ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVIII, 3 (1984), p. 473-486

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STABILITY ANALYSIS OF A DIFFERENCE SCHEME FOR THE VIBRATION EQUATION WITH A FINITE NUMBER OF DEGREES OF FREEDOM

0. INTRODUCTION

Approximate solutions of the boundary or initial value problems can be obtained by using the method of finite differences provided it is correctly posed. According to the fundamental theorem of Lax and Filipov (cf. [3], [7], [8]), approximation and stability properties of a difference scheme are sufficient conditions for its convergence.

The subject of the present paper is stability analysis of a weighed difference scheme which approximates the linear matrix equation of vibrations of a system with a finite number of degrees of freedom. The problem of stability of finite-difference methods for non-stationary problems, in particular for vibration problems, is discussed in the monographs of Samarskiĭ [8], Samarskiĭ and Gulin [9], and Richtmyer and Morton [7]. In [8] some energetic criteria for stability of a three-level scheme (see (21) below) are given. The energetic criteria for stability of the equivalent two-level scheme are derived in [9]. In the present work we study the stability of the explicit two-level scheme (24). In the analysis we use some general criteria for stability of the two-level schemes. One of our purposes is to compare the bounds for the time-integration step, which can be obtained by using different stability criteria.

The stability of difference schemes for vibration problems was also discussed by engineers ([6], [10]). We show that stability criteria given in the above-cited books are equivalent to the necessary and sufficient conditions for stability written in energetic norms.

At the beginning of the paper we recall some basic information about stability in Lyapunov's sense. We give also some energetic "a priori" estimations, the difference analogs of which form the energetic criteria of stability of the finite-difference method (52).

1. REMARKS ON THE STABILITY OF VIBRATIONS OF A SYSTEM WITH A FINITE NUMBER OF DEGREES OF FREEDOM

Linear vibrations of a system with n $(n < \infty)$ degrees of freedom are described by the second-order differential equation

$$M\frac{d^2x}{dt^2}+T\frac{dx}{dt}+Kx(t) = f(t) \tag{1}$$

with the initial conditions

$$x(0) = x_0, \qquad \left.\frac{dx}{dt}\right|_{t=0} = v_0. \tag{2}$$

Here M, T, $K \in \mathscr{R}_{n\times n}$ denote symmetric, real, and time-independent matrices. They are referred to as the *mass matrix*, the *damping matrix*, and the *stiffness matrix*, respectively. $x(t)$, $f(t) \in \mathscr{R}_n$ are the column vectors of the generalized coordinates and the external forces, respectively, and x_0, $v_0 \in \mathscr{R}_n$ are given scalar vectors.

In the case of free vibrations, the damping matrix T (called also *Rayleigh's matrix*) vanishes identically. When damping occurs, T is positive definite (cf. [4]). The mass and the stiffness matrices are then also positive definite, as the potential energy E_p and the kinetic energy E_k,

$$E_p = \tfrac{1}{2}(Kx, x), \qquad E_k = \tfrac{1}{2}(Mv, v) \qquad (v = dx/dt) \tag{3}$$

are positive. Thus

$$M = M^T > 0, \qquad T = T^T > 0, \qquad K = K^T > 0. \tag{4}$$

Notice that K is a singular matrix in the case where there are no stiff boundary conditions.

We assume also that all free vibration frequencies ω_j $(j = 1, 2, \ldots, n)$, calculated from the characteristic equation

$$\det(-\omega_j^2 M+K) = 0, \tag{5}$$

are pairwise different, i.e.

$$\omega_i \neq \omega_j \quad \text{for } i \neq j. \tag{6}$$

The vectors $a_1, a_2, \ldots, a_n \in \mathscr{R}_n$ of free vibration amplitudes are therefore linearly independent. Property (4) implies the existence of real and positive roots of (5) (cf. [2], p. 115).

Problem (1)-(2) is equivalent to the initial-value problem

$$dy/dt+Ay = p(t), \qquad y(0) = y_0, \tag{7}$$

where

$$y = \operatorname{colon}(x, v), \quad y_0 = \operatorname{colon}(x_0, v_0),$$
$$A = W^{-1}Z, \quad p(t) = W^{-1}q(t), \quad q(t) = \operatorname{colon}(f(t), 0),$$
$$W = \begin{bmatrix} 0 & M \\ E & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} K & T \\ 0 & -E \end{bmatrix}.$$

The vectors y, y_0, p, and q belong to the phase space $\mathscr{R}_n \oplus \mathscr{R}_n$. The existence of W^{-1} follows from the assumption about non-singularity of the mass matrix. We assume that problem (7) has a solution. It follows from (6) that the characteristic roots of the matrix A with the real part equal to zero have simple elementary divisors. By (4), the real parts of the roots $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ of the characteristic equation

$$\det(\lambda^2 M - \lambda T + K) = 0,$$

following from the equation $\det(A - \lambda E) = 0$, are non-negative:

$$\operatorname{Re}\lambda_j \geqslant 0 \quad (j = 1, 2, \ldots, 2n).$$

We see that the assumptions of the well-known theorem on stability in Lyapunov's sense are satisfied. The "a priori" estimation

$$\|y(t)\| \leqslant C\left[\|y_0\| + \int_0^t \|p(u)\|du\right] \tag{8}$$

holds. It is interesting from the physical point of view to specify the constant C, given properties of the matrices M, K, and T. We show that for free vibrations without damping ($T = 0$, $f = 0$) the inequality

$$\|y(t)\| \leqslant \frac{\omega_{\max}}{\omega_{\min}} (\operatorname{cond} K)^{1/2} \|y_0\| \tag{9}$$

holds, where

$$\omega_{\max} = \max_i \omega_i, \quad \omega_{\min} = \min_i \omega_i.$$

Define the normal coordinates $\xi(t)$ by the equation

$$x = \Phi\xi,$$

where $\Phi = [a_1, a_2, \ldots, a_n] \in \mathscr{R}_{n\times n}$. By (6), Φ is non-singular. The system (1) takes the separated form

$$\frac{d^2\xi(t)}{dt^2} + \omega^2\xi(t) = 0,$$

where $\boldsymbol{\omega} = \operatorname{diag}\{\omega_1, \omega_2, \ldots, \omega_n\}$. Solving this system with the appropriate initial conditions following from (2), and then coming back to the variable $\boldsymbol{x}(t)$, we obtain

$$\boldsymbol{y}(t) = \boldsymbol{U}'\boldsymbol{y}_0, \quad \boldsymbol{U}' = (\boldsymbol{\varphi\Omega})\boldsymbol{U}(\boldsymbol{\varphi\Omega})^{-1}, \tag{10}$$

$$\boldsymbol{\varphi} = \operatorname{diag}\{\boldsymbol{\Phi}, \boldsymbol{\Phi}\}, \quad \boldsymbol{\Omega} = \operatorname{diag}\{\boldsymbol{E}, \boldsymbol{\omega}\}, \tag{11}$$

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{c} & \boldsymbol{s} \\ -\boldsymbol{s} & \boldsymbol{c} \end{bmatrix},$$

$$\boldsymbol{c} = \operatorname{diag}\{\cos\omega_1 t, \ldots, \cos\omega_n t\},$$

$$\boldsymbol{s} = \operatorname{diag}\{\sin\omega_1 t, \ldots, \sin\omega_n t\}.$$

The matrix $\boldsymbol{U}'$ is similar to the orthogonal matrix $\boldsymbol{U}$. Normalizing the vectors $\boldsymbol{a}_i$ according to the rule $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\Phi} = \boldsymbol{E}$, we get

$$\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{K}^{-1}, \quad \|\boldsymbol{\Phi}^{-1}\| = \|\boldsymbol{K}\|^{1/2}. \tag{12}$$

It is easy to show that the matrix $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}$ is similar to $\boldsymbol{K}^{-1}$. Hence

$$\|\boldsymbol{\Phi}\| = \|\boldsymbol{K}^{-1}\|^{1/2}. \tag{13}$$

From (11)-(13) it follows that

$$\operatorname{cond}\boldsymbol{\varphi} = (\operatorname{cond}\boldsymbol{K})^{1/2}. \tag{14}$$

We can assume that (1)

$$\operatorname{cond}\boldsymbol{\Omega} = \operatorname{cond}\boldsymbol{\omega} = \omega_{\max}/\omega_{\min}. \tag{15}$$

The obtained results enable us to give an estimation of the norm of $\boldsymbol{y}(t)$. Starting from (10), using (14), (15), and some elementary properties of norm, we obtain (9). Thus, the constant C occurring in (8) depends on the condition of the frequency matrix $\boldsymbol{\omega}$ and the stiffness matrix $\boldsymbol{K}$.

1.1. Energetic "a priori" bounds. We assume that the solution of the initial-value problem (1)-(2) exists. The total energy of the vibrating system $E_c = E_p + E_k$, where E_p and E_k are given by (3), for every $\tau \in [0, t]$ satisfies the inequality

$$E_c(\tau) \leqslant E_c(0) + \\ + 2\exp(\omega t)\Big[\sup_{0\leqslant\tau'\leqslant t}\big(\boldsymbol{D}\boldsymbol{f}(\tau'), \boldsymbol{f}(\tau')\big) + \omega^{-1}\int_0^t \big(\boldsymbol{D}\dot{\boldsymbol{f}}(\tau_1), \dot{\boldsymbol{f}}(\tau_1)\big)d\tau_1\Big]. \tag{16}$$

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(1) The vector $\boldsymbol{y}(t)$ contains coordinates of different dimensions. Let us put $\bar{\boldsymbol{y}} = \operatorname{colon}(\boldsymbol{x}, \omega^{-1}\dot{\boldsymbol{x}})$, where $\omega \in \mathscr{R}_+$. Then the role of the matrix $\boldsymbol{\Omega}$ is played by the matrix $\bar{\boldsymbol{\Omega}} = \operatorname{diag}\{\boldsymbol{E}, \omega^{-1}\boldsymbol{\omega}\}$ which has the condition number equal to $\omega_{\max}/\omega_{\min}$ provided $\omega_{\min} \leqslant \omega \leqslant \omega_{\max}$.

Here t is a fixed time interval, ω is a certain vibration frequency, $\dot{\boldsymbol{f}} = d\boldsymbol{f}/dt$, and $\boldsymbol{D} = \boldsymbol{K}^{-1}$.

If $\boldsymbol{T} > 0$, we can choose a parameter η such that

$$\boldsymbol{T} - \eta \boldsymbol{E} > 0. \tag{17}$$

Then we get the inequality

$$E_c(\tau) \leqslant E_c(0) + \eta^{-1} \int_0^t \big(\boldsymbol{f}(\tau_1), \boldsymbol{f}(\tau_1)\big) d\tau_1 \qquad (\tau \in [0, t]) \tag{18}$$

which is different from (16), as the coefficient η does not depend on t.

Notice that (16) can be proved by applying the well-known lemma of Gronwall and Bellman (see [1]).

2. ANALYSIS OF A DIFFERENCE PROBLEM

In the sequel we analyze a difference approximation to problem (1)-(2). The matrix equation (1) is approximated by means of the symmetric weighed scheme

$$\boldsymbol{M}\boldsymbol{\delta}^k_{t\bar{t}} + \boldsymbol{T}\boldsymbol{\delta}^k_{\mathring{t}} + \boldsymbol{K}\left[\frac{1-\varkappa}{2}(\boldsymbol{\delta}^{k-1} + \boldsymbol{\delta}^{k+1}) + \varkappa \boldsymbol{\delta}^k\right] = \boldsymbol{F}^k \tag{19}$$

$$(k = 1, 2, \dots, K-1;\, Kh = T = \text{const}).$$

The initial conditions have the standard equivalents

$$\boldsymbol{\delta}^0 = \boldsymbol{x}_0, \qquad \boldsymbol{M}\boldsymbol{\delta}^1_t = \boldsymbol{M}\boldsymbol{v}_0 + h[\boldsymbol{f}(0) - \boldsymbol{T}\boldsymbol{v}_0 - \boldsymbol{K}\boldsymbol{x}_0]/2, \tag{20}$$

where

$$\boldsymbol{\delta}^k_t = (\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k-1})/h, \qquad \boldsymbol{\delta}^k_{\mathring{t}} = (\boldsymbol{\delta}^{k+1} - \boldsymbol{\delta}^{k-1})/2h,$$

$$\boldsymbol{\delta}^k_{t\bar{t}} = (\boldsymbol{\delta}^{k-1} - 2\boldsymbol{\delta}^k + \boldsymbol{\delta}^{k+1})/h^2, \qquad \boldsymbol{F}^k = \boldsymbol{f}(kh),$$

h being the time-integration step. We assume that $\varkappa \leqslant 1$. The notation used above is that of [8] or [9].

The difference scheme (19)-(20) approximates the original problem with the error $O(h^2)$. Equation (19) can be written as the three-level scheme

$$\boldsymbol{B}\boldsymbol{\delta}^{k+1} + 2\boldsymbol{A}\boldsymbol{\delta}^k + \boldsymbol{C}\boldsymbol{\delta}^{k-1} = h\boldsymbol{F}^k \qquad (k = 1, 2, \dots, K-1), \tag{21}$$

where the square matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are defined as

$$\boldsymbol{A} = \frac{\varkappa h}{2}\boldsymbol{K} - \frac{1}{h}\boldsymbol{M}, \qquad \begin{Bmatrix} \boldsymbol{B} \\ \boldsymbol{C} \end{Bmatrix} = \frac{1-\varkappa}{2} h\boldsymbol{K} + \frac{1}{h}\boldsymbol{M} + \frac{1}{2}\begin{Bmatrix} \boldsymbol{T} \\ -\boldsymbol{T} \end{Bmatrix}. \tag{22}$$

From (4) we obtain

$$B > 0, \quad B+C > 0. \tag{23}$$

The problem of stability of (19) will be discussed later.

2.1. Stability of an explicit two-level scheme. The difference problem (21) can be transformed into the explicit two-level scheme

$$Y^{k+1} = R_h Y^k + h\rho^k \quad (k = 1, 2, \ldots, K-1; Kh = T = \text{const}), \tag{24}$$

where

$$Y^k = \text{colon}(\delta^k, \tau\delta_t^k), \quad \rho^k = L_h J^k, \quad J^k = \text{colon}(F^k, F^{k+1}),$$

(25)

$$R_h = \begin{bmatrix} -2(C+B)^{-1}A & 2h'(C+B)^{-1}C \\ \frac{1}{2h}\left(4B^{-1}A(C+B)^{-1}A - B^{-1}(C+B)\right) & -2B^{-1}A(B+C)^{-1}C \end{bmatrix},$$

$$L_h = \begin{bmatrix} (B+C)^{-1} & 0 \\ -\frac{1}{h'}B^{-1}A(B+C)^{-1} & \frac{1}{2h'}B^{-1} \end{bmatrix}, \tag{26}$$

and $h' = h/\tau$. Here τ denotes a fixed time period. We assume that τ is the first value of the integration step. Hence $h \leqslant \tau$. By (23), matrices B and $B+C$ are non-singular.

A sufficient condition for stability of scheme (24) in the sense of the inequality

$$\max_{2\leqslant j\leqslant K} \|Y^j\| \leqslant C[\|Y^1\| + T \max_{1\leqslant j\leqslant K-1} \|J^j\|], \tag{27}$$

where C does not depend on h, is the existence of the constants D and L (both independent of h) such that (cf. [8])

$$\|R_h\| \leqslant 1 + Dh \quad \text{and} \quad \|L_h\| \leqslant L. \tag{28}$$

A necessary condition is the existence of the constants D' and L' (both independent of h) such that

$$\mu(R_h) \leqslant 1 + D'h \quad \text{and} \quad \mu(L_h) \leqslant L', \tag{29}$$

where $\mu(R_h)$ and $\mu(L_h)$ denote the spectral radii of matrices R_h and L_h, respectively. The first inequality of (29) is known as *Neumann's criterion*. In the next sections we analyze criteria (28) and (29) using the Euclidean norm.

2.1.1. *Analysis of Neumann's condition.* We show that the first inequality of (29) holds under the assumption

$$(1-2\varkappa)(h\,\omega_{\max})^2 + 4 > 0 \tag{30}$$

which is equivalent to

(31) $$(1-2\varkappa)h^2\boldsymbol{K}+4\boldsymbol{M}>0 \quad \text{or} \quad \boldsymbol{B}+\boldsymbol{C}-2\boldsymbol{A}>0.$$

The meaning of this condition will be explained in Section 2.2.

Consider the characteristic equation

(32) $$(\boldsymbol{R}_h-\lambda_j\boldsymbol{E})\boldsymbol{u}_j = 0,$$

where $\boldsymbol{u}_j \in \mathscr{C}_n \oplus \mathscr{C}_n$, $\lambda_j \in \mathscr{C}$, $j = 1, 2, \ldots, 2n$. Writing the vectors $\boldsymbol{u}_j$ in the form

(33) $$\boldsymbol{u}_j = \operatorname{colon}(\boldsymbol{e}_j, \boldsymbol{b}_j),$$

where $\boldsymbol{e}_j, \boldsymbol{b}_j \in \mathscr{C}_n$, by (32) and (25) we obtain the following two equations:

(34) $$(\lambda_j^2\boldsymbol{B}+2\lambda_j\boldsymbol{A}+\boldsymbol{C})\boldsymbol{e}_j = 0, \quad \tfrac{1}{2}(\lambda_j-\lambda_j^{-1})\boldsymbol{e}_j = h'\boldsymbol{b}_j.$$

Let $\boldsymbol{T}>0$. The roots λ_j of the characteristic equation

$$\det(\lambda_j^2\boldsymbol{B}+2\lambda_j\boldsymbol{A}+\boldsymbol{C}) = 0$$

may be real or complex. Scalar multiplication of the first equation of (34) by the vector $\boldsymbol{e}_j$ gives us

$$a_j\lambda_j^2+b_j\lambda_j+c_j = 0,$$

where $a_j = (\boldsymbol{B}\boldsymbol{e}_j, \boldsymbol{e}_j)$, $b_j = 2(\boldsymbol{A}\boldsymbol{e}_j, \boldsymbol{e}_j)$, and $c_j = (\boldsymbol{C}\boldsymbol{e}_j, \boldsymbol{e}_j)$. From (4) it follows that $a_j \in \mathscr{R}_+$, $b_j, c_j \in \mathscr{R}$. The matrix $\boldsymbol{C}-\boldsymbol{B} = -\boldsymbol{T}$ is negative definite. Thus we have

(35) $$c_j/a_j < 1.$$

From the second inequality of (31) it follows that

(36) $$a_j+c_j > b_j.$$

Since $\boldsymbol{B}+\boldsymbol{C}+2\boldsymbol{A} = h\boldsymbol{K}>0$, we have

(37) $$a_j+c_j > -b_j.$$

Relations (35)-(37) are necessary and sufficient conditions for the inequality $|\lambda_j|<1$ to hold (cf. [1], p. 221-222). Neumann's condition is therefore fulfilled for $D' = 0$.

In the case where no damping occurs, we have $\boldsymbol{B} = \boldsymbol{C}$ and $a_j = c_j$. The second condition of (31) simplifies to the form $\boldsymbol{B}-\boldsymbol{A}>0$. It is not difficult to show that the roots λ_j lie on the unit circle $|\lambda_j| = 1$, and Neumann's inequality is true for $D' = 0$.

Condition (30) is stronger than Neumann's criterion, as it implies the inequality $|\lambda_j| \leqslant 1$ which is stronger than the first inequality of (29).

2.1.2. *Sufficiency of the stability criterion.* We have shown in the previous section that $\mu(\boldsymbol{R}_h) = 1$ for $\boldsymbol{T} = 0$ or $\mu(\boldsymbol{R}_h) < 1$ when damping occurs, provided h satisfies (31). Notice that $\|\boldsymbol{R}_h\| \geqslant \mu(\boldsymbol{R}_h)$ ([5], Theorem 6.1.3), where equality holds when $\boldsymbol{R}_h$ is normal, which occurs in the rather rare case of $\boldsymbol{KM} = \boldsymbol{MK}$.

In the case of vibrations without damping we thus have

$$\|\boldsymbol{R}_h\| \geqslant \mu(\boldsymbol{R}_h) = 1.$$

Without loss of generality we may assume that $\|\boldsymbol{R}_h\| \geqslant 1$.

We will examine some conditions which imply the first inequality of (28). To this end we show that the matrix $\boldsymbol{R}_h^{\mathrm{T}}\boldsymbol{R}_h$ is analytic in a closed interval $[0, h_1]$ and $\boldsymbol{R}_h^{\mathrm{T}}\boldsymbol{R}_h = \boldsymbol{E}$ for $h = 0$. Assume that the integration step satisfies (cf. [5], Theorem 7.1.1)

$$\left\| \frac{1-\varkappa}{2} h^2 \boldsymbol{M}^{-1}\boldsymbol{K} + \frac{1}{2} h \boldsymbol{M}^{-1}\boldsymbol{T} \right\| < 1, \tag{38}$$

which implies the existence of the expansions

$$[h(\boldsymbol{C}+\boldsymbol{B})]^{-1} = \tfrac{1}{2}[\boldsymbol{E}-(1-\varkappa)h^2\chi + \ldots]\boldsymbol{M}^{-1},$$

$$[h\boldsymbol{B}]^{-1} = [\boldsymbol{E}-h\Psi + \ldots]\boldsymbol{M}^{-1},$$

where $\chi = \frac{1}{2}\boldsymbol{M}^{-1}\boldsymbol{K}$ and $\Psi = \frac{1}{2}\boldsymbol{M}^{-1}\boldsymbol{T}$. Using these expansions, we can represent the submatrices of $\boldsymbol{R}_h$ (see (25)) in the form of power series

$$-2(\boldsymbol{C}+\boldsymbol{B})^{-1}\boldsymbol{A} = \boldsymbol{E} - h^2\chi + \ldots,$$

$$2h'(\boldsymbol{C}+\boldsymbol{B})^{-1}\boldsymbol{C} = \frac{h}{\tau}\boldsymbol{E} + \ldots,$$

$$-2\boldsymbol{B}^{-1}\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})^{-1}\boldsymbol{C} = \boldsymbol{E} - 2h\Psi + \ldots,$$

$$\frac{\tau}{2h}[4\boldsymbol{B}^{-1}\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})^{-1}\boldsymbol{A} - \boldsymbol{B}^{-1}(\boldsymbol{B}+\boldsymbol{C})] = -2h\tau\chi + \ldots$$

We thus have

$$\boldsymbol{R}_h = \boldsymbol{E} + h\begin{bmatrix} \boldsymbol{0} & \tau^{-1}\boldsymbol{E} \\ -2\tau\chi & -2\Psi \end{bmatrix} + \ldots,$$

$$\boldsymbol{R}_h^{\mathrm{T}}\boldsymbol{R}_h = \boldsymbol{E} + h'\begin{bmatrix} \boldsymbol{0} & \boldsymbol{E} \\ \boldsymbol{E} & \boldsymbol{0} \end{bmatrix} + \ldots$$

Hence the matrix $\boldsymbol{R}_h^{\mathrm{T}}\boldsymbol{R}_h$ is analytic in the interval $[0, h_1]$ defined implicitly by (38). The equality

(39) $$R_h^{\mathrm{T}} R_h = E$$

holds for $h = 0$. By Rellich's theorem ([5], Section 7.9), all the eigenvalues of $R_h^{\mathrm{T}} R_h$ are analytic. The function

$$f(h) = \|R_h\|^2 = \max_{1 \leqslant j \leqslant 2n} \lambda_j(R_h^{\mathrm{T}} R_h)$$

is analytic in a certain interval $[0, h_2]$. For some positive constants M and N the inequalities $f(h) < M$ and $f'(h) < N$ $(0 \leqslant h \leqslant h_2)$ hold. From (39) it follows, by the continuity of the norm, that $f(0) = 1$. Now, it is easy to obtain the estimation $f(h) < 1 + Nh$ $(0 \leqslant h \leqslant h_2)$ which implies

(40) $$\|R_h\| < 1 + \frac{N}{2} h.$$

In the remaining part of this section we show that the first part of (28) is satisfied under the assumption

(41) $$(1 - 2\varkappa) h^2 \omega_{\max}^2 + 4\sigma^2 > 0 \qquad (\sigma < 1),$$

which is weaker than that obtained above and a little stronger than (30). Therefore, it will be clear that the first part of (28) holds no matter how large is the interval of analyticity of $f(h)$. We restrict ourselves to the case of non-damped vibrations.

An upper bound for $\|R_h\|$ follows from (41). To see this, we introduce the normal coordinates for the scheme (21) in a manner analogous to that used in Section 1. Let

(42) $$\delta^k = \Phi_h \xi^k.$$

where $\Phi_h = [e_1, e_2, \ldots, e_n]$, and e_i are the vectors appearing in (33). One can assume that $e_i \in \mathscr{R}_n$. The first equation of (34) takes the form ($T = 0$!)

(43) $$(B \cos\varphi_j + A) e_j = 0, \qquad \lambda_j = \exp(i\varphi_j).$$

From (6) it follows that the roots λ_j are simple, which implies the linear independence of the vectors e_j. We normalize e_j by

(44) $$\Phi_h^{\mathrm{T}} B \Phi_h = E, \qquad (Be_i, e_j) = \delta_{ij}.$$

Making a change of variables in (24) according to the rule (42), we obtain the system

$$\begin{bmatrix} \xi^{k+1} \\ \tau \xi_{\bar{t}}^{k+1} \end{bmatrix} = V_h \begin{bmatrix} \xi^k \\ \tau \xi_{\bar{t}}^k \end{bmatrix}, \qquad V_h = \begin{bmatrix} -c_h & h'E \\ \dfrac{1}{h'}(c_h^2 - E) & -c_h \end{bmatrix},$$

where $c_h = \operatorname{diag}\{\cos\varphi_1, \ldots, \cos\varphi_n\}$. This system can be also written in the form

$$\begin{bmatrix} \xi^{k+1} \\ \tau\omega_h^{-1}\dot{\xi}^{k+1} \end{bmatrix} = U_h \begin{bmatrix} \xi^{k} \\ \tau\omega_h^{-1}\dot{\xi}^{k} \end{bmatrix}, \quad U_h = \begin{bmatrix} -c_h & s_h \\ -s_h & -c_h \end{bmatrix},$$

where

$$s_h = \operatorname{diag}\{\sin\varphi_1, \ldots, \sin\varphi_n\}, \quad \omega_h = s_h/h'.$$

The matrix U_h is orthogonal. Let

$$\varphi_h = \operatorname{diag}\{\Phi_h, \Phi_h\}, \quad \Omega_h = \operatorname{diag}\{E, \omega_h\}. \tag{45}$$

We have

$$Y^{k+1} = (\varphi_h\Omega_h)^{-1}U_h(\varphi_h\Omega_h)Y^k,$$

which by (24) implies

$$R_h = (\varphi_h\Omega_h)^{-1}U_h(\varphi_h\Omega_h). \tag{46}$$

Thus, the matrices R_h and U_h are similar. Using (44) we can show that

$$\operatorname{cond}\varphi_h = \big(\operatorname{cond}(hB)\big)^{1/2}. \tag{47}$$

Moreover,

$$\min_{1\leqslant j\leqslant n} \lambda_j(hB) \geqslant \min_{1\leqslant j\leqslant n} \lambda_j(M),$$

$$\max_{1\leqslant j\leqslant n} \lambda_j(hB) \leqslant \frac{1-\varkappa}{2}\tau^2\|K\| + \|M\|.$$

This follows from (22) and a theorem on the eigenvalues of matrices P and Q such that $P-Q > 0$ (cf. [2], p. 129). Using (31) we obtain

$$\operatorname{cond}(hB) \leqslant a_1, \tag{48}$$

where

$$a_1 = \begin{cases} \dfrac{3\varkappa-1}{2(2\varkappa-1)}\operatorname{cond}M & (1/2 < \varkappa < 1), \\ \left(\dfrac{1-\varkappa}{2}\tau^2\dfrac{\|K\|}{\|M\|}+1\right)\operatorname{cond}M & (\varkappa < 1/2). \end{cases}$$

We give an upper bound for the condition number of the matrix ω_h. Comparing (43) and (5) we obtain

$$\begin{aligned} \sin\varphi_j &= h\omega_j[1+(h\omega_j/2)^2(1-2\varkappa)]^{1/2}/d, \\ \cos\varphi_j &= (1-h^2\omega_j^2/2)/d, \end{aligned} \tag{49}$$

where $d = 1+(1-\varkappa)h^2\omega_j^2/2$. By (41) we have

(50) $$\beta_1 \leqslant |h^{-1}\sin\varphi_j| \leqslant \beta_2,$$

where

$$\beta_1 = \begin{cases} [(1-\sigma)(2\varkappa-1)]^{1/2}\omega_{\min} & (1/2 < \varkappa < 1), \\ \omega_{\min}/[1+(1-\varkappa)\tau^2\omega_{\max}^2/2]^{1/2} & (\varkappa < 1/2), \end{cases}$$

$$\beta_2 = \begin{cases} \omega_{\max} & (1/2 < \varkappa < 1), \\ [1+\tau^2(1-2\varkappa)\,\omega_{\max}^2/4]^{1/2} & (\varkappa < 1/2). \end{cases}$$

Hence $\operatorname{cond}\omega_h \leqslant \beta_2/\beta_1$ and

(51) $$\operatorname{cond}\boldsymbol{\Omega}_h \leqslant a_2, \qquad a_2 \neq a_2(h).$$

From (46)-(48) and (51) we obtain $\|\boldsymbol{R}_h\| \leqslant M = a_2\sqrt{a_1}$, $M \neq M(h)$.

Notice that (40) holds whenever $h < h_2$. For h satisfying $h > h_2$ and (41) we have

$$\|\boldsymbol{R}_h\| \leqslant 1+\frac{M-1}{h_2}h.$$

2.1.3. *Stability relative to the right-hand sides.* We shall show that there exists a constant L such that $\|\boldsymbol{L}_h\| \leqslant L$. Starting from (26) and using known relations between the spectral and Euclidean norms, we obtain

$$\|\boldsymbol{L}_h\|^2 \leqslant r_2^2\|\boldsymbol{L}_h\|_E^2 = r_2^2\Big[\|(\boldsymbol{B}+\boldsymbol{C})^{-1}\|_E^2+\frac{1}{4h'^2}\|\boldsymbol{B}^{-1}\|_E^2+ \\ +(h')^{-2}\|\boldsymbol{B}^{-1}\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})^{-1}\|_E^2\Big],$$

$$\|\boldsymbol{L}_h\|^2 \leqslant (r_2/r_1)^2\big[\|(\boldsymbol{B}+\boldsymbol{C})^{-1}\|^2+\|(2h'\boldsymbol{B})^{-1}\|^2+ \\ +4\|(2h'\boldsymbol{B})^{-1}\|^2\|h'\boldsymbol{A}\|^2\big\|\big(h'(\boldsymbol{B}+\boldsymbol{C})\big)^{-1}\big\|^2\big].$$

Now, the existence of the upper bound L for the norm $\|\boldsymbol{L}_h\|$ follows from the inequalities

$$\max\big[\|(\boldsymbol{B}+\boldsymbol{C})^{-1}\|,\ \|(2h'\boldsymbol{B})^{-1}\|,\ \big\|\big(h'(\boldsymbol{B}+\boldsymbol{C})\big)^{-1}\big\|\big] \leqslant \frac{\tau}{2}\big(\min_{1\leqslant j\leqslant n}\lambda_j(\boldsymbol{M})\big)^{-1}$$

(cf. the theorem on the eigenvalues of matrices with positive definite difference [2], p. 129) and

$$\|h'\boldsymbol{A}\|^2 \leqslant \tau^{-2}\Big(\frac{|\varkappa|\tau^2}{2}\|\boldsymbol{K}\|+\|\boldsymbol{M}\|\Big)^2.$$

Thus, the second inequality of (28) (and, obviously, the second inequality of (29)) holds independently of the choice of the time-integration step.

2.2. Energetic criteria of stability. Samarskiĭ derived difference analogs of the energetic "a priori" estimations given in Section 1.1. The analog of (16) is

$$\max_{1\leqslant k\leqslant K} E_c^{k-1,k} \leqslant E_c^{01} + C \max_{1\leqslant k\leqslant K} [(\boldsymbol{D}\boldsymbol{F}^k, \boldsymbol{F}^k) + \tau^2(\boldsymbol{D}\boldsymbol{F}_{\bar{t}}^k, \boldsymbol{F}_{\bar{t}}^k)], \tag{52}$$

where

$$2E_c^{k-1,k} = \tfrac{1}{4}\big(\boldsymbol{K}(\boldsymbol{\delta}^{k-1}+\boldsymbol{\delta}^k), \boldsymbol{\delta}^{k-1}+\boldsymbol{\delta}^k\big) + (\boldsymbol{M}_h\boldsymbol{\delta}_{\bar{t}}^k, \boldsymbol{\delta}_{\bar{t}}^k), \tag{53}$$

$$\boldsymbol{M}_h = (1-2\varkappa)h^2\boldsymbol{K}/4 + \boldsymbol{M}, \qquad \boldsymbol{\delta}_{\bar{t}}^k = (\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k-1})/h, \tag{54}$$

$$\boldsymbol{F}_{\bar{t}}^k = (\boldsymbol{F}^k - \boldsymbol{F}^{k-1})/h, \qquad C \neq C(h). \tag{55}$$

The analog of (18) is

$$\max_{1\leqslant k\leqslant K} E_c^{k-1,k} \leqslant E_c^{01} + \frac{1}{2\eta}\sum_{k=1}^{K} h(\boldsymbol{F}^k, \boldsymbol{F}^k), \tag{56}$$

where η is the constant satisfying (17).

The above inequalities can be considered as the stability conditions if and only if the matrix $\boldsymbol{M}_h$, given by the first equation of (54), is positive definite. This is equivalent to the restrictions discussed previously (see (31)). Thus, (31) is the necessary and sufficient condition for the energetic inequalities to hold.

In [9] Samarskiĭ and Gulin analyze the stability using the method of reduction of the three-level scheme to the two-level one. The stability of the scheme obtained is examined by the method of energetic inequalities. The results are identical to those presented here.

3. CONCLUDING REMARKS

In Section 1 we have briefly discussed the problem of stability of linear vibrations of a system with a finite number of degrees of freedom. The use of normal coordinates in the case of free vibrations without damping enabled us to obtain the "a priori" estimation

$$\|\boldsymbol{y}(t)\| \leqslant \frac{\omega_{\max}}{\omega_{\min}} (\operatorname{cond} \boldsymbol{K})^{1/2} \|\boldsymbol{y}_0\|.$$

The greater is the ratio $\omega_{\max}/\omega_{\min}$, the weaker becomes the estimation. The condition of the stiffness matrix $\boldsymbol{K}$ is less important.

In Section 2.1 we examined the stability of the two-level explicit scheme, equivalent to the three-level scheme (19). The analysis showed that the conditions

$$(1-2\varkappa)(h\omega_i)^2+4\sigma^2>0, \quad \sigma<1, \; i=1,2,\dots,n,$$

are sufficient for stability in the sense of (27), while the necessary conditions of stability (29) are satisfied whenever the inequality

(57) $$(1-2\varkappa)(h\omega_i)^2+4>0 \quad (i=1,2,\dots,n)$$

holds.

In Section 2.2 we showed, using some results of Samarskiĭ, that (57) is the necessary and sufficient condition for stability in the sense of the energetic estimations (52)-(56). These estimations imply that the damping strengthens the stability relative to the right-hand sides of the difference scheme used.

Finally, we show that in the case where no damping occurs the considered difference scheme is unstable with respect to the initial conditions in the sense of the inequality

(58) $$\|\delta^k\| \leqslant C(\|\delta^0\|+\|\delta^1\|), \quad C \neq C(h) \;\; (k=1,2,\dots,K).$$

Changing the variables in (21) as described in (42), we obtain the equation

$$\tfrac{1}{2}(\xi^{k+1}+\xi^{k-1})+c_h\xi^k=0.$$

Solving this system of recurrence equations with appropriate initial conditions we obtain the formula

$$\delta^k=\Phi_h c_{hk}\Phi_h^{-1}\delta^0+\Phi_h s_{hk}(s_h\Phi_h^{-1}\delta^1-c_h s_h^{-1}\Phi_h^{-1}\delta^0),$$

where

$$c_{hk}=\operatorname{diag}\{\cos\varphi_1 k,\dots,\cos\varphi_n k\},$$
$$s_{hk}=\operatorname{diag}\{\sin\varphi_1 k,\dots,\sin\varphi_n k\}.$$

Assume that $\delta^0=0$. Then $\delta^k=(\Phi_h s_{hk}s_h^{-1}\Phi_h^{-1})\delta^1$. Now, we obtain easily the estimation

$$\|\delta^k\|_{\Phi_h^{-1}} \geqslant \min_{1\leqslant j\leqslant n} |\lambda_j(s_{hk}s_h^{-1})| \, \|\delta^1\|_{\Phi_h^{-1}},$$

where

(59) $$\|\delta^k\|_{\Phi_h^{-1}}=\|\Phi_h^{-1}\delta^k\|$$

is the properly defined vector norm (see [5], Section 6.2, Problem 7). Using theorems about the upper and the lower bound for the norm (59) ([5], Sections 6.3 and 6.5), we get

(60) $$\|\delta^k\|\operatorname{cond}\Phi_h \geqslant \min_{1\leqslant j\leqslant n} |\lambda_j(s_{hk}s_h^{-1})| \, \|\delta^1\|.$$

The matrix $s_{hk}s_h^{-1}$ takes the form

$$s_{hk}s_h^{-1} = \operatorname{diag}\{\sin\varphi_1 k/\sin\varphi_1, \ldots, \sin\varphi_n k/\sin\varphi_n\}.$$

From the second part of (50) we obtain

$$\min_{1\leqslant j\leqslant n} |\lambda_j(s_{hk}s_h^{-1})| \geqslant \beta_2^{-1}|\sin k\varphi_i|/h \qquad (i = 1, 2, \ldots, n).$$

For $k = K$ we have

$$(61) \qquad \beta_2^{-1}|\sin K\varphi_i|/h = \beta_2^{-1}h^{-1}\sin\left[K\left(h\omega_i + O(h^3)\right)\right] \to \infty \qquad (h\to 0)$$

because $Kh = T$ and, by (49), $\varphi_j = h\omega_j + O(h^3)$. Using the first part of (45), (47), (48), (60), and (61), we obtain

$$\sqrt{a_1}\,\|\delta^K\| \geqslant \operatorname{cond}\Phi_h\|\delta^K\| \to \infty \qquad (h\to 0),$$

which means the instability of the difference scheme (21) in the sense of (58).

Let us remark that the criteria of the choice of the integration step given in [6], [10], and in (57) are the same. However, it is easier to examine whether a matrix is positive definite than to calculate its eigenvalues. Therefore, it should be more convenient to use (31) rather than (57).

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Received on 29. 12. 1979