f_1(x) = g(x) + \varepsilon_1 \frac{\operatorname{dist}(x, Z)}{\operatorname{dist}(x, Z) + \operatorname{dist}(x, D)} \operatorname{sign}[h(x)-g(x)],$$

where $0 < \varepsilon_1 < 0.5\min(\lambda_1, \delta_1)$.

Since $[f_1(x)-g(x)][h(x)-g(x)] > 0$ for all $x \in D$ and $D = M_{f_1-g}$, it follows from the Kolmogorov criterion that there exists a better approximation $g_1 \in G$ to f_1 than g, i.e. $\|f_1-g_1\| < \|f_1-g\| = \varepsilon_1$. Hence we have

$$\|g-g_1\| \leqslant \|f_1-g_1\| + \|f_1-g\| < \lambda_1.$$

Additionally, since $|f_1(x)-g_1(x)| < |f_1(x)-g(x)|$ for all $x \in D$, we can easily show that $g_1(x)$ lies strictly between $g(x)$ and $h(x)$ for all $x \in D$. Put

$$\delta_2 = \min\{|g_1(x)-g(x)|:\ x \in D\} > 0.$$

Now, replacing g_{i-2} $(g_0 = h)$ by g_{i-1}, λ_{i-1} by λ_i, and δ_{i-1} by δ_i, we can construct — by induction — functions f_i $(i = 2, 3, \ldots)$ such that $D = M_{f_i-g}$ and that g is not the best approximation to f_i in G. Finally, denoting a better approximation to f_i by g_i, we may prove that conditions (i) and (ii) in Definition 1 are satisfied for these g_i. Thus the proof of the necessity is completed.

Sufficiency. Since X is a compact space and $f, g \in C(X)$, we obtain

$$M_{f-g} = \bigcap_{\varepsilon>0} M_{f-g}(\|f-g\|-\varepsilon)$$

and, consequently, the Kolmogorov criterion holds in the case where G has the weak betweenness property (see also [7], Theorem 5).

Finally, we give an example of a non-linear approximating family G which does not have the weak betweenness property.

Example 3. Let x_i, $a = x_0 < x_1 < \ldots < x_{s+1} = b$ $(s \geqslant 0)$, be arbitrary knots and let P_i $(i = 0, 1, \ldots, s)$ be n_i-dimensional Haar subspaces on intervals $[x_i, x_{i+1}]$. Let us set

$$P[x_1, \ldots, x_s] = \{p \in C[a, b]:\ p\,|\,[x_i, x_{i+1}] \in P_i,\ i = 0, 1, \ldots, s\},$$

where $p\,|\,[c, d]$ denotes the restriction of the function p defined on $[a, b]$ to the subinterval $[c, d]$ of $[a, b]$. It is known [6] (see also [1]) that

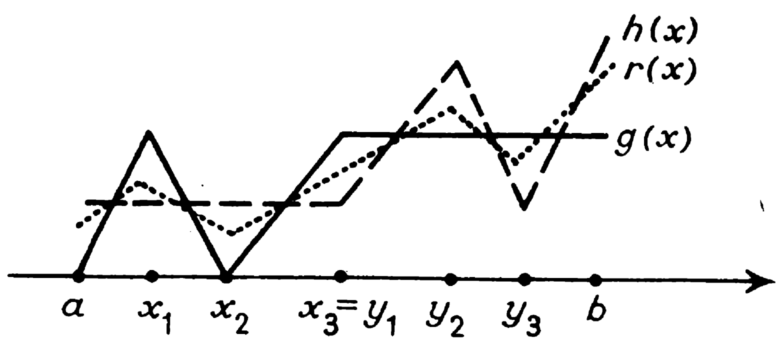

Fig. 1

$P[x_1, \ldots, x_s]$ is a $\left(\sum_{i=0}^{s} n_i - s\right)$-dimensional linear subspace. We denote by P^0 the $\left(\sum_{i=0}^{s} n_i\right)$-parameter non-linear family of the functions from $P[x_1, \ldots, x_s]$, where knots x_i, $a \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_s \leqslant b$, are free, i.e. x_i are unknown parameters. In general, the set P^0 has not the weak betweenness property.

Indeed, setting $s = 3$, $P_i = \operatorname{span}\{1, x\}$ and $D = \{x_1, x_2, x_3, y_2, y_3\}$ (see Fig. 1), it is obvious that there exists no polygonal line $r(x)$ with at most three vertices, lying between $g(x)$ and $h(x)$ for all $x \in D$. Consequently, condition (ii) in Definition 1 is not satisfied for this family P^0.

Note that the family P^0 contains the important family of splines with free knots [5]. From Example 3 and Theorem 2 it follows that we cannot use the theorem of Kolmogorov type to the characterization of best approximations by elements of P^0 in the whole space $C[a, b]$. Therefore, the following question is interesting:

What is a necessary and sufficient condition for g to be a best approximation in P^0 to an arbitrary function $f \in C[a, b]$? (**P 1169**)

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