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THE FEFERMAN-VAUGHT THEOREM REVISITED

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1. Introduction. Feferman and Vaught in their paper [7] on products of algebraic systems investigated the relation between the first order properties of a generalized product of structures and those of its factors. In its simplest form their basic result can be described as follows.

If $\langle\langle A_i, R_i\rangle | i \in I\rangle$ is a set of structures, then the product $\prod\langle A_i | i \in I\rangle$ may be considered as 2^I-valued structure, if the truth value $[\psi(f_1, \ldots, f_n)]$ for a formula $\psi(x_1, \ldots, x_n)$ and elements $f_1, \ldots, f_n$ of the product is defined by $[\psi(f_1, \ldots, f_n)] = \{i \in I \mid A_i \models \psi(f_1(i), \ldots, f_n(i))\}$. Feferman and Vaught proved that for every formula φ one can find a finite sequence $\psi_1, \ldots, \psi_m$ of formulas and a formula Φ of the language of boolean algebras such that $\varphi(f_1, \ldots, f_n)$ holds in $\prod\langle A_i | i \in I\rangle$ iff

$$\Phi\big([\psi_1(f_1, \ldots, f_n)], \ldots, [\psi_m(f_1, \ldots, f_n)]\big)$$

holds in the subset algebra 2^I. Their generalized products are obtained by admitting additional relations in the subset algebra 2^I and by relativizing the product with respect to a definable subset of the product.

Replacing the direct product formation by a more general product formation, different authors have extended the Feferman-Vaught result to reduced products (cf. [1], [2], [13]), limit powers (cf. [16], [17], [19]) and boolean powers (cf. [1]). Quite recently, Comer [3] extended the result to structures of sections of a sheaf of structures over a boolean space. Moreover, every boolean power can be considered as a structure of sections in the sense of Comer as we will show in the appendix (cf. [5]).

In this paper we want to show that all the various generalizations can be obtained as special cases of a general result. We shall consider boolean-valued structures which satisfy a maximum principle and a finite completeness property. Then we infer as before that $\varphi(a_1, \ldots, a_n)$ holds in the B-valued structure A considered as 2-valued structure iff

$$\Phi\big([\psi_1(a_1, \ldots, a_n)], \ldots, [\psi_m(a_1, \ldots, a_n)]\big)$$

holds in B.

In addition, we shall introduce the notion of a boolean-valued power of a structure, which contains as special cases all the different powers which have been considered so far. Thus we are able to study the relations between the different types of powers by a uniform method. In particular, an abstract characterization of certain boolean-valued powers yields isomorphisms of powers of different type in many cases.

2. Main result. We consider a first order language L with equality $\equiv$, negation $\neg$, conjunction $\wedge$ and existential quantification $\exists$. For the sake of simplicity we assume that L has only one n-ary relation symbol r. Besides L we have to consider L_{BA}, the first order language of boolean algebras, with equality and the usual operations.

As Feferman and Vaught did, we introduce the notion of an acceptable sequence. A sequence $\langle\Phi;\ \psi_1, \ldots, \psi_m\rangle$ is called *acceptable* if $\psi_1, \ldots, \psi_m$ are formulas of L and Φ is a formula of L_{BA} with the first m variables as free variables. An acceptable sequence $\langle\Phi;\ \psi_1, \ldots, \psi_m\rangle$ is called *partitioning* if the formulas $\psi_1 \vee \ldots \vee \psi_m$ and $\neg(\psi_i \wedge \psi_j)$ for $i \neq j$ are propositional tautologies. The following procedure associates with every acceptable sequence $\langle\Phi;\ \psi_1, \ldots, \psi_m\rangle$ a partitioning sequence $\langle\Phi';\ \psi_1', \ldots, \psi_l'\rangle$, where $l = 2^m$. Let $\{s_k \mid 1 \leqslant k \leqslant l\}$ be an enumeration of the subsets of $\{1, \ldots, m\}$ and let ψ_k' be the conjunction of the formulas ψ_i with i in s_k and $\neg\psi_j$ with j not in s_k, and let $\Phi'(x_1, \ldots, x_l)$ be the formula which states that there exist $z_1, \ldots, z_m$ such that $\Phi(z_1, \ldots, z_m)$ and each z_i is the disjunction of those x_k's with i in s_k.

Now we can associate with every formula φ in L an acceptable sequence φ^*, the companion sequence of φ:

(1) $r(x_1, \ldots, x_n)^* = \langle y_1 \equiv 1;\ r(x_1, \ldots, x_n)\rangle$,
$(x_1 \equiv x_2)^* = \langle (y_1 \vee y_2) \rightarrow y_3 \equiv 1;\ x_1 \equiv x_1, x_2 \equiv x_2, x_1 \equiv x_2\rangle$;

(2) if $\varphi_1^* = \langle\Phi_1;\ \psi_{1,1}, \ldots, \psi_{1,m_1}\rangle$ and $\varphi_2^* = \langle\Phi_2;\ \psi_{2,1}, \ldots, \psi_{2,m_2}\rangle$, then
$(\varphi_1 \wedge \varphi_2)^* = \langle\Phi_1(y_1, \ldots, y_{m_1}) \wedge \Phi_2(y_{m_1+1}, \ldots, y_{m_1+m_2});$
$\psi_{1,1}, \ldots, \psi_{1,m_1}, \psi_{2,1}, \ldots, \psi_{2,m_2}\rangle$;

(3) if $\varphi^* = \langle\Phi;\ \psi_1, \ldots, \psi_m\rangle$, then $(\neg\varphi)^* = \langle\neg\Phi;\ \psi_1, \ldots, \psi_m\rangle$;

(4) if $\varphi^* = \langle\Phi; \psi_1, \ldots, \psi_m\rangle$, then $(\exists x\varphi)^* = \langle\Phi''; \exists x\psi_1', \ldots, \exists x\psi_k'\rangle$, where $k = 2^m$ and $\langle\Phi';\ \psi_1', \ldots, \psi_k'\rangle$ is the partitioning sequence obtained from $\langle\Phi;\ \psi_1, \ldots, \psi_m\rangle$, and Φ'' is the formula which states that there exist $z_1, \ldots, z_k$ such that $z_1, \ldots, z_k$ form a partition, $z_i \leqslant x_i$ for $i = 1, \ldots, k$, and that $\Phi'(z_1, \ldots, z_k)$ holds.

It can be verified easily that the free variables of the ψ_j's in the companion sequence of φ are contained in the free variables of φ.

We now turn to the semantical notions. A *boolean-valued structure* over a set A is a quadruple $\langle A, B, E, R\rangle$, where B is a boolean algebra and $E: A^2 \to B$ and $R: A^n \to B$ are B-valued relations such that the following conditions are satisfied:

(i) $E(a_1, a_2) = E(a_2, a_1)$, $E(a_1, a_2) \wedge E(a_2, a_3) \leqslant E(a_1, a_3)$;

(ii) $E(a_1, a_1) = E(a_1, a_2) = E(a_2, a_2)$ implies $a_1 = a_2$;

(iii) $E(a_1, a_1') \wedge \ldots \wedge E(a_n, a_n') \wedge R(a_1, \ldots, a_n) \leqslant R(a_1', \ldots, a_n')$;

and B has sufficiently many suprema such that for every sentence ψ with constants from A the *truth value* $[\psi]$ in B can be defined as follows:

$$(1) \qquad [r(a_1, \ldots, a_n)] = R(a_1, \ldots, a_n), \qquad [a_1 \equiv a_2] = E(a_1, a_2),$$

$$(2) \qquad [\neg\varphi(a_1, \ldots, a_m)] = -[\varphi(a_1, \ldots, a_m)],$$

$$(3) \qquad [\varphi_1 \wedge \varphi_2] = [\varphi_1] \wedge [\varphi_2],$$

$$(4) \qquad [\exists x\varphi(x, a_1, \ldots, a_m)] = \sup\{[\varphi(a, a_1, \ldots, a_m)] \mid a \in A\}.$$

It can be verified easily that (iii) is still valid for arbitrary sentences ψ with constants from A. The elements a with $E(a, a) = 1$ are called *global.* $\langle A, B, E, R\rangle$ is called *global* iff every element in A is global.

In a boolean-valued structure $\langle A, B, E, R\rangle$ we can define an ordering as follows: $a_1 \leqslant a_2$ iff $[a_1 \equiv a_1] = [a_1 \equiv a_2]$. Two elements a_1, a_2 are called *compatible* iff $[a_1 \equiv a_1] \wedge [a_2 \equiv a_2] = [a_1 \equiv a_2]$. $\langle A, B, E, R\rangle$ is said to satisfy the *sheaf condition* if it satisfies the following two conditions:

(1) It has restrictions, i.e. for every a and b in B with $b \leqslant [a \equiv a]$ there exists a unique $a|b$ in A such that $a|b \leqslant a$ and $b = [a \equiv a|b]$.

(2) It has suprema of pairwise compatible elements, i.e., for every set $\langle a_i | i \in I\rangle$ of pairwise compatible elements such that $\sup\{[a_i \equiv a_i] | i \in I\}$ exists, there exists an element a such that $a_i \leqslant a$ for i in I and $[a \equiv a] = \sup\{[a_i \equiv a_i] | i \in I\}$.

$\langle A, B, E, R\rangle$ is said to satisfy the *finite sheaf condition* if in (2) we require I to be finite. If B is complete, then $\langle A, B, E, R\rangle$ satisfies the sheaf condition iff it has restrictions and, for every set $\langle a_i | i \in I\rangle$ of elements of A and every set $\langle b_i | i \in I\rangle$ of pairwise disjoint elements such that $b_i \leqslant [a_i \equiv a_i]$, there exists a in A such that $b_i \leqslant [a \equiv a_i]$ for i in I and $[a \equiv a] = \sup\{b_i | i \in I\}$. The finite sheaf condition can be rephrased similarly for arbitrary B.

$\langle A, B, E, R\rangle$ is called *complete* (cf. [15]) if, for every set $\langle a_i | i \in I\rangle$ of elements in A and for every set $\langle b_i | i \in I\rangle$ of pairwise disjoint elements such that $b_i \leqslant [a_i \equiv a_i]$ for i in I, there exists a in A such that $b_i \leqslant [a \equiv a_i]$. $\langle A, B, E, R\rangle$ is called *finitely compleie* iff it satisfies this condition for finite I. If $\langle A, B, E, R\rangle$ is complete and has restrictions and if B is complete, then $\langle A, B, E, R\rangle$ satisfies the sheaf condition. The same holds true with no assumptions on B if "complete" is replaced by "finitely complete".

$\langle A, B, E, R\rangle$ is said to satisfy the *maximum principle* (cf. [15]) if for every formula $\varphi(x, a_1, \ldots, a_m)$ there exists a in A such that $[\exists x\varphi(x, a_1, \ldots, a_m)] = [\varphi(a, a_1, \ldots, a_m)]$. Moreover, if $\langle A, B, E, R\rangle$ is complete and if B is complete, then $\langle A, B, E, R\rangle$ satisfies the maximum principle. To obtain this result one replaces the supremum determining $[\exists x\varphi(x, a_1, \ldots, a_m)]$ by a supremum over a disjoint refinement. Here the axiom of choice is used.

If $\langle A, B, E, R\rangle$ is a boolean-valued structure and D is a filter on B, then the *quotient* $\langle A, B, E, R\rangle/D$ is defined by $\langle A/D, B/D, E/D, R/D\rangle$. A/D is obtained by identifying a_1, a_2 in A iff $\big(E(a_1, a_1)\vee E(a_2, a_2)\big)\to\to E(a_1, a_2)$ is in D. R/D and E/D are defined as B/D-valued relations by $R/D(a_1/D, \ldots, a_n/D) = R(a_1, \ldots, a_n)/D$ and $E/D(a_1/D, a_2/D) = = E(a_1, a_2)/D$. It should be noted that finite completeness and the finite sheaf condition are preserved under taking quotients. Moreover, if $\langle A, B, E, R\rangle$ satisfies the maximum principle, then $[\varphi(a_1/D, \ldots, a_m/D)] = [\varphi(a_1, \ldots, a_m)]/D$ for arbitrary φ and $\langle A, B, E, R\rangle/D$ satisfies the maximum principle.

The *2-valued reduct* $\langle A, B, E, R\rangle|_2$ of $\langle A, B, E, R\rangle$ is given by $\langle A, 2, E|_2, R|_2\rangle$, where $R|_2(a_1, \ldots, a_n) = 1$ iff $R(a_1, \ldots, a_n) = 1$ and $E|_2(a_1, a_2) = 1$ iff $E(a_1, a_1) = E(a_2, a_2) = E(a_1, a_2)$ iff $a_1 = a_2$. Hence $\langle A, R|_2\rangle$ is a structure for L in the usual sense.

Now the main result can be formulated as follows:

THEOREM. *If $\langle \Phi; \psi_1, \ldots, \psi_m\rangle$ is the companion sequence of the formula $\varphi(x_1, \ldots, x_n)$ in L and if $\langle A, B, E, R\rangle$ is a boolean-valued structure which is finitely complete and satisfies the maximum principle, then the following two statements are equivalent:*

(i) $$\langle A, B, E, R\rangle|_2 \models \varphi(a_1, \ldots, a_n),$$

(ii) $$B \models \Phi\big([\psi_1(a_1, \ldots, a_n)], \ldots, [\psi_m(a_1, \ldots, a_n)]\big).$$

The proof proceeds by induction on φ. If φ is the atomic formula $r(x_1, \ldots, x_n)$, then $\langle A, B, E, R\rangle|_2 \models r(a_1, \ldots, a_n)$ iff $R(a_1, \ldots, a_n) = 1$ iff $B \models [r(a_1, \ldots, a_n)] \equiv 1$. If φ is the atomic formula $x_1 \equiv x_2$, then $\langle A, B, E, R\rangle|_2 \models a_1 \equiv a_2$ iff $E(a_1, a_1) = E(a_2, a_2) = E(a_1, a_2)$ iff $B \models \models ([a_1 \equiv a_1]\vee[a_2 \equiv a_2]) \to [a_1 \equiv a_2] \equiv 1$. The induction steps are straightforward except for the case of existential quantification.

If $\langle \Phi; \psi_1, \ldots, \psi_m\rangle$ is the companion sequence of the formula φ, then $\langle \Phi''; \exists x\psi_1', \ldots, \exists x\psi_k'\rangle$ with $k = 2^m$ is the companion sequence of $\exists x\varphi$. The formula $\Phi''(x_1, \ldots, x_k)$ states that there exist $z_1, \ldots, z_k$ such that $z_1, \ldots, z_k$ form a partition, $\Phi'(z_1, \ldots, z_k)$ holds and $z_i \leqslant x_i$ for $i = 1, \ldots, k$. $\langle \Phi'; \psi_1', \ldots, \psi_k'\rangle$ is the partitioning sequence obtained from $\langle \Phi; \psi_1, \ldots, \psi_m\rangle$.

If $\exists x\varphi(x, a_1, \ldots, a_n)$ holds in $\langle A, B, E, R\rangle|_2$, then there exists a in A with $\langle A, B, E, R\rangle|_2 \models \varphi(a, a_1, \ldots, a_n)$ and hence, by induction,

$$B \models \Phi([\psi_1(a, a_1, \ldots, a_n)], \ldots, [\psi_m(a, a_1, \ldots, a_n)]).$$

This yields $B \models \Phi'([\psi_1'(a, a_1, \ldots, a_n)], \ldots, [\psi_k'(a, a_1, \ldots, a_n)])$ and hence

$$B \models \Phi''([\exists x\psi_1'(x, a_1, \ldots, a_n)], \ldots, [\exists x\psi_k'(x, a_1, \ldots, a_n)])$$

as required.

Conversely, if in B the formula

$$\Phi''([\exists x\psi_1'(x, a_1, \ldots, a_n)], \ldots, [\exists x\psi_k'(x, a_1, \ldots, a_n)])$$

holds, then there exist $b_1, \ldots, b_k$ in B such that $b_1, \ldots, b_k$ form a partition of B, $\Phi'(b_1, \ldots, b_k)$ holds in B and $b_i \leqslant [\exists x\psi_i'(x, a_1, \ldots, a_n)]$ for $i = 1, \ldots, k$. The maximum principle yields elements a_i' with

$$[\exists x\psi_i'(x, a_1, \ldots, a_n)] = [\psi_i'(a_i', a_1, \ldots, a_n)].$$

The finite completeness yields an element a' with $b_i \leqslant [a' \equiv a_i']$ and hence

$$b_i \leqslant [a' \equiv a_i'] \wedge [\psi_i'(a_i', a_1, \ldots, a_n)] \leqslant [\psi_i'(a', a_1, \ldots, a_n)] \quad \text{for } i = 1, \ldots, k.$$

This implies $b_i = [\psi_i'(a', a_1, \ldots, a_n)]$ since $\langle \Phi'; \psi_1', \ldots, \psi_k' \rangle$ is partitioning. Therefore we obtain

$$B \models \Phi'([\psi_1'(a', a_1, \ldots, a_n)], \ldots, [\psi_k'(a', a_1, \ldots, a_n)])$$

and hence

$$B \models \Phi([\psi_1(a', a_1, \ldots, a_n)], \ldots, [\psi_m(a', a_1, \ldots, a_n)]).$$

By induction this implies $\langle A, B, E, R \rangle|_2 \models \varphi(a', a_1, \ldots, a_n)$ and thus

$$\langle A, B, E, R \rangle|_2 \models \exists x\varphi(x, a_1, \ldots, a_n)$$

as required. This completes the proof.

It should be noted that the class of boolean-valued structures to which the theorem can be applied is closed under quotients with respect to a filter, since finite completeness and the maximum principle are preserved under taking quotients.

3. Applications to products and powers. In the following we want to show how the various generalizations of the Feferman-Vaught theorem can be obtained from our theorem.

A product $\prod \langle A_i | i \in I \rangle$ of structures $\langle A_i, R_i \rangle$ can be made into a complete 2^I-valued structure, if R and E are defined by

$$R(f_1, \ldots, f_n) = \{i \in I \mid A_i \models r(f_1(i), \ldots, f_n(i))\}$$

and

$$E(f_1, f_2) = \{i \in I \mid f_1(i) = f_2(i)\}$$

for $f_1, \ldots, f_n$ in $\prod \langle A_i | i \in I \rangle$. Hence the *reduced product* $\prod \langle A_i | i \in I \rangle / D$, where D is a filter on I, is a finitely complete $2^I/D$-valued structure satisfying the maximum principle. This yields the results of Chang and Keisler in [2], Pacholski in [13] and Galvin in [9].

Let P be a sheaf of structures over a boolean space X in the sense of Comer [3], where $\langle P_x, R_x\rangle$ is the structure at the stalk P_x for x in X (cf. [5]). Let B be the dual algebra of X. Then $\Gamma(X, P)$, the structure of global sections, can be made into a finitely complete B-valued structure if R and E are defined by

$$R(f_1, \ldots, f_n) = \{x \in X \mid P_x \models r(f_1(x), \ldots, f_n(x))\}$$

and

$$E(f_1, f_2) = \{x \in X \mid f_1(x) = f_2(x)\}$$

for $f_1, \ldots, f_n$ in $\Gamma(X, P)$. Comer's condition (C) ensures that

$$[\psi(f_1, \ldots, f_m)] = \{x \in X \mid P_x \models \psi(f_1(x), \ldots, f_m(x))\}$$

is in B and that the maximum principle holds. This yields the result of Comer in [3] for *structures of sections*. $\Gamma(X, P)$ could be called a *generalized product*. More generally, quotients of such structures might be considered.

In the literature various types of powers have been considered, e.g. reduced powers, limit powers and boolean powers. The following notion presents a generalization of all of these. A boolean-valued structure $\langle A, B, E, R\rangle$ is said to be a *boolean-valued power* of a structure $\langle C, S\rangle$ if there exists a map d from C to A such that

$$E(d(c_1), d(c_2)) = \begin{cases} 1 & \text{if } c_1 = c_2, \\ 0 & \text{if } c_1 \neq c_2, \end{cases} \tag{0}$$

$$E(a_1, a_2) = \sup\{E(a_1, d(c)) \wedge E(a_2, d(c)) \mid c \in C\}, \tag{1}$$

$$\begin{aligned} &R(a_1, \ldots, a_n) \\ &\quad = \sup\{E(a_1, d(c_1)) \wedge \ldots \wedge E(a_n, d(c_n)) \mid \langle C, S\rangle \models r(c_1, \ldots, c_n)\}. \end{aligned} \tag{2}$$

By induction we obtain for arbitrary $\varphi(a_1, \ldots, a_m)$

$$\begin{aligned} &[\varphi(a_1, \ldots, a_m)] \\ &\quad = \sup\{[a_1 \equiv d(c_1)] \wedge \ldots \wedge [a_m \equiv d(c_m)] \mid \langle C, S\rangle \models \varphi(c_1, \ldots, c_m)\}. \end{aligned} \tag{3}$$

In particular, we have

$$[\varphi(d(c_1), \ldots, d(c_m))] = 1 \quad \text{iff} \quad \langle C, S\rangle \models \varphi(c_1, \ldots, c_m). \tag{4}$$

The quotient $\langle A, B, E, R\rangle/D$ is said to be a *boolean-valued reduced power* of $\langle C, S\rangle$. Because of (4) the elementary type of $\langle A, B, E, R\rangle/D|_2$, where $\langle A, B, E, R\rangle$ is a finitely complete boolean-valued power of $\langle C, S\rangle$ satisfying the maximum principle, depends only on the elementary type of $\langle C, S\rangle$ and the elementary type of B/D. This follows by a twofold application of the main theorem.

Any boolean-valued power $\langle A, B, E, R\rangle$ of a structure $\langle C, S\rangle$ can be represented as follows. Let $C^{\langle B\rangle}$ be the set of functions g from C to B such that $g(c)\wedge g(c') = 0$ for $c \neq c'$ and that $\sup\{g(c)|c\in C\}$ exists. Let $e\colon A\to C^{\langle B\rangle}$ be defined by $e(a)\,(c) = E\big(a, d(c)\big)$. It follows from (1) that e is injective. Let B' be the minimal completion of B. Then $C^{\langle B\rangle}$ becomes a B'-valued structure, if R' and E' are defined by

$$R'(g_1, \ldots, g_n) = \sup\{g_1(c_1)\wedge\ldots\wedge g_n(c_n)\,|\,\langle C, S\rangle \models r(c_1, \ldots, c_n)\}$$

and

$$E'(g_1, g_2) = \sup\{g_1(c)\wedge g_2(c)\mid c\in C\} \quad \text{for } g_1, \ldots, g_n \text{ in } C^{\langle B\rangle}.$$

Then e will be a truth value preserving map.

Under certain conditions we can characterize boolean-valued powers up to isomorphism. A boolean-valued power $\langle A, B, E, R\rangle$ of $\langle C, S\rangle$ is called *bounded* if every element a in A is bounded, i.e. the set $\{c\,|\,[a \equiv d(c)] \neq 0\}$ is finite. With the help of the above representation we can easily verify that any two global, complete boolean-valued powers $\langle A_1, B, E_1, R_1\rangle$ and $\langle A_2, B, E_2, R_2\rangle$ of $\langle C, S\rangle$ are isomorphic as boolean-valued structures. The same holds true if "complete" is replaced by "bounded and finitely complete". Another version can be obtained using the sheaf condition instead of the global completeness. More generally, we could consider k-bounded elements and k-completeness, where k is an infinite cardinal.

If $\langle A, B, E, R\rangle$ is boolean-valued power of $\langle C, S\rangle$, then any subset A' containing $d(C)$ determines a new boolean-valued power $\langle A', B, E', R'\rangle$ of $\langle C, S\rangle$, where E' and R' are the restrictions of E and R to A'. Moreover, the inclusion is a truth value preserving map. Hence $\langle A', B, E', R'\rangle|_2$ will be an elementary substructure of $\langle A, B, E, R\rangle|_2$ if both structures are finitely complete and satisfy the maximum principle. In particular, the boolean-valued power determined by A_ω, the subset of bounded elements, is finitely complete and satisfies the maximum principle. The latter can be seen as follows.

Any boolean-valued power $\langle A, B, E, R\rangle$ of $\langle C, S\rangle$, which is bounded and finitely complete, satisfies the maximum principle. The proof relies on the existence of an element a' for every formula $\varphi(x, a_1, \ldots, a_m)$ which satisfies

$$[a' \equiv d(c)] = \sup\{[a_1 \equiv d(c_1)]\wedge\ldots\wedge[a_m \equiv d(c_m)]\mid c = f(c_1, \ldots, c_m)\},$$

where f is a Skolem function for φ. I am indebted to S. Koppelberg for this idea. The axiom of choice is being used in this proof. Finally, it should be noted that the quotient with respect to a filter of a bounded boolean-valued power of a structure $\langle C, S\rangle$ is again a bounded boolean-valued power of $\langle C, S\rangle$.

In the following we will study the various powers which have been considered in the literature. First we shall discuss the complete powers A^I and $A^{[B]}$ of a structure $\langle A, R\rangle$, where I is a set and B is a complete boolean algebra. The boolean power $A^{[B]}$ is defined by

$$A^{[B]} = \{g \in B^A \mid g(a) \wedge g(a') = 0 \text{ for } a \neq a', \sup\{g(a) \mid a \in A\} = 1\}$$

(cf. [12]).

In particular, we have $A^I \simeq A^{[B]}$ for $B = 2^I$. A^I becomes a complete 2^I-valued power of $\langle A, R\rangle$, if E' and R' are defined by

$$E'(f_1, f_2) = \bigcup \langle f_1^{-1}(a) \wedge f_2^{-1}(a) \mid a \in A\rangle = \{i \in I \mid f_1(i) = f_2(i)\}$$

and

$$R'(f_1, \ldots, f_n) = \bigcup \langle f_1^{-1}(a_1) \wedge \ldots \wedge f_n^{-1}(a_n) \mid \langle A, R\rangle \models r(a_1, \ldots, a_n)\rangle$$
$$= \{i \in I \mid \langle A, R\rangle \models r(f_1(i), \ldots, f_n(i))\} \quad \text{for } f_1, \ldots, f_n \text{ in } A^I.$$

$A^{[B]}$ becomes a complete B-valued power of $\langle A, R\rangle$ if R' and E' are defined by

$$E'(g_1, g_2) = \sup\{g_1(a) \wedge g_2(a) \mid a \in A\}$$

and

$$R'(g_1, \ldots, g_n) = \sup\{g_1(a_1) \wedge \ldots \wedge g_n(a_n) \mid \langle A, R\rangle \models r(a_1, \ldots, a_n)\}$$

for $g_1, \ldots, g_n$ in $A^{[B]}$.

Hence the reduced power A^I/D resp. $A^{[B]}/D$, where D is a filter on I resp. B, can be viewed as a finitely complete boolean-valued reduced power with values in $2^I/D$ resp. B/D which satisfies the maximum principle.

The limit power $A^I|F$ of A, where F is a filter on $I \times I$, is defined by

$$A^I|F = \{f \in A^I \mid \{\langle i, j\rangle \mid f(i) = f(j)\} \in F\}$$

(cf. [11]).

$A^I|F$ can be represented as the directed union $\bigcup \langle A^{I/Q} \mid Q \in F, Q \in \mathrm{Eq}(I)\rangle$, where $\mathrm{Eq}(I)$ is the set of equivalence relations on I. The boolean limit power $A^{[B]}|\Gamma$ in the sense of Potthoff [14], where Γ is a directed set of complete subalgebras with complete inclusions, consists of those elements of $A^{[B]}$, whose range is contained in some algebra C in Γ. $A^{[B]}|\Gamma$ can be represented as the directed union $\bigcup \langle A^{[C]} \mid C \in \Gamma\rangle$. Since finite completeness and the maximum principle are preserved under the directed union of boolean-valued powers, $A^I|F$ resp. $A^{[B]}|\Gamma$ can be viewed as finitely complete boolean-valued power of $\langle A, R\rangle$ with values in $2^I|F$ resp. $2^{[B]}|\Gamma$ which satisfies the maximum principle if E' and R' are defined as for the full powers. As above, the limit reduced powers $A^I|F/D$ and $A^{[B]}|\Gamma/D$ have values in $2^I|F/D$ and $2^{[B]}|\Gamma/D$ and are again finitely complete and satisfy the maximum principle. Hence the theorem yields the results of Waszkiewicz and Węglorz in [16] on limit reduced powers, Wojciechowska's result on limit powers in [19] and Ash's result on boolean powers in [1].

The bounded powers A^I_ω resp. $A^{[B]}_\omega$, where B is an arbitrary boolean algebra, can be considered as special limit powers.

A^I_ω resp. $A^{[B]}_\omega$ is defined by $A^I_\omega = \{f \in A^I \mid \{a \in A \mid f(i) = a$ for some i in $I\}$ is finite$\}$ resp. $A^{[B]}_\omega = \{g \in B^A \mid g(a) \wedge g(a') = 0$ for $a \neq a'$, $\{a \in A \mid g(a) \neq 0\}$ is finite, $\sup\{g(a) \mid a \in A\} = 1\}$. A^I_ω is the subset of bounded elements of A^I. Hence it determines a finitely complete 2^I-valued power of $\langle A, R\rangle$ satisfying the maximum principle, if E' and R' are defined as for the full power. Moreover, A^I_ω becomes an elementary substructure of A^I. Similarly, $A^{[B]}_\omega$ becomes a finitely complete, bounded B-valued power of $\langle A, R\rangle$ satisfying the maximum principle if E' and R' are defined as in $A^{[B]}$. If B is complete, then $A^{[B]}_\omega$ is the set of bounded elements of $A^{[B]}$ and $A^{[B]}_\omega$ becomes an elementary substructure of $A^{[B]}$. As above, these results can be extended to the bounded limit reduced powers $A^I_\omega | F/D$ and $A^{[B]}_\omega | \Gamma/D$, where Γ is a directed set of subalgebras of B. In particular, this yields Ash's results in [1] on bounded reduced powers and bounded boolean powers. More generally, we could have considered k-bounded powers, where k is an infinite cardinal.

It follows from our results on characterizations of boolean-valued powers of a structure $\langle A, R\rangle$ that $A^{[B]}_\omega$ is the prototype of the bounded, finitely complete, global powers with values in B. Therefore we have the following isomorphisms:

(1) $A^I_\omega | F/D \simeq A^{[B]}_\omega$, where $B = 2^I | F/D$,

(2) $A^{[B]}_\omega /D \simeq A^{[B/D]}_\omega$,

(3) $A^{[B]}_\omega | \Gamma \simeq A^{[B']}_\omega$, where $B' = 2^{[B]} | \Gamma$,

(4) $A^{[B]}_\omega \simeq A^X_\omega | F$, where X is the dual space of B and F is the filter on $X \times X$ generated by the equivalence relations with finite clopen partitions.

The special case of (1) concerning reduced powers is mentioned in Ash [1]. The results in (2) and (3) seem to be new. The result in (4) can be found in Waszkiewicz and Węglorz [16]. It relies on the observation that B is isomorphic to $2^X | F$ in this case.

If B is complete, then $A^{[B]}$ is the prototype of the complete, global powers of $\langle A, R\rangle$ with values in B. However, we cannot replace in (1)-(3) the bounded powers by the corresponding full powers unless all boolean algebras involved are complete. With respect to (4) we refer to the appendix. But the corresponding full powers in (1)-(4) will at least be elementarily equivalent.

The results can be summarized as follows. All the various powers which have been studied so far can be considered as finitely complete boolean-valued powers satisfying the maximum principle. Hence the bounded elements of any such power determine an elementary substructure which is isomorphic to a bounded boolean power. The same holds true if "bounded boolean power" is replaced by "bounded limit power". Thus any such power is up to elementary equivalence a bounded boolean power resp. bounded limit power resp. limit power. Moreover, any such power is up to elementary equivalence a bounded reduced power resp. reduced power. This can be seen as follows. By a theorem of Ershov [6] there exists for every boolean algebra B a filter D on a set I such that B and $2^I/D$ are elementarily equivalent. Hence the bounded powers $A_\omega^{[B]}$ and A_ω^I/D are elementarily equivalent by our earlier remark on the elementary type of powers. This yields the required result.

4. Appendix. It is well known that the bounded boolean power $A_\omega^{[B]}$ of a structure $\langle A, R\rangle$ can be viewed as the structure of global sections in the sense of Comer [3] of the constant sheaf $A\times X$ over X, where X is the dual space of B and A is considered as discrete space. With every global section, i.e. continuous function from X to A, we associate an element g of $A_\omega^{[B]}$ as follows: $g(a) = f^{-1}(a)$ for a in A.

In order to describe the full boolean power $A_\omega^{[B]}$ of a structure $\langle A, R\rangle$ as a structure of global sections of a sheaf over X, we have to consider the following sheaf. Let P be the associated sheaf of the constant sheaf $A\times X$ over X with respect to the double negation (= interior of the closure) topology. Since B is complete, its dual space X is extremally disconnected. In particular, a subset is clopen iff it is regular open. The set of sections $\Gamma(U, P)$ of P over an open subset U of X can be described as the direct limit of the sets of continuous functions from V to A, where V is dense open in U. Hence we can associate an element g of $A^{[B]}$ with every global section f of P, i.e. continuous function f into A, which is defined on a dense open subset of X, as follows: $g(a) = \mathrm{int}\big(\mathrm{cl}\,(f^{-1}(a))\big)$. However, two such functions have to be identified if they agree on a dense open subset of X (cf. [4]). Moreover, the bijection preserves truth values. Comer's condition (C) is satisfied since X is extremally disconnected.

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