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Remark on a paper of M. A. McKiernan

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Abstract. In this note we give an elementary proof for a result of McKiernan [2] concerning functions with vanishing n-th differences, without using Hamel-bases. Our proof is based on a theorem of Hosszu [1] and it can be applied in more general situations than the case of the real line.

We use the following notation and terminology. Let $n \geqslant 1$ be an integer and let G, S be Abelian groups. A function $A_n\colon G^n \to S$ is called *n-additive* if it is a homomorphism in each of its variable. A function A_n is called *symmetric* if $A_n(x_1, \ldots, x_n) = A_n(x_{i_1}, \ldots, x_{i_n})$ for $x_1, \ldots, x_n \in G$ and for every permutation $\{i_1, \ldots, i_n\}$ of $\{1, \ldots, n\}$. If $A_n\colon G^n \to S$ is an n-additive symmetric function, then we write

$$A^{(n)}(x) = A_n(x, \ldots, x) \qquad (x \in G).$$

If $f\colon G \to S$ is a function and if $y, y_1, \ldots, y_n \in G$, we shall use the notation

$$\mathop{\Delta}\limits_{(y_1,\ldots,y_n)}^{n} = \prod_{i=1}^{n} (E_{y_i} - I)$$

and

$$\Delta_y^n = \mathop{\Delta}\limits_{(y,\ldots,y)}^{n},$$

where

$$(E_y f)(x) = f(x+y) \qquad (x \in G),$$

and

$$(If)(x) = f(x) \qquad (x \in G).$$

The above-mentioned result of M. Hosszu is the following theorem ([1]): Let $n \geqslant 0$ be an integer, G, S Abelian groups and suppose that S is divisible; further, let $f\colon G \to S$ be a function. Then for arbitrary element $y_1, \ldots, y_{n+1}$ of G the equality $(\mathop{\Delta}\limits_{(y_1,\ldots,y_{n+1})}^{n+1} f)(x) = 0$ holds for every x in G

if and only if there exist $A_k\colon G^k \to S$ $(k = 1, \ldots, n)$ k-additive symmetric functions and $A^{(0)} \in S$ such that

$$f(x) = A^{(n)}(x) + \ldots + A^{(1)}(x) + A^{(0)} \qquad (x \in G).$$

Our result is the following

THEOREM. *Let G, S be divisible Abelian groups, S is torsion-free, $n \geqq 0$ an integer and let $f\colon G \to S$ be a function. Then*

$$(\Delta_y^{n+1} f)(x) = 0$$

holds for each x, $y \in G$ if and only if

$$(\underset{y_1, \ldots, y_{n+1}}{\overset{n+1}{\Delta}} f)(x) = 0$$

holds for each x, $y_1, \ldots, y_{n+1} \in G$; equivalently, if and only if there exist $A_k\colon G^k \to S$ $(k = 1, 2, \ldots, n)$ k-additive symmetric functions and $A^{(0)} \in S$ such that

$$f(x) = A^{(n)}(x) + \ldots + A^{(1)}(x) + A^{(0)} \qquad (x \in G).$$

Proof. For brevity we use the following terminology. If G, S are divisible Abelian groups and $n \geqq 1$ is an integer, then a function f from G into S will be called of *degree n* if there are $p_2, \ldots, p_n$, $r_2, \ldots, r_n$ rationals different from zero and functions f_i from G into S $(i = 1, 2, \ldots, n+1)$ such that the equality

$$f_1(x+y) + \sum_{i=2}^{n} f_i(p_i x + r_i y) + f_{n+1}(y) + f(x) = 0 \tag{1}$$

holds for x, $y \in G$. It is obvious that a function f from G into S is of degree one if and only if for arbitrary x, $y \in G$ the value $(\Delta_y f)(x)$ does not depend on x.

Returning to the proof of the theorem we see that the "if" part is obvious. To prove the "only if" part we note that any function f which satisfies $\Delta_y^{n+1} f(x) = 0$ for every $x, y \in G$ is of degree $n+1$, because for $x, y \in G$

$$\Delta_y^{n+1} f(x) = \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i f[x + (n+1-i)y].$$

Thus, if we show that for each function f of degree n the function $\Delta_t f$ is of degree $n-1$ for arbitrary $t \in G$, then by induction on n we get

$$\underset{(t_1, \ldots, t_{n+1})}{\Delta} f(x) = \underset{t_{n+1}}{\Delta} \left[\Delta_{t_n}[\ldots \Delta_{t_1} f] \ldots\right](x) = 0 \quad \text{for every } t_1, \ldots, t_{n+1} \in G$$

and by Hosszu's theorem the statement follows.

Suppose that f satisfies (1) with suitable f_i, p_i, q_i, then substituting $x+t$, $y-t$ in place of x, y in (1) and subtracting (1) from the new equation, we obtain

$$(2) \qquad \sum_{i=2}^{n} \psi_i(p_i x + r_i y) + \psi_{n+1}(y) + (\Delta_t f)(x) = 0 \qquad (x, y \in G),$$

where

$$\psi_i(s) = (\underset{(p_i - r_i)t}{\Delta} f_i)(s) \qquad (s \in G;\, i = 2, \ldots, n)$$

and

$$\psi_{n+1}(s) = (\Delta_{-t} f_{n+1})(s) \qquad (s \in G).$$

Let $\varphi_1(s) = \psi_2(p_2 s)(s \in G)$ and substitute $\frac{p_2}{r_2}\, y$ in place of y in (2); then we obtain

$$\varphi_1(x+y) + \sum_{i=2}^{n-1} \varphi_i\left(p_i x + \frac{p_2 r_i}{r_2}\, y\right) + \varphi_n(y) + (\Delta_t f)(x) = 0 \qquad (x, y \in G),$$

where

$$\varphi_i(s) = \psi_{i+1}(s) \qquad (s \in G;\, i = 2, \ldots, n-1),$$

$$\varphi_n(s) = \psi_{n+1}\left(\frac{p_2}{r_2}\, s\right) \qquad (s \in G).$$

Hence $\Delta_t f$ is of degree $n-1$, and so, according to our preceding remarks, the theorem is proved.

References

[1] M. Hosszu, *On the Fréchet's functional equation*, Bul. Inst. Pol. Iasi 10 (1964) p. 27–28.

[2] M. A. McKiernan, *On vanishing n-th ordered differences and Hamel-bases*, Ann. Polon. Math. 19 (1967), p. 331–336.

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