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On Lakshmikantham's comparison method for Gronwall inequalities

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Abstract. We deal with Gronwall type integral inequalities of the form

$$f(x(t)) \leqslant a(t)+H\left(t, \int_{a}^{t} W(t, s, x(s))\,ds\right), \quad t \in J = [a, \beta]. \tag{1}$$

The main result is that, under appropriate continuity and monotonicity assumptions, any solution x of (1) must satisfy

$$x(t) \leqslant f^{-1}[a(t)+H(t, \tilde{r}(t))], \quad a \leqslant t \leqslant \beta_1 \ (\leqslant \beta), \tag{2}$$

with $\tilde{r}(t) = r(t, t, a)$, where $r(t, T, a)$ is the maximal solution of the differential system

$$\frac{dr}{dt} = W(T, t, f^{-1}[a(t)+H(t, r)]), \quad a \leqslant t \leqslant T \leqslant \beta_1;\ r(a) = 0. \tag{3}$$

Corresponding results are obtained when $\leqslant$ is replaced by $\geqslant$ in one or both of (1), (2), or with minimal rather than maximal solutions of (3). The results extend, and in some cases clarify or correct, earlier results of V. Lakshmikantham. To this end we also include a detailed analysis leading to a correct domain of existence of maximal or minimal solutions of differential systems of the form

$$\frac{dr}{dt} = k(t)g(r)+\sigma(t), \quad r(a) = r_0, \tag{4}$$

to which (3) often reduces. As special cases, we also obtain (or extend) earlier results of Bihari, Gollwitzer, Stachurska and Beesack. Finally the method is also applied to obtain more general results than those obtained recently by S. G. Deo and U. D. Dhongade for inequalities involving two integral terms, and to eliminate a certain defect in those results.

1. Introduction. One of the most useful and pervasive techniques used in the theory of ordinary and partial differential equations and integral equations consists in applying so-called *Gronwall type inequalities*. Gronwall's inequality appeared in 1919 in [13], although a special case of it had occurred in Peano [22] as early as 1885. These inequalities, and their numerous extensions and generalizations are, in fact, implications of the form

$$(1) \qquad f(x(t)) \leqslant (Tx)(t), \quad t \in J \Rightarrow x(t) \leqslant X(t), \quad t \in J_1 \subset J,$$

where T is an operator involving integrals, f is a given increasing function, J and J_1 are intervals, and X is a function which is an upper bound for all functions x satisfying the hypothesis. (An abstract version for general partially ordered spaces has been considered by A. Pelczar [23]; see also F. Chandra and B. A. Fleishman [6] and L. Losonczi [18].) Examples of such inequalities with $f(x) \equiv x$ are given in Bellman [3], Babkin [1], Satō [24], Bihari [4], Li [17], Viswanatham [27], Jones [14], Willett [30], Chu and Metcalf [7], Maroni [19], Beesack [2], Deo and Murdeshwar [11] and Stachurska [25]. Examples with general (or special) f have been considered by Willett and Wong [31], Gollwitzer [12], Butler and Rogers [5], Lakshmikantham [15], and Deo and Dhongade [8], [9]. In addition, results involving systems of inequalities have been given by Ważewski [29], Opial [21], Olech [20], Ziebur [32], and Deo and Murdeshwar [10], among others.

The best results of the form (1) are those in which X is the solution (or the maximal solution in the absence of uniqueness) of the corresponding functional equation

$$(2) \qquad f(X(t)) = (TX)(t), \quad t \in J_1.$$

(The concept of the maximal solution of an ordinary differential equation also dates back at least to Peano [22].) Of the papers listed above, best upper bounds of this form are found in [3], [4], [14], [30], [7], and the maximal solution was explicitly introduced in [1], [24], [27], [29], [20], [21], [32]. In many cases, especially for $f(x) \not\equiv x$, the solution or maximal solution of (2) can not be explicitly obtained.

V. Lakshmikantham [15] considered inequalities of the form

$$(3) \qquad f(x(t)) \leqslant a(t) + b(t) h\Big[c(t) + \int_a^t k(t, s) w(s, x(s))\, ds\Big], \quad t \in J,$$

for which solutions of the corresponding functional equation are not readily obtained. Nevertheless, by using a comparison theorem for differential inequalities together with estimates of the maximal solutions of related differential equations, a number of the earlier results were obtained in a uniform manner. It is the purpose of this paper to extend and clarify Lakshmikantham's comparison method somewhat, and to correct some of the defects and errors in [15], [8], [9].

Our main results will deal with the more general inequality

$$(4) \qquad f(x(t)) \leqslant a(t) + H\Big(t, \int_a^t W(t, s, x(s))\, ds\Big), \quad t \in J,$$

and its special cases. We shall also apply these results to inequalities of the form

$$(5)\qquad f\big(x(t)\big) \leqslant a(t)+\int_a^t k_1(s)g_1\big(x(s)\big)\,ds+h\Big(\int_a^t k(s)g\big(x(s)\big)\,ds\Big),$$

considered by Deo and Dhongade [8], [9]. Many of the results obtained are either extensions or improvements of earlier results obtained on an ad hoc basis.

2. The main theorem. We need the following well-known comparison theorem, proofs of which may be found in Szarski ([26], Theorem 9.5, p. 27), Lakshmikantham and Leela ([16], Theorem 1.4.1, p. 15), and Walter ([28], Theorem X, p. 68).

THEOREM 2.1. *Let $F(t, x)$ be continuous in an open set D containing the point (a, x_0), and suppose that the initial value problem*

$$(1)\qquad \frac{dr}{dt} = F(t, r), \qquad r(a) = x_0,$$

has a maximal solution $r = r(t)$ with domain $[a, \beta_1]$. If x is a differentiable function on $[a, \beta_1]$ such that $\big(t, x(t)\big) \in D$ for $t \in [a, \beta_1]$ and

$$(2)\qquad x'(t) \leqslant F\big(t, x(t)\big), \qquad a \leqslant t \leqslant \beta_1,\ x(a) \leqslant x_0,$$

then

$$(2')\qquad x(t) \leqslant r(t), \qquad a \leqslant t \leqslant \beta_1.$$

Moreover, the result remains valid if "maximal" is replaced by "minimal" and $\leqslant$ is replaced by $\geqslant$ in both (2) *and* (2′).

Notation. In what follows we shall be dealing with monotonic functions of several variables. If $F_1(t, s, u)$ is defined on $I_1 \times J_1 \times K_1$, where I_1, J_1, K_1 are intervals, then by saying that $F_1(u)$ is monotonic (or non-decreasing, increasing, etc.) we shall mean that for each $(t, s) \in I_1 \times J_1$, the function $g(u) \equiv F_1(t, s, u)$ is monotonic (or non-decreasing, increasing, etc.) on K_1. Similarly, if $F_2(t, s, v)$ is another such function defined on $I_2 \times J_2 \times K_2$, then we say, for example, that $F_1(u)$, $F_2(t)$ have the *same parity* (or the *opposite parity*) if $F_1(u)$, $F_2(t)$ are both non-increasing or both non-decreasing (or that one is non-increasing and the other is non-decreasing). We shall also use the symbol $\vee$ to denote the logical "or".

THEOREM 2.2. *Let $f(x)$ be continuous and strictly monotonic on an interval I, let $H(t, v)$ be continuous on $J \times K$, where $J = [a, \beta]$ and K is an interval containing* 0, *with $H(v)$ monotonic, and let $W(t, s, u)$ be continuous and of one sign on $T_0 \times I$, where $T_0 = \{(t, s)\colon a \leqslant s \leqslant t \leqslant \beta\}$, with*

$W(t)$ and $W(u)$ both monotonic. Suppose also that the functions x and a are both continuous on J with

$$a(t)+H(t,v)\in f(I) \quad \text{for } t\in J,\ v\in K,\ |v|\leqslant b, \tag{3}$$

for some constant $b>0$. Let

$$f\big(x(t)\big)\leqslant a(t)+H\Big(t,\int_0^t W\big(t,s,x(s)\big)\,ds\Big),\qquad t\in J, \tag{4}$$

and let $r(t,T,a)$ be the maximal (minimal) solution of

$$\begin{aligned}&\frac{dr}{dt}=W\big(T,t,f^{-1}[a(t)+H(t,r)]\big),\qquad a\leqslant t\leqslant T\leqslant\beta_1\ (\leqslant\beta),\\ &r(a)=0,\end{aligned} \tag{5}$$

if $W(u)$, f have the same (opposite) parity, where $\beta_1>a$ is chosed so that the maximal (minimal) solution exists on $[a,\beta_1]$. Then, provided $W(t)$, $H(v)$ have the same parity,

$$x(t)\lesseqgtr f^{-1}\big[a(t)+H\big(t,\tilde r(t)\big)\big],\qquad a\leqslant t\leqslant\beta_1, \tag{4'}$$

where $\tilde r(t)=r(t,t,a)$ and $\leqslant$ holds if (i) *f is increasing and $W(u)$, $H(v)$ have the same parity, while $\geqslant$ holds if* (ii) *f is decreasing and $W(u)$, $H(v)$ have the opposite parity.*

Proof. The function $F(T,t,r)\equiv W\big(T,t,f^{-1}[a(t)+H(t,r)]\big)$ is continuous on the compact set $T_0\times[-b,b]$, so there is a constant $M>0$ such that $|F(T,t,r)|\leqslant M$ on this set. It follows from a standard existence theorem that, *independent of* $T\in[a,\beta]$, there exists $\beta_1>a$ (in fact, $\beta_1-a\geqslant\min(\beta-a,bM^{-1})$) such that *all* solutions of (5) exist on $[a,\beta_1]$.

Fix $T\in[a,\beta]$ and let $a\leqslant t\leqslant T$. Then

$$v(t,t)\equiv\int_a^t W\big(t,s,x(s)\big)\,ds\lesseqgtr\int_a^t W\big(T,s,x(s)\big)\,ds\equiv v(t,T), \tag{6}$$

where $\leqslant$ or $\geqslant$ holds according as $W(t)$ is increasing or decreasing. (Observe that (4) implies that $v(\tau,\tau)\in K$ for all $\tau\in J$; since K is an interval with $0\in K$, it follows (by setting $\tau=T$ or $\tau=t$) that $v(t,T)\in K$ in both cases of (6), whether $W\geqslant 0$ or $W\leqslant 0$.) By (4), we obtain

$$x(t)\lesseqgtr f^{-1}\big[a(t)+H\big(t,v(t,t)\big)\big], \tag{7}$$

where $\leqslant$ or $\geqslant$ holds according as f is increasing or decreasing. Hence, since $v'(t,T)=W\big(T,t,x(t)\big)$ for $a\leqslant t\leqslant T\leqslant\beta$,

$$v'(t,T)\lesseqgtr W\Big(T,t,f^{-1}\big[a(t)+H\big(t,v(t,t)\big)\big]\Big), \tag{8}$$

where $\leqslant$ or $\geqslant$ holds according as $W(u), f$ have the same or the opposite parity. Now by (6),

$$H\big(t, v(t, t)\big) \lesseqgtr H\big(t, v(t, T)\big), \quad a \leqslant t \leqslant T, \tag{6'}$$

where $\leqslant$ or $\geqslant$ holds according as $W(t)$, $H(v)$ have (a) the same parity, or (b) the opposite parity. Thus

$$f^{-1}\big[a(t) + H\big(t, v(t, t)\big)\big] \lesseqgtr f^{-1}\big[a(t) + H\big(t, v(t, T)\big)\big], \quad a \leqslant t \leqslant T,$$

where $\leqslant$ or $\geqslant$ holds according as (a'): $\big(f$ increasing, (a)$\big) \vee \big(f$ decreasing, (b)$\big)$, or (b'): $\big(f$ increasing, (b)$\big) \vee \big(f$ decreasing, (a)$\big)$. This in turn implies that

$$W\Big(T, t, f^{-1}\big[a(t) + H\big(t, v(t, t)\big)\big]\Big) \lesseqgtr W\Big(T, t, f^{-1}\big[a(t) + H\big(t, v(t, T)\big)\big]\Big), \tag{9}$$

where $\leqslant$ or $\geqslant$ holds according as (a''): $\big(W(u)$ increasing, (a')$\big) \vee \big(W(u)$ decreasing, (b')$\big)$, or as (b''): $\big(W(u)$ increasing, (b')$\big) \vee \big(W(u)$ decreasing, (a')$\big)$. Combining this with (8) we see that *provided* $W(t)$, $H(v)$ *have the same parity*, then

$$v'(t, T) \lesseqgtr W\Big(T, t, f^{-1}\big[a(t) + H\big(t, v(t, T)\big)\big]\Big), \quad a \leqslant t \leqslant T \leqslant \beta, \tag{10}$$

where $\leqslant$ or $\geqslant$ holds according as $W(u), f$ have the same or the opposite parity.

Since $v(a, T) = 0$ the preceding comparison theorem shows that *if* $W(t)$, $H(v)$ *have the same parity*, and if $r(t, T, a)$ is the maximal or the minimal solution of (5) as specified, then

$$v(t, T) \lesseqgtr r(t, T, a), \quad a \leqslant t \leqslant T \leqslant \beta_1.$$

Setting $t = T$ and changing notation we obtain

$$v(t, t) \lesseqgtr \tilde{r}(t), \quad a \leqslant t \leqslant \beta_1, \tag{11}$$

where $\leqslant$ or $\geqslant$ holds according as $W(u), f$ have (A) the same parity, or (B) the opposite parity. As in the analysis of (6), (6'), for $a \leqslant t \leqslant \beta_1$ we have

$$H\big(t, v(t, t)\big) \lesseqgtr H\big(t, \tilde{r}(t)\big),$$

where $\leqslant$ or $\geqslant$ holds according as (A'): $\big(H(v)$ increasing, (A)$\big) \vee \big(H(v)$ decreasing, (B)$\big)$, or as (B'): $\big(H(v)$ increasing, (B)$\big) \vee \big(H(v)$ decreasing, (A)$\big)$. Moreover,

$$f^{-1}\big[a(t) + H\big(t, v(t, t)\big)\big] \lesseqgtr f^{-1}\big[a(t) + H\big(t, \tilde{r}(t)\big)\big],$$

where $\leqslant$ or $\geqslant$ holds according as (A''): $\big(f$ increasing, (A')$\big) \vee \big(f$ decreasing, (B')$\big)$, or as (B''): $\big(f$ increasing, (B')$\big) \vee \big(f$ decreasing, (A')$\big)$. Analyzing the various cases we see that *if* $W(t)$, $H(v)$ *have the same parity*, then

$$f^{-1}\big[a(t) + H\big(t, v(t, t)\big)\big] \lesseqgtr f^{-1}\big[a(t) + H\big(t, \tilde{r}(t)\big)\big], \quad a \leqslant t \leqslant \beta_1, \tag{12}$$

where $\leqslant$ or $\geqslant$ holds according as $W(u)$, $H(v)$ have the same or the opposite parity. The conclusion (4′) now follows in cases (i), (ii) from (7) and (12).

In the same way one can prove

THEOREM 2.3. *Under the hypotheses of Theorem 2.2 suppose that*

$$f(x(t)) \geqslant a(t) + H\Big(t, \int_a^t W(t, s, x(s))\,ds\Big), \quad t \in J,$$

and that $W(t)$, $H(v)$ have the opposite parity. Now let $\tilde{r}(t) = r(t, t, a)$, where $r(t, T, a)$ is the maximal (minimal) solution of (5) *according as $W(u), f$ have the opposite (the same) parity. Then*

$$x(t) \gtreqless f^{-1}\big[a(t) + H(t, \tilde{r}(t))\big], \quad a \leqslant t \leqslant \beta_1,$$

where $\geqslant$ or $\leqslant$ holds according as case (i) *or* (ii) *holds in Theorem* 2.2.

Remark 1. Similar results can be obtained for inequalities of the form

$$f(x(t)) \lesseqgtr a(t) + H\Big(t, \int_t^\beta W(t, s, x(s))\,ds\Big), \quad t \in J. \tag{13}$$

We do not state the results but only note that with essentially the same hypotheses on the functions, the change of variable $t = -u$, $x_1(t) = x(-t)$ transforms (13) into

$$f(x_1(u)) \lesseqgtr a_1(u) + H_1\Big(u, \int_{-\beta}^{u} W_1(u, \sigma, x_1(\sigma))\,d\sigma\Big), \quad -\beta \leqslant u \leqslant -a,$$

with $H_1(t, v) = H(-t, v)$, $W_1(t, s, u) = W(-t, -s, u)$, to which the preceding theorems apply. Note that now, if $\leqslant$ holds in (13), then $W(t)$ and $H(v)$ must have opposite parity in order to apply Theorem 2.2.

Remark 2. In case $K = [0, t_0]$, then $W \geqslant 0$ necessarily holds (since $v(t, t) \in K$); in this case hypothesis (3) reduces to

$$a(t) + H(t, v) \in f(I), \quad t \in J,\ 0 \leqslant v \leqslant b,$$

for some constant $b \in (0, t_0]$. A similar remark applies if $K = [-t_0, 0]$. We also note that the assumption that W was of one sign was only used in the proof of the theorem at (6) to ensure $v(t, T) \in K$ for $a \leqslant t \leqslant T \leqslant \beta$. If this condition is added as a hypothesis, then W may change sign.

In order to see that Theorem 2.2 actually includes Lakshmikantham's result ([15], Theorem 3.1 (i)), we require the following elementary lemma. We shall also make extensive use of this lemma in later sections.

LEMMA 2.1. *Let the functions F, G be continuous and of one sign in a domain D containing the point (a, r_0), and suppose that the initial value* *problem*

$$\frac{dr_1}{dt} = G(t, r_1), \quad r_1(a) = r_0, \tag{14}$$

has a maximal (if $G \geqslant 0$) or a minimal (if $G \leqslant 0$) solution $r_1 = r_1(t)$ on $[a, \beta_1]$ such that

$$S_+ = \{(t, r)\colon a \leqslant t \leqslant \beta_1,\ r_0 \leqslant r \leqslant r_1(t)\} \subset D \quad \text{if } G \geqslant 0,$$
or
$$S_- = \{(t, r)\colon a \leqslant t \leqslant \beta_1,\ r_0 \geqslant r \geqslant r_1(t)\} \subset D \quad \text{if } G \leqslant 0.$$

If $0 \leqslant \pm F(t, r) \leqslant \pm G(t, r)$ for $(t, r) \in S_\pm$, then all solutions r of the initial value problem

$$\frac{dr}{dt} = F(t, r), \quad r(a) = r_0, \tag{15}$$

exist on $[a, \beta_1]$ and, moreover, $r_0 \leqslant r(t) \leqslant r_1(t)$ on $[a, \beta_1]$ if $G \geqslant 0$, or $r_0 \geqslant r(t) \geqslant r_1(t)$ on $[a, \beta_1]$ if $G \leqslant 0$.

The proof follows from the fact that if r is any (local) solution of (15) defined on some subinterval $[a, \beta_r]$ of $[a, \beta_1]$ then, by Theorem 2.1,

$$r_0 \leqslant r(t) \leqslant r_1(t) \quad \text{or} \quad r_0 \geqslant r(t) \geqslant r_1(t), \quad t \in [a, \beta_r],$$

in the maximal or minimal case respectively. That is, the graph of r lies in the set S_+ (or S_-), and hence r can be continued to all of $[a, \beta_1]$.

COROLLARY 2.1 (Lakshmikantham [15], Theorem 3.1 (i)). *Let x, a, b, c be continuous non-negative functions on $J = [a, \beta]$, and f, h be continuous non-negative functions on $\mathbf{R}^+ = [0, \infty)$ with f strictly increasing and h non-decreasing. In addition, suppose $k(t, s)$ is continuous and non-negative on $T_0 = \{(t, s)\colon a \leqslant s \leqslant t \leqslant \beta\}$, and that $w(s, u)$ is non-negative and continuous on $J \times \mathbf{R}^+$ with $w(u)$ non-decreasing. Define*

$$C(t) = \max_{a \leqslant s \leqslant t} c(s), \quad K(t, s) = \max_{s \leqslant \sigma \leqslant t} k(\sigma, s) \quad (a \leqslant s \leqslant t \leqslant \beta).$$

If

$$f\big(x(t)\big) \leqslant a(t) + b(t)h\Big(c(t) + \int_a^t k(t, s)w\big(s, x(s)\big)\,ds\Big), \quad t \in J, \tag{16}$$

then

$$x(t) \leqslant f^{-1}\Big[a(t) + b(t)h\Big(\tilde{r}_1\big(t, C(t)\big)\Big)\Big], \quad t \in J_1, \tag{16'}$$

with $\tilde{r}_1(t; r_{10}) = r_1(t, t, r_{10})$, where $r_1 = r_1(t, T, r_{10})$ is the maximal solution of

$$\frac{dr_1}{dt} = K(T, t)w\big(t, f^{-1}[a(t) + b(t)h(r_1)]\big), \quad r_1(a) = r_{10},$$

existing on some $[a, \beta_1] = J_1 \subset J_0$.

Here it is assumed that $J_1 \subset J_0$ is chosen so that the right-hand side of (16′) is defined. We shall actually prove a slightly better conclusion than (16′). In order to apply Theorem 2.2 we note that (16) implies

$$(17)\qquad f(x(t)) \leqslant a(t)+b(t)h\Big(c(t)+\int_a^t K(t,s)w\big(s,x(s)\big)\,ds\Big),\qquad t\in J.$$

The functions H, W defined by

$$H(t,v)\equiv b(t)h\big(c(t)+v\big),\qquad W(t,s,u)\equiv K(t,s)w(s,u)$$

are continuous on $J\times \boldsymbol{R}^+$, $T_0\times\boldsymbol{R}^+$ respectively, with $H(v)$, $W(t)$, and $W(u)$ non-decreasing and $W\geqslant 0$. Now restrict t to a subinterval $J' = [a,\beta']$ of J chosen so that

$$a(t)+H(t,v) = a(t)+b(t)h\big(c(t)+v\big)\in f(I),\qquad t\in J',\ 0\leqslant v\leqslant b,$$

for some constant $b>0$. (This is, of course, an unstated hypothesis of [15].) Let $r = r(t,T,a)$ be the maximal solution of the system

$$(18)\qquad \frac{dr}{dt} = K(T,t)w\Big(t,f^{-1}\big[a(t)+b(t)h\big(c(t)+r\big)\big]\Big),\qquad r(a) = 0,$$

to which (5) reduces. If we apply Theorem 2.2 (i) to (17), with J replaced by J', $K = I = \boldsymbol{R}^+$, we obtain

$$(17')\qquad x(t)\leqslant f^{-1}\big[a(t)+b(t)h\big(c(t)+\tilde r(t)\big)\big],\qquad a\leqslant t\leqslant\beta_2,$$

for some β_2 with $[a,\beta_2]\subset J'$. To see that this estimate is better than that in (16′), observe that $r_2(t,T)\equiv r_1\big(t,T,C(T)\big)-C(T)$ satisfies the system

$$\frac{dr_2}{dt} = K(T,t)w\Big(t,f^{-1}\big[a(t)+b(t)h\big(C(T)+r_2\big)\big]\Big),\qquad r_2(a) = 0$$

for $a\leqslant t\leqslant T\leqslant\beta_1$. Since $c(t)+r\leqslant C(T)+r$ for $a\leqslant t\leqslant T$, it follows that

$$K(T,t)w\Big(t,f^{-1}\big[a(t)+b(t)h\big(c(t)+r\big)\big]\Big)\leqslant K(T,t)w\Big(t,f^{-1}\big[a(t)+b(t)h\big(C(T)+r\big)\big]\Big)$$

for $a\leqslant t\leqslant T\leqslant\beta_1$ and all $r\geqslant 0$. By Lemma 2.1, it follows that $\beta_2\geqslant\beta_1$ and that $r(t,T,a)\leqslant r_2(t,T)$ for $a\leqslant t\leqslant T$. Hence, on setting $t = T$ and changing notation, $\tilde r(t)\leqslant\tilde r_1\big(t,C(t)\big)-C(t)$, from which it follows that

$$c(t)+\tilde r(t)\leqslant C(t)+\tilde r(t)\leqslant \tilde r_1\big(t,C(t)\big),$$

and

$$f^{-1}\big[a(t)+b(t)h\big(c(t)+\tilde r(t)\big)\big]\leqslant f^{-1}\Big[a(t)+b(t)h\Big(\tilde r_1\big(t,C(t)\big)\Big)\Big],$$

for $a\leqslant t\leqslant\beta_1$, proving the assertion.

Remark 3. There are alternative versions of this corollary which one could obtain using the alternative hypotheses of Theorems 2.2 and

2.3. Similarly, the other parts of Theorem 3.1 of [15] can be obtained in greater generality from Theorems 2.2, 2.3. We shall not do so, but instead shall obtain either better or more general versions of some of the results in [4], [12], [2], [25] contained in Theorem 3.1 of [15], and of those in [8], [9].

3. Bounds for solutions of a differential equation. In [15], V. Lakshmikantham obtained an upper bound for solutions of the initial value problem

$$\frac{dr}{dt} = k(t)g(r) + \sigma(t), \quad r(a) = r_0, \tag{1}$$

as a corollary of a variation of parameters formula of a somewhat more general equation. Unfortunately, the domain of validity for the bound is incorrect in [15], and indeed is not discussed adequately at all. We deal with this problem in

THEOREM 3.1. *Let σ and k be continuous and of one sign on an interval $J = [a, \beta]$, and let g be continuous, monotonic, and never zero on an interval I containing a point r_0. Suppose that either g is non-decreasing and $k \geqslant 0$ or g is non-increasing and $k \leqslant 0$. If r is any solution of* (1) *existing on a subinterval $[a, \beta_1) \subset J$, then*

$$r(t) \lesseqgtr G^{-1}\Big[\int_a^t k\,ds + G\Big(r_0 + \int_a^t \sigma\,ds\Big)\Big], \quad a \leqslant t < \bar{\beta}_1, \tag{1'}$$

where $\leqslant$ or $\geqslant$ holds according as $\sigma \geqslant 0$ or $\sigma \leqslant 0$, and

$$G(u) \equiv \int_{u_0}^{u} dy/g(y), \quad u \in I \ (u_0 \in I),$$

$$\bar{\beta}_1 = \min(\beta_1, \beta_2), \quad \beta_2 = \min(u_1, u_2)$$

with

$$u_1 = \sup\Big\{u \in J : r_0 + \int_a^u \sigma\,ds \in I\Big\},$$

$$u_2 = \sup\Big\{u \in J : \int_a^t k\,ds + G\Big(r_0 + \int_a^t \sigma\,ds\Big) \in G(I),\ a \leqslant t \leqslant u\Big\}.$$

Moreover, if $\sigma \geqslant 0$ and k, g have the same sign ($\sigma \leqslant 0$ and k, g have the opposite sign), then (1) *has a unique solution, $\beta_1 \geqslant \beta_2$, and we also have*

$$G^{-1}\Big(\int_a^t k\,ds + G(r_0)\Big) \leqslant r(t) \quad \Big(G^{-1}\Big(\int_a^t k\,ds + G(r_0)\Big) \geqslant r(t)\Big) \tag{2}$$

for $a \leqslant t < \beta_2$.

Proof. We first prove the results when $g > 0$. In this case G is increasing, and we shall show that if r is any solution of (1) on $[\alpha, \beta_1)$, then for $\alpha \leqslant t < \beta_1$ we have

$$\int_\alpha^t k\,ds + G(r_0) \lesseqgtr G\big(r(t)\big) \lesseqgtr \int_\alpha^t k\,ds + G\Big(r_0 + \int_\alpha^t \sigma\,ds\Big), \tag{3}$$

where $\leqslant$ or $\geqslant$ holds according as $\sigma \geqslant 0$ or $\sigma \leqslant 0$. Note that when $\sigma \geqslant 0$ and $kg \geqslant 0$, or $\sigma \leqslant 0$ and $kg \leqslant 0$, the left-hand inequalities of (3) can be written in the form (2). For, in all such cases, $G(r_0) + \int_\alpha^t k\,ds$ lies between $G(r_0)$ and $G(r_0 + \int_\alpha^t \sigma\,ds) + \int_\alpha^t k\,ds$ both of which points lie on the interval $G(I)$.

To prove (3) when $g > 0$, observe that by (1), for $\alpha \leqslant t < \beta_1$,

$$r(t) \lesseqgtr r_0 + \int_\alpha^t \sigma\,ds,$$

where $\leqslant$ or $\geqslant$ holds according as $k \leqslant 0$ or $k \geqslant 0$. Since g is non-increasing for $k \leqslant 0$ and non-decreasing for $k \geqslant 0$ it follows that in either case,

$$g\big(r(t)\big) \geqslant g\Big(r_0 + \int_\alpha^t |\sigma\,ds|\Big).$$

For any solution, $r(t)$, of (1) we also have

$$\frac{r'(s)}{g\big(r(s)\big)} = k(s) + \frac{\sigma(s)}{g\big(r(s)\big)},$$

so integrating over $[\alpha, t]$, we obtain

$$G\big(r(t)\big) \lesseqgtr \int_\alpha^t k\,ds + \int_\alpha^t \frac{\sigma(s)\,ds}{g\big(r_0 + \int_\alpha^s \sigma\,dx\big)},$$

where $\leqslant$ or $\geqslant$ holds according as $\sigma \geqslant 0$ or $\sigma \leqslant 0$. This reduces to the right-hand inequalities of (3). As for the other, it follows in the same way from

$$\frac{r'(s)}{g\big(r(s)\big)} \lesseqgtr k(s),$$

where $\geqslant$ or $\leqslant$ holds according as $\sigma \geqslant 0$ or $\sigma \leqslant 0$.

We note that as in [4], the bounds in (1′), (2), (3) are independent of the choice of $u_0 \in I$. For simplicity, we take $u_0 = r_0$ in the rest of the proof, so that $G(r_0) = 0$. From now on, we deal (primarily) with the case that either (a) $\sigma \geqslant 0$, $k \geqslant 0$, $g > 0$ and g is non-decreasing, or (b) $\sigma \leqslant 0$,

$k \leqslant 0$, $g < 0$ and g is non-increasing. Following Lakshmikantham [15], we consider the initial value problem

$$\frac{dv_0}{dt} = \frac{g(v_0)\sigma(t)}{g\{G^{-1}[G(v_0) + \int_a^t k\,ds]\}}, \quad v_0(a) = r_0. \tag{4}$$

It was shown formally in [15] that if v_0 is a solution of (4), then the function r defined by

$$r(t) = G^{-1}\Big[G\big(v_0(t)\big) + \int_a^t k\,ds\Big] \tag{5}$$

is a solution of (1).

We first prove that all solutions of (4) exist on $[a, \beta_2)$. (This is true even if $\sigma \geqslant 0$ and k, g have opposite sign, or $\sigma \leqslant 0$ and k, g have the same sign.) To obtain this result from Lemma 2.1, we set

$$F(t, r) = g(r)\sigma(t)/g\Big\{G^{-1}\Big[G(r) + \int_a^t k\,ds\Big]\Big\}$$

and use the comparison equation

$$\frac{dr_1}{dt} = \sigma(t), \quad r_1(a) = r_0, \tag{4'}$$

which has the unique solution $r_1(t) = r_0 + \int_a^t \sigma\,ds$ for all $t \geqslant a$. The asserted existence will follow provided F is continuous on a domain D such that

$$S_+ = \Big\{(t, r):\ a \leqslant t < \beta_2,\ r_0 \leqslant r \leqslant r_0 + \int_a^t \sigma\,ds\Big\} \subset D \quad \text{if } \sigma \geqslant 0,$$

or

$$S_- = \Big\{(t, r):\ a \leqslant t < \beta_2,\ r_0 \geqslant r \geqslant r_0 + \int_a^t \sigma\,ds\Big\} \subset D \quad \text{if } \sigma \leqslant 0,$$

provided that $0 \leqslant \pm F(t, r) \leqslant \pm\sigma(t)$ for $(t, r) \in S_\pm$. The continuity and inclusion relations are clearly satisfied for

$$D = \Big\{(t, r):\ G(r) + \int_a^t k\,ds \in G(I)\Big\}.$$

Moreover, for $(t, r) \in D$, recalling that $g > 0$, one easily verifies that

$$g(r)/g\Big\{G^{-1}\Big[G(r) + \int_a^t k\,ds\Big]\Big\} \leqslant 1$$

holds whether g is non-decreasing and $k \geqslant 0$, or g is non-increasing and $k \leqslant 0$. Hence $F(t, r) \leqslant \sigma(t)$ if $\sigma \geqslant 0$, or $F(t, r) \geqslant \sigma(t)$ if $\sigma \leqslant 0$, proving that all solutions of (4) exist on $[\alpha, \beta_2)$.

If v_0 is any solution of (4) then by Lemma 2.1,

$$r_0 \lesseqgtr v_0(t) \lesseqgtr r_0 + \int_a^t \sigma\, ds = r_1(t),$$

where $\leqslant$ or $\geqslant$ holds according as $\sigma \geqslant 0$ or $\sigma \leqslant 0$. Hence $r(t)$ is well-defined by (5) on $[\alpha, \beta_2)$ and, as shown in [15] (or by direct differentiation), r is a solution of (1).

In cases (a), (b) we note that (4) has a *unique* solution on $[\alpha, \beta_2)$. To see this, let v_m and v_M be the minimal and maximal solutions of (4) and let r_m and r_M be the related solutions of (1) defined by (5). Since G is increasing, it follows that $r_m(t) \leqslant r_M(t)$. Moreover, by (4),

$$\frac{v_m'(s)}{g(v_m(s))} = \frac{\sigma(s)}{g(r_m(s))}, \qquad \frac{v_M'(s)}{g(v_M(s))} = \frac{\sigma(s)}{g(r_M(s))}.$$

Hence

$$G(v_M(t)) = \int_a^t \frac{\sigma(s)}{g(r_m(s))}\, ds \geqslant \int_a^t \frac{\sigma(s)}{g(r_M(s))}\, ds = G(v_M(t))$$

follows in both cases (a), (b). Since G is increasing, $v_m(t) \geqslant v_M(t)$ and so $v_m(t) = v_M(t)$, proving uniqueness.

Let v_0 be the unique solution of (4) and let r be the corresponding solution of (1) defined by (5) on $[\alpha, \beta_2)$. To prove that r is the only solution of (1) on $[\alpha, \beta_2)$ in case (a) we show in this and the next two paragraphs that r is the maximal solution of (1). To this end, let $\bar{r}(t)$ denote the maximal solution of (1), with domain $[\alpha, \beta_1)$ say, and set

$$\bar{v}_0(t) \equiv G^{-1}\Big[G(\bar{r}(t)) - \int_a^t k\, ds\Big].$$

It follows from inequality (3), which holds with $r = \bar{r}$, that for $\alpha \leqslant t < \bar{\beta}_1$

$$0 = G(r_0) \leqslant G(\bar{r}(t)) - \int_a^t k\, ds \leqslant G\Big(r_0 + \int_a^t \sigma\, ds\Big).$$

Since $G(I)$ is an interval, $\bar{v}_0(t)$ is therefore defined at least for $\alpha \leqslant t < \bar{\beta}_1$. By direct differentiation, $\bar{v}_0$ is a solution of (4), whence $\bar{v}_0(t) = v_0(t)$ by uniqueness. But then

$$\bar{r}(t) = G^{-1}\Big[G(\bar{v}_0(t)) + \int_a^t k\, ds\Big] = G^{-1}\Big[G(v_0(t)) + \int_a^t k\, ds\Big] = r(t)$$

for $a \leqslant t < \bar{\beta}_1$, so that r coincides with the maximal solution of (1) on $[a, \bar{\beta}_1)$ at least.

Let τ be the supremum of values $u \in J$ such that r is the maximal solution of (1) on $[a, u]$. If $\tau \geqslant \beta_2$ we are through, so we assume $\bar{\beta}_1 \leqslant \tau < \beta_2$. The solution r is maximal on $[a, \tau]$ and if $r(\tau) = \bar{r}_0$, we may apply the preceding analysis to the initial value problems (1), (4) with the new initial conditions $r(\tau) = \bar{r}_0$, $v_0(\tau) = \bar{r}_0$, and to the new initial value problem (4′) with r_0 replaced by $\bar{r}_0$; in (4), G is also replaced by $\bar{G}$, where

$$\bar{G}(u) = \int_{\bar{r}_0}^{u} dy/g(y), \qquad u \in I.$$

We obtain a solution $\bar{r}$ of the new problem (1) which is maximal on some interval $[\tau, \tau+\varepsilon]$ and having

$$\bar{r}(t) \equiv \bar{G}^{-1}\Big[\bar{G}\big(\bar{v}_0(t)\big)+\int_{\tau}^{t} k\,ds\Big] = G^{-1}\Big[G\big(\bar{v}_0(t)\big) + \int_{\tau}^{t} k\,ds\Big]$$

for $t \in [\tau, \tau+\varepsilon]$, where $\bar{v}_0(t)$ is the (unique) solution of

$$\bar{v}_0' = g(\bar{v}_0)\sigma(t)/g\Big\{\bar{G}^{-1}\Big[\bar{G}(\bar{v}_0)+\int_{\tau}^{t} k\,ds\Big]\Big\}, \qquad \bar{v}_0(\tau) = \bar{r}_0.$$

We now show that $\bar{r}(t) \leqslant r(t)$ on $[\tau, \tau+\varepsilon]$ whence, since $\bar{r}$ is maximal, $r(t) = \bar{r}(t)$ on $[\tau, \tau+\varepsilon]$. But then r is maximal on $[a, \tau+\varepsilon]$, contradicting the definition of τ and proving that $\tau \geqslant \beta_2$ must hold in case (a).

To prove that $\bar{r}(t) \leqslant r(t)$, it suffices to prove that

$$G\big(\bar{v}_0(t)\big)+\int_{\tau}^{t} k\,ds \leqslant G\big(v_0(t)\big)+\int_{a}^{t} k\,ds. \tag{6}$$

By (4), (5) and (1), we have

$$\frac{v_0'(s)}{g\big(v_0(s)\big)} = \frac{\sigma(s)}{g\big(r(s)\big)} = -k(s)+\frac{r'(s)}{g\big(r(s)\big)},$$

and similarly,

$$\frac{\bar{v}_0'(s)}{g\big(\bar{v}_0(s)\big)} = \frac{\sigma(s)}{g\big(\bar{r}(s)\big)}.$$

Hence,

$$G\big(v_0(t)\big)-G\big(v_0(\tau)\big) = \int_{\tau}^{t} \frac{\sigma(s)}{g\big(r(s)\big)}\,ds = -\int_{\tau}^{t} k\,ds + G\big(r(t)\big)-G(\bar{r}_0),$$

$$G\big(\bar{v}_0(t)\big)-G(\bar{r}_0) = \int_{\tau}^{t} \frac{\sigma(s)}{g\big(\bar{r}(s)\big)}\,ds.$$

Since $r(s) \leqslant \bar{r}(s)$ certainly holds and g is non-decreasing, $\sigma \geqslant 0$ in case (a), it follows that

$$G\big(\bar{v}_0(t)\big) - G(\bar{r}_0) \leqslant G\big(v_0(t)\big) - G\big(v_0(\tau)\big) = -\int_\tau^t k\,ds + G\big(r(t)\big) - G(\bar{r}_0),$$

or

$$G\big(\bar{v}_0(t)\big) + \int_\tau^t k\,ds \leqslant G\big(r(t)\big).$$

By (5), this reduces to (6).

In the same way, one shows that r is also the minimal solution of (1) in case (a), and hence that (1) has a unique solution in this case. Similarly, in case (b) the solution r of (1) defined by (5) on $[\alpha, \beta_2)$ is both maximal and minimal, hence the unique solution of (1).

Finally, if $g < 0$ we replace k, g in (1) by $k_1 = -k$, $g_1 = -g$, and note that the hypotheses of the theorem apply to k_1, g_1 and σ with $g_1 > 0$. All of the existence and domain conclusions now follow. Moreover, so do inequalities (3), with k replaced by k_1 and G by G_1, where

$$G_1(u) = \int_{u_0}^u dy/g_1(y) = -G(u), \qquad G_1^{-1}(v) = G^{-1}(-v).$$

These inequalities reduce to (3) as stated, completing the proof of the theorem.

Remark. Under certain circumstances, a better (but more complicated) bound than (1′) may be obtained. To illustrate, we consider only the case that σ, k, g are all non-negative and g is non-decreasing. In this case, (1) has a unique solution $r = r(t)$ on $[\alpha, \beta_2)$ and from (1),

$$\frac{r'(s)}{g\big(r(s)\big)} \geqslant k(s) \Rightarrow G\big(r(t)\big) \geqslant G(r_0) + \int_\alpha^t k\,ds,$$

so

$$r(t) \geqslant G^{-1}\Big[G(r_0) + \int_\alpha^t k\,ds\Big] \qquad \text{for } t \in [\alpha, \beta_2).$$

Then

$$\frac{r'(s)}{g\big(r(s)\big)} = k(s) + \frac{\sigma(s)}{g\big(r(s)\big)} \leqslant k(s) + \sigma(s)/g\Big\{G^{-1}\Big[G(r_0) + \int_\alpha^s k\,dn\Big]\Big\},$$

and integrating over $[\alpha, t]$ again,

$$G\big(r(t)\big) \leqslant G(r_0) + \int_\alpha^t k\,ds + \int_\alpha^t \Big(\sigma(s)/g\Big\{G^{-1}\Big[G(r_0) + \int_\alpha^s k\,dx\Big]\Big\}\Big)\,ds. \tag{7}$$

It is easy to verify that this is a better bound than (1′) if $\sigma(t) \leqslant k(t)g(r_0 + \int_a^t \sigma ds)$ on $[a, \beta_2)$. In the linear case, $g(r) \equiv r$, if $r_0 > 0$ the estimates (1′), (7) reduce to

$$r(t) \leqslant \Big(r_0 + \int_a^t \sigma\, ds\Big)\exp\Big(\int_a^t k\, ds\Big) \equiv r_1(t),$$

$$r(t) \leqslant r_0 \exp\Big(\int_a^t k\, ds + r_0^{-1}\int_a^t \sigma(s)\exp\Big(-\int_a^s k\, dx\Big)ds\Big) \equiv r_2(t),$$

and the exact solution of (1) is

$$r(t) = \Big\{r_0 + \int_a^t \sigma(s)\exp\Big(-\int_a^s k\, dx\Big)\Big\}\exp\Big(\int_a^t k\, dx\Big).$$

In this case, $r_2(t) \leqslant r_1(t)$ if $\sigma(t) \leqslant r_0 k(t)\exp\big(\int_a^t k\, ds\big)$ for all t.

4. Direct applications of the main theorem. In this section we shal use Theorem 2.2 (together with Lemma 2.1 o r Theorem 3.1 in some cases to obtain extensions, generalizations, or improvements of some known results.

THEOREM 4.1 (cf. Bihari [4]). *Let g be a continuous monotonic function which is never zero on an interval I containing a point u_0. Let x, k be continuous functions on an interval $J = [a, \beta]$ such that $x(J) \subset I$ and k does not change sign on I. Let $a \in I$ and*

$$x(t) \leqslant a + \int_a^t k(s) g\big(x(s)\big) ds, \qquad t \in J. \tag{1}$$

If either g is non-decreasing and $k \geqslant 0$, or g is non-increasing and $k \leqslant 0$, then

$$x(t) \leqslant G^{-1}[G(a) + \int_a k\, ds], \qquad a \leqslant t < \beta_1, \tag{1′}$$

where $G(u) \equiv \int_{u_0}^u dy/g(y)$, $u \in I$, and

$$\beta_1 = \sup\{u \in J\colon\ G(a) + \int_a^t k(s) ds \in G(I),\ a \leqslant t \leqslant u\}.$$

Moreover, the result remains valid if $\leqslant$ is replaced by $\geqslant$ in both (1) *and* (1′). *Finally, both results are valid if $\int_a^t$ is replaced throughout by $\int_t^\beta$, and* $[a, \beta_1)$

is replaced by $(\alpha_1, \beta]$, *where*

$$\alpha_1 = \inf\left\{v \in J:\ G(a)+\int_t^\beta k(s)\,ds \in G(I),\ v \leqslant t \leqslant \beta\right\}.$$

As for the proof, it suffices to take

$$f(x) \equiv x, \quad H(t, v) \equiv v, \quad W(t, s, u) \equiv k(s)g(u), \quad K = \boldsymbol{R},$$

in Theorem 2.2 or 2.3 (or Remark 1, Section 2). The comparison equation (5), Section 2, in this case is

$$\frac{dr}{dt} = k(t)g(a+r), \quad r(a) = 0,$$

and has the (unique) solution

$$\tilde{r}(t) = G^{-1}\left[G(a)+\int_a^t k\,ds\right], \quad a \leqslant t < \beta_1.$$

The conclusion (1′) now follows from (4′), Section 2, since case (i) holds there.

THEOREM 4.2 (cf. Gollwitzer, [12], Theorem 2). *Let the functions* x, k *be continuous, with* k *non-negative, on an interval* $J = [\alpha, \beta]$, *and let* g *be a function which does not change sign and has a continuous derivative which is never zero on an interval* I *such that* $x(J) \subset I$. *Suppose hat* a *is a non-zero constant and* g *satisfies either* (i) g *is concave and* g, t *have the same sign, or* (ii) g *is convex and* g, a *have the opposite sign. If* $a+g^{-1}(0) \in I$ *and*

$$x(t) \leqslant a+g^{-1}\left(\int_a^t k(s)g\big(x(s)\big)\,ds\right), \quad t \in J, \tag{2}$$

then

$$x(t) \leqslant a+g^{-1}\left[G^{-1}\left(\int_a^t k\,ds\right)\right], \quad a \leqslant t < \beta_1, \tag{2′}$$

where $G(u) = \int_0^u dy/g[a+g^{-1}(y)]$, $u \in g(I) = K$, *and*

$$\beta_1 = \sup\left\{t \in J:\ \int_a^t k\,ds \in G(K)\right\}.$$

If (i) *or* (ii) *is replaced by* (i′) g *is concave and* g, a *have the opposite sign, or* (ii′) g *is convex and* g, a *have the same sign, then the result remains valid provided* $\leqslant$ *is replaced by* $\geqslant$ *in both* (2) *and* (2′). *Finally both inequalities remain valid if* $\int_a^t$ *is replaced by* $\int_t^\beta$ *throughout and* $[a, \beta_1)$ *is replaced by* $(\alpha_1, \beta]$, *where*

$$\alpha_1 = \inf\Big\{t \in J\colon \int_t^\beta k\,ds \in G(K)\Big\}.$$

To prove this, we take

$$f(x) \equiv x, \quad H(t, v) \equiv g^{-1}(v), \quad W(t, s, u) \equiv k(s)g(u), \quad K = g(I),$$

in Section 2. The comparison system is

$$\frac{dr}{dt} = k(t)g[a+g^{-1}(r)], \quad r(\alpha) = 0, \tag{3}$$

and by Theorem 2.2, (2) implies

$$x(t) \leqslant a+g^{-1}\big(r(t)\big), \quad \alpha \leqslant t < \beta_1,$$

where $r(t)$ is the (unique) solution of (3) on $[\alpha, \beta_1)$. This reduces to (2′). The other parts follow in the same way from Theorem 2.3 or Remark 1, Section 2.

We observe that the hypotheses and (2) imply the existence of a unique $u_0 \in I$ such that $g(u_0) = 0$. One can show that if g satisfies hypothesis (i) above, then either

(ia) g' is non-increasing, $g \leqslant 0$, $a < 0$, I has u_0 as a right end-point, and $g' > 0$ on I, so g is increasing, or

(ib) g' is non-increasing, $g \geqslant 0$, $a > 0$, I has u_0 as a left end-point, and $g' > 0$ on I, so g is increasing.

Similarly, if g satisfies (ii), then either

(iia) g' is non-decreasing, $g \leqslant 0$, $a > 0$, I has u_0 as a left end-point, and $g' < 0$ on I, so g is decreasing, or

(iib) g' is non-decreasing, $g \geqslant 0$, $a < 0$, I has u_0 as a right end-point, and $g' < 0$ on I, so g is decreasing.

In [12], Theorem 2, Gollwitzer considered case (ib) with $I = [0, \infty)$, $g(0) = 0$, $g(\boldsymbol{R}^+) = \boldsymbol{R}^+$ and, instead of (2′), obtained

$$x(t) \leqslant g^{-1}\Big[g(a)\exp\Big(\int_\alpha^t k\,ds\Big)\Big], \quad \alpha \leqslant t \leqslant \beta. \tag{4}$$

In this case, $\beta_1 = \beta$ is clear, and we show that (2′) is better than (4). To see this, make the change of variable $y = g[a+g^{-1}(r)]$ in the right-hand integral of the equation

$$\int_\alpha^t k\,ds = G\big(r(t)\big) = \int_0^{r(t)} dr/g[a+g^{-1}(r)]$$

to which (3) is equivalent. This gives

$$\int_a^t k\,ds = G(r(t)) = \int_{g(a)}^{g[a+g^{-1}(r(t))]} \{g'[g^{-1}(r)]/g'[a+g^{-1}(r)]\}\frac{dy}{y}.$$

Since (ib) holds,

$$\frac{g'[g^{-1}(r)]}{g'[a+g^{-1}(r)]} \geqslant 1, \quad \text{and} \quad g[a+g^{-1}(r(t))] > g(a) > 0,$$

so

$$\int_a^t k\,ds = G(r(t)) \leqslant \log\{g[a+g^{-1}(r(t))]/g(a)\},$$

$$g[a+g^{-1}(r(t))] \leqslant g(a)\exp\Big(\int_a^t k\,ds\Big),$$

$$a+g^{-1}\Big[G^{-1}\Big(\int_a^t k\,ds\Big)\Big] = a+g^{-1}(r(t)) \leqslant g^{-1}\Big[g(a)\exp\Big(\int_a^t k\,ds\Big)\Big],$$

proving that (2′) is better than (4).

As the simple analysis leading from (3) to (2′) suggests, we may obtain a similar result for a broader class of inequalities. We merely state the result as

THEOREM 4.3. *Let x, k be continuous functions with k non-negative, on an interval $J = [\alpha, \beta]$. Let g be continuous and monotonic on an interval I such that $x(J) \subset I$ and g is non-zero on I except perhaps at an end point of I. Let h be continuous and monotonic on an interval K such that $0 \in K$, and let a be a constant such that $a+h(0) \in I$. If g and h are both non-increasing or both non-decreasing, and*

$$x(t) \leqslant a+h\Big(\int_a^t k(s)g(x(s))\,ds\Big), \qquad t \in J,$$

then

$$x(t) \leqslant a+h\Big[G^{-1}\Big(\int_a^t k\,ds\Big)\Big], \qquad \alpha \leqslant t < \beta_1,$$

where $G(u) \equiv \int_0^u dr/g[a+h(r)]$, $u \in K$, *and*

$$\beta_1 = \sup\Big\{t \in J\colon \int_a^t k\,ds \in G(K)\Big\}.$$

THEOREM 4.4 (cf. Beesack [2], Theorem 2). *Let x, a, b, k be continuous functions such that $b \geqslant 0$ and k does not change sign on an interval $J = [\alpha, \beta]$.*

Let g be continuous, positive, non-decreasing, subadditive and submultiplicative on an interval I such that $x(J) \subset I$, $a(J) \subset I$ and $b(J) \subset I$. Suppose also that the function h is continuous and monotonic on an interval K such that $0 \in K$ and $h(K) \subset I$. If either (i) *h is non-decreasing and $k \geqslant 0$, or* (ii) *h is non-increasing and $k \leqslant 0$, and*

$$x(t) \leqslant a(t) + b(t)h\Big(\int_a^t k(s)g\big(x(s)\big)ds\Big), \qquad t \in J, \tag{5}$$

then

$$x(t) \leqslant a(t) + b(t)h\Big\{G^{-1}\Big[\int_a^t k(s)g\big(b(s)\big)ds + G\Big(\int_a^t k(s)g\big(a(s)\big)ds\Big)\Big]\Big\} \tag{5'}$$

for $a \leqslant t < \beta_1$, where $G(u) \equiv \int_{u_0}^{u} dy/g\big(h(y)\big)$ for $u \in K$ ($u_0 \in K$), and $\beta_1 = \min(u_1, u_2, u_3, u_4)$ with

$$u_1 = \sup\Big\{u \in J:\ \int_a^u k(s)g\big(a(s)\big)ds \in K\Big\},$$

$$u_2 = \sup\Big\{u \in J:\ G\Big(\int_a^u k(s)g\big(a(s)\big)ds\Big) + \int_a^u k(s)g\big(b(s)\big)ds \in G(K)\Big\},$$

$$u_3 = \sup\big\{u \in J:\ a(t) + b(t)h\big(R(t)\big) \in I,\ a \leqslant t \leqslant u\big\}$$

$$R(t) \equiv G^{-1}\Big[\int_a^t k(s)g\big(b(s)\big)ds + G\Big(\int_a^t k(s)g\big(a(s)\big)ds\Big)\Big],$$

$$u_4 = \sup\big\{u \in J:\ b(t)h\big(R(t)\big) \in I,\ a \leqslant t \leqslant u\big\}.$$

The result is valid if $\leqslant$ is replaced by $\geqslant$ in (5) *and* (5′) *provided g is now non-increasing on I, and conditions* (i), (ii) *are replaced by* (i′) *h is non-increasing and $k \geqslant 0$, or* (ii′) *h is non-decreasing and $k \leqslant 0$. Finally, both of these results remain valid if $\int_a^t$ and $[a, \beta_1)$ are replaced throughout by $\int_t^\beta$ and $(\bar{a}_1, \beta]$, where $\bar{a}_1 = \max(v_1, v_2, v_3, v_4)$ with the v_i defined in the obvious way.*

This follows from Theorems 2.2, 2.3 and Remark 1, Section 2, by taking

$$f(x) \equiv x, \qquad H(t, v) \equiv b(t)h(v), \qquad W(t, s, u) \equiv k(s)g(u).$$

The comparison equation is

$$\frac{dr}{dt} = k(t)g[a(t) + b(t)h(r)], \qquad r(a) = 0, \tag{6}$$

and (5) implies that

$$(7) \qquad x(t) \leqslant a(t)+b(t)h\big(r(t)\big), \quad a \leqslant t < \beta_1,$$

where $r(t)$ is the maximal or minimal solution of (6) on $[a, \beta_1)$ according as case (i) or (ii) of Theorem 4.4 holds. Since g is subadditive and submultiplicative on I,

$$(8) \qquad k(t)g[a(t)+b(t)h(r)] \lesseqgtr k(t)g\big(a(t)\big)+k(t)g\big(b(t)\big)g\big(h(r)\big),$$

where $\leqslant$ or $\geqslant$ holds according as (i) or (ii) holds. We now apply Theorem 3.1 to the comparison system

$$(6') \qquad \frac{dr_1}{dt} = k(t)g\big(b(t)\big)g\big(h(r_1)\big)+k(t)g\big(a(t)\big), \quad r_1(a) = 0,$$

to obtain, in case (i) or (ii) respectively,

$$(9) \qquad r_1(t) \leqslant R(t), \quad a \leqslant t < \beta_2 = \min(u_1, u_2).$$

By Lemma 2.1, it follows that $0 \leqslant r(t) \leqslant r_1(t)$ or $0 \geqslant r(t) \geqslant r_1(t)$ in case (i) or (ii) respectively, for $t \in J_1 = [a, \beta_1)$ with $\beta_1 = \min(u_3, u_4)$. (For, $t \in J_1$ implies (8) is valid for $a \leqslant t < \beta_1$, $0 \leqslant r \leqslant r_1(t)$, or $a \leqslant t < \beta_1$, $0 \geqslant r \geqslant r_1(t)$ respectively.) Since h is non-decreasing in case (i) and non-increasing in case (ii), it follows from this and (7), (9) that in both cases we obtain (5′).

THEOREM 4.5 (cf. Stachurska [25]). *Let the functions x, a, b, k be continuous and non-negative on $J = [a, \beta]$, let $p \neq 1$ be a positive constant, suppose that a/b is non-decreasing and*

$$(10) \qquad x(t) \leqslant a(t)+b(t)\int_a^t k(s)x^p(s)\,ds, \quad a \leqslant t \leqslant \beta.$$

Then

$$(10') \qquad x(t) \leqslant a(t)/\Big\{1-(p-1)\,[a(t)/b(t)]^{p-1}\int_a^t kb^p\,ds\Big\}^{1/(p-1)}, \quad a \leqslant t < \beta_1,$$

where $\beta_1 = \beta$ if $0 < p < 1$, and

$$\beta_1 = \sup\Big\{t \in J\colon\ (p-1)\,[a(t)/b(t)]^{p-1}\int_a^t kb^p\,ds < 1\Big\}$$

if $p > 1$.

This follows from Theorem 2.2 with

$$f(x) \equiv x, \quad H(t, v) \equiv b(t)v, \quad W(t, s, u) \equiv k(s)u^p, \quad I = K = \mathbf{R}^+.$$

The comparison equation is

$$(11) \qquad \frac{dr}{dt} = k(t)\,[a(t)+b(t)r]^p, \quad r(a) = 0.$$

To obtain an upper bound for solutions of (11), we write

$$\frac{dr}{dt} \leqslant k(t)b^p(t)\left[\frac{a(T)}{b(T)}+r\right]^p, \quad a \leqslant t \leqslant T \leqslant \beta,\ r \geqslant 0.$$

By direct integration and Lemma 2.1, the solution of this elementary differential inequality has

$$a(T)+b(T)r(t) \leqslant a(T)\left\{1-(p-1)\,[a(T)/b(T)]^{p-1}\int_a^t kb^p\,ds\right\}^{-1/(p-1)}.$$

Setting $t = T$ and changing notation, the conclusion (10′) now follows from Theorem 2.2.

In the case $p \geqslant 2$ is an integer, Stachurska obtained

$$(10'') \qquad x(t) \leqslant a(t)\Big/\left\{1-(p-1)\int_a^t kba^{p-1}\,ds\right\}^{1/(p-1)}, \quad a \leqslant t < \bar{\beta}_1,$$

where $\bar{\beta}_1$ is the least value of t such that $\{\ldots\} = 0$. It is easy to see that a/b non-decreasing implies that (10″) is better than (10′).

5. Some results of Deo and Dhongade. By using Theorem 2.2 twice we may also obtain (generalizations of) results of S.G. Deo and U.D. Dhongade [8], [9]. In these papers a class $\mathscr{F}$ of functions g was defined which, among other things, satisfied the condition

$$(1) \qquad \frac{1}{v}\,g(u) \leqslant g\left(\frac{u}{v}\right), \quad u \geqslant 0,\ v > 0.$$

The authors did not notice that (1) actually implies that $g(u) \equiv g(1)u$ for $u > 0$. For, setting $v = u$ in (1) gives $g(u) \leqslant g(1)u$ for $u > 0$. On the other hand, setting $u = 1$ implies $g(1) \leqslant vg(1/v)$ for $v > 0$, or $g(1) \leqslant u^{-1}g(u)$ for $u > 0$.

To avoid trivialities, we may replace condition (1) by the condition

$$(2) \qquad \frac{1}{v}\,g(u) \leqslant g\left(\frac{u}{v}\right) \quad \text{for } u > 0,\ v \geqslant 1,$$

and observe that (2) implies

$$(3) \qquad g(u) \leqslant g(1)u, \quad u \geqslant 1; \quad g(ru) \leqslant rg(u), \quad u > 0,\ r \geqslant 1.$$

The second of (3) shows that (2) is compatible with the subadditivity of g.

We shall require the following two lemmas.

LEMMA 5.1. *Let f be continuous and strictly increasing on $[u_0, \infty)$, where $u_0 \geqslant 0$. If $f(u)/u$ is non-decreasing for $u > u_0$, then*

$$(4) \qquad a^{-1}f^{-1}(x) \leqslant f^{-1}(xa^{-1}) \quad \textit{for } af(u_0) < x,\ a \geqslant 1.$$

For, setting $u = f^{-1}(x)$, $v = f^{-1}(xa^{-1})$, we have $u_0 < v \leqslant u$, so $v^{-1}f(v) \leqslant u^{-1}f(u)$ which reduces to (4).

LEMMA 5.2. *Let g, h be continuous, with h non-decreasing on $[0, \infty)$ and $0 < h(u) \leqslant u$, $0 < g(h(u)) \leqslant u$ for $u > 0$. If*

$$(5)\qquad G(u) \equiv \int_{u_0}^{u} dy/g(h(y)), \quad u > 0 \ (u_0 > 0),$$

then

$$(5')\qquad h\{G^{-1}(x+G(a))\} \leqslant ae^x \quad \text{for } x \geqslant 0,\ a > 0.$$

If $h(u) \geqslant u$ and $g(h(u)) \geqslant u$ for $u > 0$, then (5) *holds with $\leqslant$ replaced by $\geqslant$.*

For G^{-1}, with G, is strictly increasing so that $x+G(a) \geqslant G(a)$ implies $G^{-1}(x+G(a)) \geqslant a > 0$, and hence

$$h\{G^{-1}(x+G(a))\} \leqslant G^{-1}(x+G(a)), \quad x \geqslant 0,\ a > 0.$$

Hence it suffices to prove that $G^{-1}(x+G(a)) \leqslant ae^x$, or that

$$\varphi(x) \equiv G(ae^x) - x - G(a) \geqslant 0 \quad \text{for } x \geqslant 0,\ a > 0.$$

This follows at once from the fact that $\varphi(0) = 0$ while

$$\varphi'(x) = \{ae^x/g(h(ae^x))\} - 1 \geqslant 0 \quad \text{for all } x.$$

Under the alternative hypotheses all inequalities are reversed.

COROLLARY 5.1. *Let f be continuous and strictly increasing on $[0, \infty)$ with $f(u) \geqslant u$ for $u > 0$, $f(0) = 0$, and let g_1 be continuous on $[0, \infty)$ with $0 < g_1(u) \leqslant f(u)$ for $u > 0$. If*

$$(6)\qquad F_1(u) \equiv \int_{u_0}^{u} dy/g_1(f^{-1}(y)), \quad u > 0 \ (u_0 > 0),$$

then

$$(6')\qquad (F_1 \circ f)^{-1}[x+F_1(a)] \leqslant ae^x \quad \text{for } x \geqslant 0,\ a > 0.$$

This follows from the lemma by taking $h = f^{-1}$, $g = g_1$.

The following theorem includes Theorem 1 (for $f(x) \equiv x$, $h(u) \equiv u$, $g_1(u) \equiv u$) and Theorem 2 (for $f(x) \equiv x$, $g_1(u) \equiv u$) of Deo and Dhongade [8].

THEOREM 5.1. *Let x, a, k, k_1 be non-negative continuous functions on $J = [\alpha, \beta]$, and let a be non-decreasing on J. Let g, h be continuous non-decreasing functions on $[0, \infty)$ such that g is positive, subadditive and submultiplicative on $(0, \infty)$ and $h(u) > 0$ for $u > 0$. Let f, g_1 be continuous on $[0, \infty)$ with f strictly increasing, $f(u) \geqslant u$ for $u > 0$, $f(0) = 0$, $0 < g_1(u) \leqslant f(u)$ for $u > 0$. If*

$$(7)\qquad f(x(t)) \leqslant a(t) + h\Big(\int_a^t k(s)g(x(s))\,ds\Big) + \int_a^t k_1(s)g_1(x(s))\,ds, \quad t \in J,$$

then

$$(7') \qquad x(t) \leqslant (F_1 \circ f)^{-1}\Big(\int_a^t k_1 ds + F_1\Big\{a(t) + h \circ G^{-1}\Big[\int_a^t k(s) g\big(E(s)\big) ds + G\Big(\int_a^t k(s) g\big(a(s)E(s)\big) ds\Big)\Big]\Big\}\Big),$$

for $a \leqslant t < \beta_1$, *where* F_1 *is defined by* (6), $E(t) \equiv \exp\big(\int_a^t k_1 ds\big)$, G *is defined by* (5), *and*

$$\beta_1 = \sup\Big\{t \in J:\ G\Big(\int_a^t k(s) g\big(a(s)E(s)\big) ds\Big) + \int_a^t k(s) g\big(E(s)\big) ds \in G(\boldsymbol{R}^+)\Big\}.$$

If $a(t) \equiv a > 0$, *then* g *need not be subadditive, and in this case*

$$(7'') \qquad x(t) \leqslant (F_1 \circ f)^{-1}\Big\{\int_a^t k_1 ds + F_1\Big[a + h \circ G_a^{-1}\Big(\int_a^t k(s) g\big(E(s)\big) ds\Big)\Big]\Big\}, \qquad a \leqslant t < \beta_2,$$

where $G_a(u) \equiv \int_0^u dy/g[a + h(y)]$, $u \geqslant 0$, *and*

$$\beta_2 = \sup\Big\{t \in J: \int_a^t k(s) g\big(E(s)\big) ds \in G_a(\boldsymbol{R}^+)\Big\}.$$

Proof. Denote the first two terms on the right-hand side of (7) by $a_1(t)$ and apply Theorem 2.2 with

$$H(t, v) \equiv v, \quad W(t, s, u) \equiv k_1(s) g_1(u), \quad I = K = [0, \infty),$$

noting that $f(I) = I$. The comparison equation (5), Section 2, is

$$(8) \qquad \frac{dr}{dt} = k_1(t) g_1 \circ f^{-1}[a_1(t) + r], \qquad r(a) = 0,$$

and if $r = r(t)$ is the maximal solution of this equation, then for t in the domain of r,

$$(9) \qquad x(t) \leqslant f^{-1}[a_1(t) + r(t)].$$

Since a_1 is clearly non-decreasing,

$$r'(t) \leqslant k_1(t) g_1 \circ f^{-1}[a_1(T) + r(t)], \qquad a \leqslant t \leqslant T.$$

Now, $F_1(u) \to \infty$ as $u \to \infty$ since $g_1\big(f^{-1}(y)\big) \leqslant y$, hence solving this inequality it follows from Lemma 2.1 that

$$F_1\big(a_1(T) + r(t)\big) \leqslant F_1\big(a_1(T)\big) + \int_a^t k_1\, ds,$$

$$a_1(T) + r(t) \leqslant F_1^{-1}\Big\{F_1\big(a_1(T)\big) + \int_a^t k_1\, ds\Big\}, \qquad a \leqslant t \leqslant T \leqslant \beta.$$

Setting $t = T$ and changing notation, (9) gives

$$x(t) \leqslant (F_1 \circ f)^{-1} \left\{ \int_a^t k_1\, ds + F_1\big(a_1(t)\big) \right\}, \quad a \leqslant t \leqslant \beta,$$

or

$$(F_1 \circ f)\big(x(t)\big) \leqslant \int_a^t k_1\, ds + F_1\Big[a(t) + h\Big(\int_a^t k(s) g\big(x(s)\big)\, ds\Big)\Big], \quad t \in J. \tag{10}$$

Again we apply Theorem 2.2 with f replaced by $F_1 \circ f$,

$$H(t, v) \equiv F_1\big(a(t) + h(v)\big), \quad W(t, s, u) \equiv k(s) g(u), \quad I = K = [0, \infty),$$

noting that $\int_a^t k_1\, ds + F_1\big(a(t) + h(v)\big) \in F_1\big(f(I)\big) = F_1(R^+)$ for $t \in J$, $v \in K$. The comparison equation is now

$$\frac{dr}{dt} = k(t) g \circ (F_1 \circ f)^{-1} \Big[\int_a^t k_1\, ds + F_1\big(a(t) + h(r)\big)\Big], \quad r(a) = 0, \tag{8$'$}$$

and for t in the domain of r, we have

$$(F_1 \circ f)\big(x(t)\big) \leqslant \int_a^t k_1\, ds + F_1\big[a(t) + h\big(r(t)\big)\big], \tag{9$'$}$$

where r is the maximal solution of (8′). By Corollary 5.1, for arbitrary $\varepsilon > 0$ we have

$$(F_1 \circ f)^{-1} \Big[\int_a^t k_1\, ds + F_1\big(a(t) + \varepsilon + h(r)\big)\Big] \leqslant E(t)\, [a(t) + \varepsilon + h(r)], \quad t \in J.$$

By continuity this also holds for $\varepsilon = 0$. Since g is subadditive and submultiplicative, it follows from (8′) that

$$\frac{dr}{dt} \leqslant k(t) g\big(a(t) E(t)\big) + k(t) g\big(E(t)\big) g\big(h(r)\big).$$

By Theorem 3.1, the initial value problem

$$\frac{dr_1}{dt} = k(t) g\big(E(t)\big) g\big(h(r_1)\big) + k(t) g\big(a(t) E(t)\big), \quad r_1(a) = 0,$$

has a unique solution defined on $[a, \beta_1)$ and satisfying

$$r_1(t) \leqslant G^{-1} \Big\{ \int_a^t k(s) g\big(E(s)\big)\, ds + G\Big(\int_a^t k(s) g\big(a(s) E(s)\big)\, ds\Big)\Big\}. \tag{11}$$

By Lemma 2.1, $r(t) \leqslant r_1(t)$ for $t \in [a, \beta_1)$ whence (7′) follows from (9′) and (11).

In case $a(t) \equiv a > 0$ the proof is unchanged down to (10); also (8′) and (9′) remain unchanged with $a(t) \equiv a$. The auxiliary comparison equation is now replaced by

$$\frac{dr_1}{dt} = k(t)g\big(E(t)\big)g[a+h(r_1)], \qquad r_1(a) = 0.$$

This equation has the unique solution

$$r_1(t) = G_a^{-1}\Big(\int_a^t k(s)g\big(E(s)\big)ds\Big), \qquad a \leqslant t < \beta_2,$$

and the conclusion (7″) again follows from (9′) and Lemma 2.1.

Theorems 4 (for $f(x) \equiv x$, $h(u) \equiv u$, $a(t) \equiv a$) and 5 (for $f(x) \equiv x$) of Deo and Dhongade [8] are included in the next result (modified to account for the remark concerning (1), (2)).

THEOREM 5.2. *Let x, a, k, k_1 be non-negative continuous functions on $J = [a, \beta]$ such that a is non-decreasing with $a(t) \geqslant 1$. Let g, g_1, h be continuous non-decreasing functions on $[0, \infty)$ such that g is positive, subadditive and submultiplicative on $(0, \infty)$, g_1 is positive on $(0, \infty)$ and satisfies condition (2), and $h(u) > 0$ for $u > 0$. Let f be continuous and strictly increasing on $[0, \infty)$ with $f(x)/x$ non-decreasing for $x > 0$, $f(0) = 0$, and suppose that*

$$(12) \qquad f\big(x(t)\big) \leqslant a(t) + h\Big(\int_a^t k(s)g\big(x(s)\big)ds\Big) + \int_a^t k_1(s)g_1\big(x(s)\big)ds, \qquad t \in J.$$

Then

$$(12') \qquad x(t) \leqslant A(t)\Big\{a(t) + h\circ G^{-1}\Big[\int_a^t k(s)g\big(A(s)\big)ds + G\Big(\int_a^t k(s)g\big(a(s)A(s)\big)ds\Big)\Big]\Big\},$$

for $a \leqslant t < \beta_1$, where $A(t) \equiv (F_1\circ f)^{-1}[F_1(1) + \int_a^t k_1 ds]$, G and F_1 are defined by (5) and (6), and

$$\beta_1 = \sup\Big\{t \in J:\ \int_a^t k(s)g\big(A(s)\big)ds + G\Big(\int_a^t k(s)g\big(a(s)A(s)\big)ds\Big) \in G(\mathbf{R}^+)\Big\}.$$

If $a(t) \equiv a \geqslant 1$, then g need not be subadditive and in this case,

$$(12'') \qquad x(t) \leqslant A(t)\Big\{a + h\circ G_a^{-1}\Big(\int_a^t k(s)g\big(A(s)\big)ds\Big)\Big\}, \qquad a \leqslant t < \beta_2,$$

where G_a is as defined following (7″), and

$$\beta_2 = \sup\Big\{t \in J:\ \int_a^t k(s)g\big(A(s)\big)ds \in G_a(\mathbf{R}^+)\Big\}.$$

Proof. To simplify the proof, we begin by writting (12) in the form

$$x(t) \leqslant f^{-1}\Big[a_1(t) + \int_a^t k_1(s) g_1\big(x(s)\big)\, ds\Big], \qquad t \in J,$$

noting that the hypotheses on f imply that $f(\boldsymbol{R}^+) = \boldsymbol{R}^+$, and (by Lemma 5.1) that f^{-1} satisfies condition (2). Since $a_1 \geqslant 1$, a_1 is non-decreasing, and g_1 also satisfies condition (2), we have

$$u(t) \equiv \frac{x(t)}{a_1(t)} \leqslant f^{-1}\Big\{1 + \int_a^t k_1(s) g_1\Big(\frac{x(s)}{a_1(s)}\Big) ds\Big\} = f^{-1}\Big\{1 + \int_a^t k_1(s) g_1\big(u(s)\big)\, ds\Big\},$$

or

$$f\big(u(t)\big) \leqslant 1 + \int_a^t k_1(s) g_1\big(u(s)\big)\, ds, \qquad t \in J. \tag{13}$$

Now apply Theorem 2.2 to (13) with

$$H(t, v) \equiv v, \qquad W(t, s, u) \equiv k_1(s) g_1(u), \qquad I = K = [0, \infty).$$

It follows that

$$u(t) \leqslant f^{-1}[1 + r(t)], \tag{14}$$

where $r = r(t)$ is the maximal solution of

$$\frac{dr}{dt} = k_1(t) g_1 \circ f^{-1}(1 + r), \qquad r(a) = 0. \tag{15}$$

This system has the explicit solution

$$r(t) = F_1^{-1}\Big[\int_a^t k_1\, ds + F_1(1)\Big] - 1, \qquad t \in J.$$

(Note that $g_1(u) \leqslant g_1(1) u$ and $f^{-1}(u) \leqslant f^{-1}(1) u$ by (3), whence $F_1(\boldsymbol{R}^+) \supset \boldsymbol{R}^+$ follows.) By (14),

$$\frac{x(t)}{a_1(t)} \leqslant (F_1 \circ f)^{-1}\Big[\int_a^t k_1\, ds + F_1(1)\Big] = A(t),$$

or

$$x(t) \leqslant A(t) a(t) + A(t) h\Big(\int_a^t k(s) g\big(x(s)\big)\, ds\Big), \qquad t \in J. \tag{16}$$

Again we may apply Theorem 2.2 with

$$f(x) \equiv x, \quad H(t, v) \equiv A(t) h(v), \quad W(t, s, u) \equiv k(s) g(u), \quad I = K = [0, \infty),$$

to obtain

$$(14') \qquad x(t) \leqslant A(t)a(t) + A(t)h\big(r(t)\big)$$

for t in the domain of r, where $r = r(t)$ is the maximal solution of

$$(15') \qquad \frac{dr}{dt} = k(t)g[A(t)a(t) + A(t)h(r)], \qquad r(a) = 0.$$

Since g is subadditive and submultiplicative, and $k \geqslant 0$,

$$\frac{dr}{dt} \leqslant k(t)g\big(A(t)\big)g\big(h(r)\big) + k(t)g\big(a(t)A(t)\big).$$

Precisely as in the proof of Theorem 5.1, with E replaced by A, it follows that

$$r(t) \leqslant G^{-1}\Big\{\int_a^t k(s)g\big(A(s)\big)ds + G\Big(\int_a^t k(s)g\big(a(s)E(s)\big)ds\Big)\Big\}$$

for $a \leqslant t < \beta_1$, so that (12′) follows from (14′).

If $a(t) \equiv a \geqslant 1$, the proof is unchanged down to (15′). Now, using only the fact that g is submultiplicative, we have

$$\frac{dr}{dt} \leqslant k(t)g\big(A(t)\big)g[a + h(r)].$$

This inequality can be solved explicitly and Lemma 2.1 gives

$$r(t) \leqslant G_a^{-1}\Big(\int_a^t k(s)g\big(A(s)\big)ds\Big), \qquad a \leqslant t < \beta_2.$$

The conclusion (12″) follows from this and (14′) — with $a(t) \equiv a$.

Remark 1. An application of Corollary 5.1 to (10) would give

$$x(t) \leqslant E(t)a(t) + E(t)h\Big(\int_a^t k(s)g\big(x(s)\big)ds\Big), \qquad t \in J,$$

which is (16) with A replaced by E. Proceeding now as in the proof of Theorem 5.2, one would obtain

$$(17) \qquad x(t) \leqslant E(t)\Big\{a(t) + h\circ G^{-1}\Big[\int_a^t k(s)g\big(E(s)\big)ds + G\Big(\int_a^t k(s)g\big(a(s)E(s)\big)ds\Big)\Big]\Big\}.$$

Inequality (7′) of Theorem 5.1 is, however, better than (17) according to Corollary 5.1.

Remark 2. If f and g_1 satisfy the hypotheses of Theorem 5.1 (i.e., Corollary 5.1), then $A(t) \leqslant E(t)$. Hence (7′) and (12′) are not readily comparable when f, g_1 satisfy the hypotheses of both Theorem 5.1 and Theorem 5.2, although both are better than (17). Under the hypotheses of Theorem 5.2 it is easy to prove that

$$(18)\qquad A(t) = (F_1 \circ f)^{-1}\Big[\int_a^t k_1\,ds + F_1(1)\Big] \leqslant \exp\Big(g_1(1)\int_a^t k_1\,ds\Big) = [E(t)]^{g_1(1)}.$$

We note that Theorem 5.2 always gives a better bound than Theorem 5.1 when $f(x) \equiv x$, $g_1(x) \equiv \gamma x$ with $0 < \gamma < 1$. On the other hand, the hypotheses of Theorem 5.1 are less restrictive than those of Theorem 5.2 except when f is linear.

Remark 3. Under certain additional hypotheses on h and g we may obtain somewhat simpler (but larger) upper bounds than those given in Theorems 5.1, 5.2. In fact, *if g, h satisfy the conditions of Lemma* 5.2, then the bounds (7′), (12′) imply that

$$x(t) \leqslant E(t)\Big\{a(t) + \exp\Big(\int_a^t k(s)g\big(E(s)\big)\,ds\Big)\int_a^t k(s)g\big(a(s)E(s)\big)\,ds\Big\},$$

$$x(t) \leqslant A(t)\Big\{a(t) + \exp\Big(\int_a^t k(s)g\big(A(s)\big)\,ds\Big)\int_a^t k(s)g\big(a(s)A(s)\big)\,ds\Big\},$$

respectively. An additional simplification of the last inequality may be made using (18).

The inequality

$$(19)\qquad x(t) \leqslant a + \int_a^t \big\{k(s)g\big(x(s)\big) + k_1(s)x(s)\big\}\,ds, \qquad t \in J,$$

dealt with in [8], Theorem 1, and included in Theorem 5.1 (for $f(x) \equiv x$, $h(u) \equiv u$, $g_1(u) \equiv u$, $a(t) \equiv a$) may also be dealt with by a single application of Theorem 2.2. Under the hypotheses of Theorem 5.1 for the case $a(t) \equiv a$ inequality (7″) implied by (19) becomes

$$(19')\qquad x(t) \leqslant E(t)G^{-1}\Big[G(a) + \int_a^t k(s)g\big(E(s)\big)\,ds\Big], \qquad a \leqslant t < \beta_2,$$

where $E(t) \equiv \exp\big(\int_a^t k_1\,ds\big)$, $G(u) \equiv \int_{u_0}^u dy/g(y)$, $u > 0$ $(u_0 > 0)$, and

$$\beta_2 = \sup\Big\{t \in J:\ G(a) + \int_a^t k(s)g\big(E(s)\big)\,ds \in G(\mathbf{R}^+)\Big\}.$$

We now proceed as suggested following (19), and obtain a result which is better than (19′) in some cases at least, and with weaker hypotheses.

THEOREM 5.3. *Let the functions x, k, k_1 be continuous and non-negative on an interval $J = [a, \beta]$, and let g be continuous and non-decreasing on $I = [0, \infty)$ with $g(u) > 0$ for $u > 0$. If x satisfies inequality* (19) *for some $a > 0$, then*

$$(19'') \qquad x(t) \leqslant G^{-1}\Big[G\big(aE(t)\big) + E(t)\int_a^t k(s)\big(E(s)\big)^{-1}ds\Big], \qquad a \leqslant t < \bar\beta,$$

where G is as defined following (19'), *and*

$$\bar\beta = \sup\Big\{t \in J\colon\ G\big(aE(t)\big) + E(t)\int_a^t k(s)\big(E(s)\big)^{-1}ds \in G(\boldsymbol{R}^+)\Big\}.$$

Proof. In Theorem 2.2, take

$$f(x) \equiv x, \quad H(t, v) \equiv v, \quad W(t, s, u) \equiv k(s)g(u) + k_1(s)u, \quad K = [0, \infty).$$

The comparison system is

$$\frac{dr}{dt} = k(t)g(a+r) + k_1(t)(a+r), \qquad r(a) = 0,$$

or with $u = a+r$,

$$(20) \qquad \frac{du}{dt} = k(t)g(u) + k_1(t)u, \qquad u(a) = a.$$

Moreover,

$$(21) \qquad x(t) \leqslant a + r(t) = u(t)$$

for t in the domain of the maximal solution $u(t)$ of (20). To obtain an estimate of u (and of its domain) multiply (20) by the integrating factor $\big(E(t)\big)^{-1}$ and set $v = u\big(E(t)\big)^{-1}$. Then (20) is equivalent to

$$(20') \qquad \frac{dv}{dt} = k(t)\big(E(t)\big)^{-1}g\big(vE(t)\big), \qquad v(a) = a.$$

For $a \leqslant t \leqslant T$,

$$k(t)\big(E(t)\big)^{-1}g\big(vE(t)\big) \leqslant k(t)\big(E(t)\big)^{-1}g\big(vE(T)\big),$$

and the system

$$\frac{dr_1}{dt} = k(t)\big(E(t)\big)^{-1}g\big(r_1E(T)\big), \qquad r_1(a) = a,$$

has the unique solution

$$r_1(t) = G^{-1}\Big[G\big(aE(T)\big) + E(T)\int_a^t k_1(s)\big(E(s)\big)^{-1}ds\Big], \qquad a \leqslant t \leqslant T < \bar\beta.$$

Setting $t = T$ and changing notation, (19'') follows from Lemma 2.1, (21) and the above.

In case $g(u) = u^2$ and $\alpha = 0$, (19′) and (19″) reduce to

$$x(t) \leqslant aE(t)\Big\{1 - a\int_0^t k(s)E^2(s)\,ds\Big\}^{-1} \equiv x_1(t),$$

$$x(t) \leqslant aE(t)\Big\{1 - aE^2(t)\int_0^t k(s)\big(E(s)\big)^{-1}ds\Big\}^{-1} \equiv x_2(t),$$

respectively. We have $x_2(t) \leqslant x_1(t)$ for all t in their common domain if $a \geqslant 4$, $k(s) \equiv s$, $k_1(s) \equiv 1$ for example, while $x_1(t) \leqslant x_2(t)$ if k, k_1 are both constant.

Remark 4. If, in addition to the hypotheses of Theorem 5.3 we also assume that g satisfies condition (2), then (19) implies

$$(19''') \qquad x(t) \leqslant E(t)G^{-1}\Big[G(a) + \int_\alpha^t k\,ds\Big], \quad \alpha \leqslant t < \beta',$$

where $\beta' = \sup\big\{t \in J\colon\ G(a) + \int_\alpha^t k\,ds \in G(\boldsymbol{R}^+)\big\}$. For, since $E(t) \geqslant 1$, (20′) and condition (2) imply that

$$\frac{dv}{dt} \leqslant k(t)g(v), \quad v(\alpha) = a.$$

Hence by Lemma 2.1,

$$v(t) = u(t)\big(E(t)\big)^{-1} \leqslant G^{-1}\Big[G(a) + \int_\alpha^t k\,ds\Big], \quad \alpha \leqslant t < \beta'.$$

Inequality (19‴) now follows from (21). If $g(u) \geqslant 1$ for $u \geqslant 1$, (19‴) is clearly better than (19′).

Our final result of this section includes two more theorems of S.G. Deo and U. D. Dhongade [8], Theorem 3 and [9], Lemma 2.

THEOREM 5.4. *Let the functions x, a, b, k be continuous and non-negative on $J = [\alpha, \beta]$ with $a \geqslant 1$, $b \geqslant 1$ and a non-decreasing on J. Let g, h be continuous, non-negative and non-decreasing on $I = [0, \infty)$ with $g(u) > 0$ for $u > 0$. Suppose that g, h both satisfy condition* (2) *on I and that g is submultiplicative. If*

$$(22) \qquad x(t) \leqslant a(t) + b(t)h\Big(\int_\alpha^t k(s)g\big(x(s)\big)ds\Big), \quad t \in J,$$

then

$$(22') \qquad x(t) \leqslant a(t) + b(t)h\circ G_1^{-1}\Big(\int_\alpha^t k(s)g\big(b(s)\big)ds\Big), \quad \alpha \leqslant t < \beta_1,$$

where $G_1(u) \equiv \int_0^u dy/g[1+h(y)],\ u \geqslant 0,$ *and*

$$\beta_1 = \sup\Big\{t \in J:\ \int_a^t k(s)g\big(b(s)\big)ds \in G_1(\mathbf{R}^+)\Big\}:$$

Moreover, if $b(t) \equiv 1$, *g need not be submultiplicative and in this case,*

$$x(t) \leqslant a(t)\Big\{1+h\circ G_1^{-1}\Big(\int_a^t k(s)ds\Big)\Big\}, \quad a \leqslant t < \beta_2, \tag{22''}$$

where $\beta_2 = \sup\{t \in J:\ \int_a^t k\,ds \in G_1(\mathbf{R}^+)\}$.

Note. In [9], (22′) was proved with the first term on the right-hand side replaced by $a(t)b(t)$. The hypothesis $a(t) \geqslant 1$ was omitted but the assumptions on g, h implied that these functions were linear. (See remarks at (1), (2).)

Proof. In order to simplify the proof in applying Theorem 2.2, we note first that (22) and the hypotheses on a, g, h imply that

$$u(t) \equiv \frac{x(t)}{a(t)} \leqslant 1+b(t)h\Big(\int_a^t k(s)g\big(u(s)\big)ds\Big), \quad t \in J.$$

The rest of the proof is more or less as in the first part of the proof of Theorem 5.2, but with $f(x) \equiv x$ and $H(t, v) \equiv b(t)h(v)$. The comparison equation is now

$$\frac{dr}{dt} = k(t)g[1+b(t)h(r)], \quad r(a) = 0,$$

and this is dominated by

$$\frac{dr_1}{dt} = k(t)g\big(b(t)\big)g[1+h(r_1)], \quad r_1(a) = 0,$$

which has the unique solution

$$r_1(t) = G_1^{-1}\Big(\int_a^t k(s)g\big(b(s)\big)ds\Big), \quad a \leqslant t < \beta_1.$$

Since

$$u(t) \leqslant 1+b(t)h\big(r(t)\big), \tag{23}$$

the conclusion (22′) now follows from Lemma 2.1.

If $b(t) \equiv 1$, the comparison equation can be solved explicitly to give

$$r(t) = G_1^{-1}\Big(\int_a^t k\,ds\Big), \quad a \leqslant t < \beta_2,$$

and this, with (23), implies (22″).

Remark 5. The case $b(t) \equiv 1$, $h(v) \equiv v$ corresponds to Theorem 3 of [8], and in this case (22″) can be written in the form

$$x(t) \leqslant a(t) G^{-1}\Big[G(1) + \int_a^t k\,ds\Big], \qquad a \leqslant t < \beta_2,$$

where $G(u) \equiv \int_{u_0}^{u} dy/g(y)$, $u > 0$ $(u_0 > 0)$, and

$$\beta_2 = \sup\Big\{t \in J:\ G(1) + \int_a^t k\,ds \in G(\boldsymbol{R}^+)\Big\}.$$

This follows from the fact that $G_1(u) = G(u+1) - G(1)$.

Remark 6. The bound (22′) is not readily comparable with the bound (5′) of Theorem 4.4 (i). By applying Theorem 2.2 directly to (22), still other bounds may be obtained under a variety of hypotheses. We merely state two such results:

With the continuity and non-negativity assumptions of Theorem 5.4, *suppose that* $a \geqslant 1$, $b \geqslant 1$, g *is submultiplicative, and* g, h *are both non-decreasing. Then* (22) *implies*

$$(24) \qquad x(t) \leqslant a(t) + b(t) h \circ G_1^{-1}\Big(\int_a^t k(s) g\big(a(s)\big) g\big(b(s)\big)\,ds\Big), \qquad a \leqslant t < \beta_1,$$

where G_1 *is as in Theorem* 5.4, *and*

$$\beta_1 = \sup\Big\{t \in J:\ \int_a^t k(s) g\big(a(s)\big) g\big(b(s)\big)\,ds \in G_1(\boldsymbol{R}^+)\Big\}.$$

(Inequality (24) is clearly better than (22′) when $g\big(a(s)\big) \leqslant 1$, but not, in general, when $g \circ a$ is large.)

With the continuity and non-negativity assumptions of Theorem 5.4, *suppose that* $b \geqslant 1$, g *is subadditive and satisfies condition* (2), *and* g, h *are both non-decreasing. Then* (22) *implies*

$$(25) \qquad x(t) \leqslant a(t) + b(t) h \circ G^{-1}\Big[\int_a^t kb\,ds + G\Big(\int_a^t kbg(a/b)\,ds\Big)\Big], \qquad a \leqslant t < \beta_3,$$

where $G(u) \equiv \int_{u_0}^{u} dy/g\big(h(y)\big)$, $u > 0$ $(u_0 > 0)$ *and*

$$\beta_3 = \sup\Big\{t \in J:\ \int_a^t kb\,ds + G\Big(\int_a^t kbg(a/b)\,ds\Big) \in G(\boldsymbol{R}^+)\Big\}.$$

If g is also submultiplicative, both bounds (25) and (5′), Section 4 apply. Since $bg(a/b) \geqslant g(a)$ by (2) we see that (5′) is the better estimate whenever $g(b) \leqslant b$ for $b \geqslant 1$.

6. An inversion procedure. We conclude with some observations concerning pairs of inequalities of the form

$$f(x(t)) \leqslant (Tx)(t), \quad t \in J, \tag{1}$$

$$x(t) \leqslant (Tx)(t), \quad t \in J, \tag{2}$$

where f is continuous and strictly monotonic on an interval $I \supset x(J)$, and T is an operator on $C(J)$, the class of continuous functions on J. For each inequality obtained as a consequence of (2), one may obtain a corresponding result from (1) merely by writing (1) in the form

$$u(t) \leqslant (T_1 u)(t), \quad t \in J, \tag{1'}$$

where $u(t) \equiv f(x(t))$ and $T_1 u \equiv (Tf^{-1})u$ for $u \in C(J)$. We illustrate by obtaining the following corollary of Theorem 5.4.

THEOREM 6.1. *In addition to the hypotheses of Theorem* 5.4, *let f be continuous, strictly increasing and supermultiplicative on $I = [0, \infty)$ with $f(u)/u$ non-decreasing and positive for $u > 0$ and $f(0) = 0$. If*

$$f(x(t)) \leqslant a(t) + b(t)h\Big(\int_a^t k(s)g(x(s))\,ds\Big), \quad t \in J, \tag{3}$$

then

$$x(t) \leqslant f^{-1}\Big\{a(t) + a(t)b(t)h\circ\tilde{G}_1^{-1}\Big(\int_a^t k(s)g\circ f^{-1}(b(s))\,ds\Big)\Big\}, \tag{3'}$$

$$\alpha \leqslant t < \beta_1,$$

where $\tilde{G}_1(u) \equiv \int_0^u dy/g\circ f^{-1}[1+h(y)]$, $u > 0$, *and*

$$\beta_1 = \sup\Big\{t \in J\colon \int_a^t k(s)g\circ f^{-1}(b(s))\,ds \in \tilde{G}_1(\mathbf{R}^+)\Big\}.$$

The result follows from Theorem 5.4 and the above remarks on noting the following facts. The hypotheses on f, g imply that $\tilde{g} = g\circ f^{-1}$ is continuous and non-decreasing on I; observe that $f(u) \geqslant f(1)u$ for $u \geqslant 1$ so that $f(I) = I$ and the domain of $\tilde{g}$ is I. Moreover, $\tilde{g}(u) > 0$ for $u > 0$, $\tilde{g}$ is submultiplicative (since f supermultiplicative $\Rightarrow f^{-1}$ submultiplicative), and $\tilde{g}$ satisfies condition (2), Section 5 (on using Lemma 5.1).

We note that the main Theorem 2.2 is invariant under this process. That is, if the result of Theorem 2.2 is assumed only for the case $f(x) \equiv x$, and the above process is applied to this case, the result is precisely the general Theorem 2.2 as stated. On the other hand, some of the results of Section 5 are not so invariant. This can be seen at once by noting that the functions $g\circ f^{-1}$, $g_1\circ f^{-1}$ do not appear uniformly in the conclusions. As an example, we illustrate by applying the inversion process to Theorem 5.1.

Let x, a, k, k_1, g, h satisfy the hypotheses of Theorem 5.1, *and let g_1 be continuous on $[0, \infty)$ with $0 < g_1(u) \leqslant u$ for $u > 0$, and*

$$(4)\qquad x(t) \leqslant a(t) + h\Big(\int_a^t k(s) g\big(x(s)\big)\,ds\Big) + \int_a^t k_1(s) g_1\big(x(s)\big)\,ds, \qquad t \in J.$$

By Theorem 5.1, *with $f(x) \equiv x$, it follows that*

$$(4')\qquad x(t) \leqslant F_1^{-1}\Big[\int_a^t k_1\,ds + F_1\Big\{a(t) + h\circ G^{-1}\Big[\int_a^t k(s) g\big(E(s)\big)\,ds + G\Big(\int_a^t k(s) g\big(a(s)E(s)\big)\,ds\Big)\Big]\Big\}\Big], \qquad a \leqslant t < \beta_1,$$

where $F_1(u) \equiv \int_{u_0}^u dy/g_1(y)$, $u > 0$ $(u_0 > 0)$, $G(u) \equiv \int_{u_0}^u dy/g\big(h(y)\big)$, $u > 0$, and

$$\beta_1 = \sup\Big\{t \in J\colon\ G\Big(\int_a^t k(s) g\big(a(s)E(s)\big)\,ds\Big) + \int_a^t k(s) g\big(E(s)\big)\,ds \in G(\mathbf{R}^+)\Big\}.$$

One can verify that *if f is continuous and strictly increasing on $\mathbf{R}^+$ with $f(0) = 0$, and f is both superadditive and supermultiplicative on $\mathbf{R}^+$, and if now g_1 satisfies $0 < g_1(v) \leqslant f(v)$ for $v > 0$, then the functions $\tilde{g} = g\circ f^{-1}$ and $\tilde{g}_1 = g_1\circ f^{-1}$ satisfy the hypotheses listed above for* (4) (along with x, a, k, k_1, h as above). (The superadditivity of f implies that $f(x) \to \infty$ as $x \to \infty$.) Applying the above result to

$$(5)\qquad f\big(x(t)\big) \leqslant a(t) + h\Big(\int_a^t k(s) g\big(x(s)\big)\,ds\Big) + \int_a^t k_1(s) g_1\big(x(s)\big)\,ds, \qquad t \in J,$$

or

$$u(t) \leqslant a(t) + h\Big(\int_a^t k(s)\,\tilde{g}\big(u(s)\big)\,ds\Big) + \int_a^t k_1(s)\,\tilde{g}_1\big(u(s)\big)\,ds, \qquad t \in J,$$

we obtain

$$(5')\qquad x(t) \leqslant f^{-1}\circ \tilde{F}_1^{-1}\Big[\int_a^t k_1\,ds + \tilde{F}_1\Big\{a(t) + h\circ \tilde{G}^{-1}\Big[\int_a^t k(s)\,\tilde{g}\big(E(s)\big)\,ds + \tilde{G}\Big(\int_a^t k(s)\,\tilde{g}\big(a(s)E(s)\big)\,ds\Big)\Big]\Big\}\Big], \qquad a \leqslant t < \tilde{\beta}_1,$$

where $\tilde{F}_1$, $\tilde{G}$ and $\tilde{\beta}_1$ are defined as F_1, G and β_1 above, but with g, g_1 replaced by $\tilde{g} = g\circ f^{-1}$ and $\tilde{g}_1 = g_1\circ f^{-1}$.

In those cases for f, where both Theorem 5.1 and the above result applies, it is not clear whether the results are comparable.

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