On series of homogeneous polynomials
and their partial sums

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Abstract. Let \( \sum_{r=0}^{\infty} Q_r(z) \) be a series of homogeneous polynomials on \( \mathbb{C}^N \) with \( Q_r(z) = \sum_{|\alpha|=r} c_{\alpha} z^{\alpha} \), \( \alpha \in \mathbb{Z}_+^N \). Let \( S_n := Q_0 + \ldots + Q_n \) be the \( n \)-th sum of the series, \( E \) a subset of \( \mathbb{C}^N \), and \( \{n_j\} \) an increasing sequence of positive integers. We say that a pair \((E, \{n_j\})\) has \( A \)-Property if each series \( \sum_{n=0}^{\infty} Q_n(z) \) with \( \{S_{n_j}(z)\} \) converging for every \( z \) in \( E \) has a positive radius of convergence. The paper delivers a characterization of pairs \((E, \{n_j\})\) with \( A \)-Property in terms of global extremal plurisubharmonic functions and of related capacities as well as of the rate of convergence \( \varkappa := \limsup_{j \to \infty} n_{j+1}/n_j \) of the sequence \( \{n_j\} \). In particular, it is shown that if \( \varkappa \) is finite and \( \{S_{n_j}(z)\} \) converges at each point \( z \) of a nonpluriharmonic set \( E \) in \( \mathbb{C}^N \), then the series converges in the ball \( \|z\| < \varkappa^r \), where \( \alpha = \alpha(E) \) is a capacity of \( E \).

1. Introduction

Let

(1.1) \[ \sum_{r=0}^{\infty} Q_r(z) \quad \text{with} \quad Q_r(z) = \sum_{|\alpha|=r} c_{\alpha} z^{\alpha}, \quad z \in \mathbb{C}^N \]

be a series of homogeneous polynomials of \( N \) complex variables. If \( N = 1 \) then (1.1) is a power series,

(1.2) \[ \sum_{r=0}^{\infty} c_r z^r, \quad z \in \mathbb{C}. \]

It is the classical result due to Abel that if (1.2) converges at a point \( z_0 \neq 0 \) then it converges in the disk \( |z| < r = |z_0| \). About 60 years ago F. Leja observed that, given any countable set \( E \) in \( \mathbb{C}^2 \), one can construct a series (1.1) convergent on \( E \) but having no positive radius of convergence.

Definition 1.1. We say that a subset \( E \) of \( \mathbb{C}^N \) has \emph{Abel's Property} if there
is a positive number \( R = R(E) \) such that every series (1.1) converging pointwise on \( E \) converges in the ball \( |z| < R \).

**Definition 1.2.** We say that a subset \( E \) of \( C^N \) has *Weak Abel's Property* if every series (1.1) converging on \( E \) has a positive radius of convergence.

**Problem 1** (due to Leja). Characterize subsets \( E \) of \( C^N \) with Abel's Property (resp. with Weak Abel's Property).

**Definition 1.3.** Let \( E \) be a subset of \( C^N \) and let \( \{n_j\} \) be an increasing sequence of positive integers. Let \( S_n = Q_0 + \ldots + Q_n \) be the \( n \)-th partial sum of series (1.1). We say that a pair \( (E, \{n_j\}) \) has *A-Property* if each series (1.1) with the subsequence \( \{S_{n_j}\} \) converging on \( E \) has a positive radius of convergence.

**Problem 2.** Characterize pairs \((E, \{n_j\})\) with *A-Property*.

Problem 2 reduces to Problem 1 by taking \( n_j = j, j \in N \).

Problem 1 for compact sets \( K \) in \( C^2 \) was solved by Leja ([3], [4]) in terms of his "triangular" transfinite diameter. His method does not work in \( C^N \) for \( N \geq 3 \), because it is based on the well-known fact that each homogeneous polynomial of two variables can be written as a product of linear factors. Such a property is no more true for homogeneous polynomials of \( N \) variables with \( N \geq 3 \). A partial solution to Problem 1 in \( C^N \) (\( N \geq 2 \)) was given in [7] (for compact subsets of \( C^N \)) and in [10] (for arbitrary subsets of \( C^N \)) in terms of the extremal homogeneous function \( \Psi_E \) associated with subsets \( E \) of \( C^N \) (see Chapter 2). Here we give a complete solution to Problem 1 in terms of \( \Psi_E \) (or equivalently, in terms of a projective capacity \( g(E) \) of \( E \); see next chapter for the definitions). Problem 2 for the plane case has been already studied by Naftalevich [6]. Our solution to Problem 2 given in this paper contains results of [6] as special cases. We characterize pairs \((E, \{n_j\})\) with *A-Property* in terms of the rate of convergence of \( \{n_j\} \),

\[
\alpha = \limsup N_{j+1}/n_j,
\]

and of the capacity \( \alpha(E) \) of \( E \) defined by 2.2 below.

**2. Global extremal plurisubharmonic functions and capacities in \( C^N \)**

(Reminder of definitions and main properties)

Let \( \mathcal{L} \) be the set of all plurisubharmonic (plsh) functions \( u \) in \( C^N \) with \( \sup \{u(z) - \log(1 + |z|); z \in C^N \} < +\infty \). Let \( \mathcal{H} \) be the set of all functions \( h \) plurisubharmonic in \( C^N \) such that \( h(\lambda z) = |\lambda|h(z), \lambda \in C, z \in C^N \). It is well known that \( \log h \in \mathcal{L} \) for every \( h \in \mathcal{H}, h \neq 0 \).

2.1. For every subset \( E \) of \( C^N \), we define two extremal functions \( \Phi_E(z) \equiv \Phi(z, E), \Psi_E(z) \equiv \Psi(z, E) \) in \( C^N \) by the formulas

\[
\Phi(z, E) := \sup \{\exp u(z); u \in \mathcal{L}, u \leq 0 \text{ on } E\},
\]

\[
\Psi(z, E) := \sup \{h(z); h \in \mathcal{H}, h \leq 1 \text{ on } E\}
\]
if $E \subseteq C^N$ is bounded; and
\[
\Phi(z, E) = \inf \{ \Phi(z, F); F \subseteq E, \, F \text{ is bounded} \}, \\
\Psi(z, E) = \inf \{ \Psi(z, F); F \subseteq E, \, F \text{ is bounded} \}
\]
if $E$ is arbitrary.

2.2. Let $\alpha$ and $\varphi$ be set functions defined for all $E \subseteq C^N$ by the formulas
\[
\alpha(E) := 1/\sup \{ \Phi(z, E); z \in B \}, \quad \varphi(E) := 1/\sup \{ \Psi(z, E); z \in B \},
\]
where $B := \{ z \in C^N; \| z \| < 1 \}$ is the unit ball with respect to a fixed norm $\| \cdot \|$ in $C^N$.

2.3. Let $\beta$ be a set function defined for all $E \subseteq C^N$ by the formula
\[
\beta(E) := \varphi(\varphi(E)),
\]
where $\varphi(E) = S^{2N+1} \cap C \cdot \{ \{ 1 \} \times E \}$ denotes the intersection of the cone $C \cdot \{ \{ 1 \} \times E \} = \{ (t, tz); t \in C, \, z \in E \}$ with the Euclidean unit sphere $S^{2N+1}$ in $C^{N+1}$.

Now we shall recall some of the properties of $\Phi, \Psi, \alpha, \varphi$ and $\beta$ (the proofs may be found in [1], [9]-[10]).

2.4. If $K$ is a compact subset of $C^N$, then
\[
\Phi(z, K) = \sup \Phi_n^{1/n}(z) = \lim_{n \to \infty} \Phi_n^{1/n}(z), \quad \Psi(z, K) = \sup \Psi_n^{1/n}(z) = \lim_{n \to \infty} \Psi_n^{1/n}(z),
\]
\[
\alpha(K) = \inf \alpha_n^{1/n} = \lim_{n \to \infty} \alpha_n^{1/n}, \quad \varphi(K) = \inf \varphi_n^{1/n} = \lim_{n \to \infty} \varphi_n^{1/n},
\]
where
\[
\Phi_n(z) := \sup \{ \| P(z); P(z) = \sum_{|x| \leq n} c_x z^x, \| P \|_K \leq 1 \},
\]
\[
\Psi_n(z) := \sup \{ \| Q(z); Q(z) = \sum_{|x| = n} c_x z^x, \| Q \|_K \leq 1 \},
\]
\[
\alpha_n := \inf \{ \| P \|_K; P(z) = \sum_{|x| \leq n} c_x z^x, \| P \|_B \geq 1 \},
\]
\[
\varphi_n := \inf \{ \| Q \|_K; Q(z) = \sum_{|x| = n} c_x z^x, \| Q \|_B \geq 1 \}.
\]

2.5. If $K$ is a non-pluripolar subset of $C^N$, then $G(z, K) = \log \Phi^*(z, K)$ (where $\Phi^*$ denotes the upper semicontinuous regularization of $\Phi$) is the unique function $u$ plurisubharmonic in $C^N$ with the following properties:

(i) $c_1 + \log(1 + |z|) \leq u(z) \leq c_2 + \log(1 + |z|)$ in $C^N$, where $c_1, c_2$ are real constants;

(ii) $(dd^c u)^N = 0$ in $C^N \setminus K$, where $d = \partial + \overline{\partial}, \, d^c = i(\overline{\partial} - \partial)$;

(iii) $u = 0$ on $K \setminus F$, where $F$ is an $F_\sigma$ pluripolar subset of $K$.

In particular, if $N = 1$ and $K \subseteq C$ is not polar, then $G(z, K)$ is the Green function for $C \setminus \hat{K}$ with pole at infinity.
2.6. Characterization of pluripolar sets in $\mathbb{C}^N$. For a subset $E$ of $\mathbb{C}^N$, the following conditions are equivalent:

(a) $E$ is globally pluripolar (i.e. there is a plsh function $u$ in $\mathbb{C}^N$ with $u = -\infty$ on $E$);

(b) $E$ is $\mathcal{L}$-polar, i.e. there is $u$ in $\mathcal{L}$ with $u = -\infty$ on $E$;

(c) $\alpha(E) = 0$;

(d) $\Phi^*(z, E) \equiv +\infty$.

2.7. Characterization of circled pluripolar sets in $\mathbb{C}^N$. For a subset $E$ of $\mathbb{C}^N$, the following conditions are equivalent:

(a) the complex cone $C\cdot E$ generated by $E$ is pluripolar;

(b) $q(E) = 0$;

(c) there is $h$ in $\mathcal{H}$ with $h = 0$ on $E$, $h \neq 0$;

(d) $\Psi^*(z, E) \equiv +\infty$.

2.8. Basic inequalities for polynomials in $\mathbb{C}^N$. If $P(z) = \sum_{|z|\leq n} c_z z^z$ and $K$ is a compact subset of $\mathbb{C}^N$, then

$$ |P(z)| \leq \|P\|_K \Phi^n(z, K), \quad |P(z)| \leq \|P\|_K (\max \{1, \|z\|/\alpha(K)\})^n, $$

$$ |Q(z)| \leq \|Q\|_K \Psi^n(z, K), \quad |Q(z)| \leq \|Q\|_K (\|z\|/q(K))^n $$

for all $z$ in $\mathbb{C}^n$, where $\|P\|_K := \max_{K} |P(z)|$.

2.9. Let $A, B, E, E_n$ be arbitrary sets and $K, K_n$ compact sets in $\mathbb{C}^N$. Then

I. $\Phi_A \leq \Phi_B, \quad \Psi_A \leq \Psi_B$ if $B \subset A$,

II. $\Phi_{K_n} \uparrow \Phi_K, \quad \Psi_{K_n} \uparrow \Psi_K$ if $K_{n+1} \subset K_n, \quad K = \bigcap_{1}^{\infty} K_n$,

III. $\Phi_{E_n}^* \downarrow \Phi_E^*, \quad \Psi_{E_n}^* \downarrow \Psi_E^*$ if $E_n \subset E_{n+1}, \quad E = \bigcup_{1}^{\infty} E_n$,

IV. $\Phi_{E \cup A}^* = \Phi_E^*$ if $A$ is pluripolar,

$\Psi_{E \cup A}^* = \Psi_E^*$ if $C\cdot A$ is pluripolar,

V. $\log \Phi_E^* \in \mathcal{L}$ if and only if $E$ is not pluripolar,

$\log \Psi_E^* \in \mathcal{L}$ if and only if $C\cdot E$ is not pluripolar.

2.10. If $c$ denotes any of the set functions $\alpha, \beta$, or $q$, then $c$ is a Choquet capacity, i.e.

(i) $c(A) \leq c(B)$ if $A \subset B$,

(ii) $c(K_n) \downarrow c(K)$ if $K_n \downarrow K$,

(iii) $c(E_n) \uparrow c(E)$ if $E_n \uparrow E$,

where $A, B, E, E_n$ are arbitrary and $K, K_n$ are compact sets in $\mathbb{C}^N$. 
2.11. **Geometric interpretation of the capacities \( \alpha \) and \( \varrho \).** \( 1/\alpha(K) \) is equal to the radius \( R \) of the smallest level domain \( \{ z \in C^N; G(z, K) < \log R \} \) containing the unit ball \( B \).

\[
\varrho(K) = \sup \{ r \geq 0; rB \subseteq \hat{K}_b \}, \quad \text{where} \quad \hat{K}_b = \{ z \in C^N; |Q(z)| \leq \|Q\|_K \text{ for every homogeneous polynomial } Q \},
\]

i.e. \( \varrho(K) \) is the radius of the maximal ball (with respect to the given norm) contained in the convex hull \( \hat{K}_b \) of \( K \) with respect to homogeneous polynomials.

2.12. \( \alpha(E) > 0 \) iff \( \log \Phi^*_E \in H \); \( \varrho(E) > 0 \) iff \( \Psi^*_E \in H \).

2.13. If \( \alpha(A) = 0 \) (resp. \( \varrho(A) = 0 \)), then \( \alpha(E \cup A) = \alpha(E) \) (resp. \( \varrho(E \cup A) = \varrho(E) \)).

2.14. \( \Phi(z, E_b) \equiv \Phi(z, E_c) \equiv \max \{ 1, \Psi(z, E) \}, \quad \alpha(E_b) = \alpha(E_c) = \min \{ 1, \varrho(E) \} \),

where \( E_b = \{ \lambda z; \lambda \in C, |\lambda| \leq 1, z \in E \}, \quad E_c = \{ \lambda z; \lambda \in C, |\lambda| = 1, z \in E \} \).

2.15. Given a subset \( E \) of \( C^N \) let \( 1 \times E \) be the subset of \( C \times C^N \) defined by \( 1 \times E: = \{(1, z); z \in E \} \). Then

\[
\Phi_E(z) = \Psi_{1 \times E}(1, z) \quad \text{in} \quad C^N \quad \text{and} \quad m\varrho(1 \times E) \leq \alpha(E) \leq M\varrho(1 \times E),
\]
m and \( M \) being positive constants depending only on the fixed norms in \( C^N \) and \( C^{N+1} \) but not on \( E \). Remember that \( \varrho(1 \times E) \) depends on the norm in \( C^{N+1} \) and \( \varrho(E) \) on the norm in \( C^N \).

2.16. If \( \beta \) is the set function defined in 2.3, then

\[
m\alpha(E) \leq m\varrho(1 \times E) \leq \beta(E) \leq \varrho(1 \times E) \leq \alpha(E), \quad E \subseteq B(0, R),
\]

where

\[
m = m(R): = \sup \{ \Psi_{\varrho(E)}(1, z); \|z\| \leq R \}.
\]

3. **A solution to Problem 1**

The aim of this section is to prove the following theorem.

**Theorem 3.1.** (i) If \( E \) is a subset of \( C^N \) such that the complex cone \( C \cdot E \) is not pluripolar, then each series of homogeneous polynomials (1.1) convergent \( \varrho \)-a.e. on \( E \) converges locally uniformly in the domain

\[
\Omega = \{ z \in C^N; |\Psi^*(z, E)| < 1 \}
\]

and consequently in the ball \( \|z\| < \varrho(E) \).

(ii) If \( E \) is any \( F_\sigma \) set in \( C^N \), then the following conditions are equivalent:

1. \( \varrho(E) > 0 \) (i.e. \( C \cdot E \) is not plp);
2. \( E \) has Abel's Property;
3. \( E \) has Weak Abel's Property.
Conditions (1) and (2) are equivalent for all sets in \( C^N \).

(iii) A subset \( E \) of \( C^N \) has Weak Abel's Property if and only if there is no \( F_\sigma \) complex cone \( F \) containing \( E \) with \( \varrho(F) = 0 \).

(iv) If \( E \) is a subset of \( C^N \) such that the cone \( C \cdot E \) is a \( G_\delta \) set dense in an open non-empty subset \( \Omega \) of \( C^N \), then \( E \) has Weak Abel's Property. In particular there are pluripolar subsets \( E \) of \( C^N \) \((N \geq 2)\) with Weak Abel's Property.

Proof. (i) Put \( E_j = \{ z \in E; |Q_k(z)| \leq j, k \geq 0, |z| \leq j \} \). Then \( E_j \subset E_{j+1} \) and \( \bigcup_{j=1}^{\infty} E_j = E \setminus A \) with \( \varrho(A) = 0 \). By 2.9, \( \Psi^*(z, E_j) \downarrow \Psi^*(z, E) \) in \( C^N \). Let \( K \) be any fixed compact subset of \( \Omega \). Put \( \theta_1 := \max_k \Psi^*(z, E_j) \). Take \( \theta \) with \( \theta_1 < \theta < 1 \). Then by Dini's argument
\[ \Psi^*(z, E_j) < \theta, \quad z \in K, \ j > j_0. \]

Hence by 2.8
\[ |Q_j(z)| \leq j(\Psi(z, E_j)^v \leq j^{\theta^v}, \quad z \in K, \ j > j_0, \ v > 0, \]
which shows that the series \( \sum Q_j \) is uniformly convergent on \( K \).

(ii) By (i), the implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are always true. Suppose now \( E \) is an \( F_\sigma \) set with \( \varrho(E) = 0 \). Then we may assume \( E = \bigcup K_j \), where \( K_j \subset K_{j+1} \) are compact sets. Moreover, there exists a plsh absolutely homogeneous function \( h \) on \( C^N \) such that \( h = 0 \) on \( E \) and \( h \neq 0 \). It is known that \( h \) can be written in the form
\[ h = v^* \quad \text{with} \quad v = \limsup_{n \to \infty} |Q_n|^{1/n}, \]
\( Q_n \) being a homogeneous polynomial of degree \( n \) and \( v^* \) denoting the upper semicontinuous regularization of \( v \). It is clear that
\[ \lim_{n \to \infty} |Q_n(z)|^{1/n} = 0 \quad \text{for all} \quad z \in E. \]

Let \( a \) be a point of \( C^N \) with \( \limsup_{n \to \infty} |Q_n(a)|^{1/n} \neq 0 \). Let \( \{n_j\} \) be an increasing sequence of positive integers such that \( |Q_{n_j}(a)|^{1/n_j} \geq m = \text{const} > 0, \ j \geq 1 \). By the Hartogs Lemma there exists an increasing sequence of positive integers \( \{j_s\} \) such that
\[ |Q_{n_{j_s}}(z)|^{1/n_{j_s}} \leq s^{-2} \quad \text{on} \quad K_s \quad \text{for all} \quad s \geq 1. \]

Put \( d_s := n_{j_s} \). The function
\[ P_s(z) = s^{d_s}Q_{d_s}(z)/Q_{d_s}(a), \quad z \in C^N \]
is a homogeneous polynomial of degree \( d_s \). For every positive integer \( t \) the series \( \sum P_s \) is uniformly convergent on \( K_t \), because
\[ \|P_s\|_{K_t}^{1/d_s} \leq s^{-1}/|Q_{d_s}(a)|^{1/d_s} \leq 1/sm, \quad s \geq t. \]
On the other hand, the series $\sum_{1}^{\infty} P_s$ is divergent at every point $t^{-1}a$, $t = 1, 2, \ldots$, because $P_s(t^{-1}a) = (s/t)^{d_s}$ for all $s \geq 1$. Therefore our series has zero radius of convergence. So we have proved that for $F_\sigma$ sets $E$ one has (3) $\Rightarrow$ (1).

Suppose now $E$ is a set in $C^N$ with $\varrho(E) = 0$ and let $h$ be a function given by (+) with $h = 0$ on $E$, $h \neq 0$. For every $r > 0$ there is a point $a$ in $C^N$ such that $|a| < r$ and $h(a) = \limsup_{n \to \infty} |Q_n(a)|^{1/n} > 0$. Without loss of generality we may assume $h(a) = 1$. The series $\sum_{n=0}^{\infty} Q_n$ converges at each point of $C \cdot E$ and diverges at each point $z = \lambda a$ with $\lambda \in C$, $|\lambda| > 1$. Therefore $E$ does not satisfy (2). We have thus proved that (2) $\Rightarrow$ (1) for every set $E$ in $C^N$.

(iii) If there is an $F_\sigma$ complex cone $F$ with $E \subset F$, $\varrho(F) = 0$, then by (ii) $E$ has not Weak Abel's Property.

Suppose now $\varrho(F) > 0$ for each $F_\sigma$ complex cone $F$ containing $E$. Given a series (1.1) converging on $E$, put $\Psi(z) = \sup_{n \geq 1} |Q_n(z)|^{1/n}$ and $F = \{z \in C^N; \Psi(z) < +\infty\}$. Then $F$ is an $F_\sigma$ cone containing $E$. Hence $\varrho(F) > 0$. The set $F_j = \{z \in C^N; \Psi(z) \leq j, |z| \leq j\}$ is compact and $\varrho(F_j)\uparrow \varrho(F)$. Take $j$ so large that $\varrho(F_j) > 0$. Then for all $v \geq 1$

$$|Q_n(z)| \leq (j \|z\|/\varrho(F_j))^v \leq 2^{-v^v} \quad \text{if} \quad \|z\| \leq (2j)^{-1} \varrho(F),$$

which implies that series (1.1) converging on $E$ has a positive radius of convergence. The proof of (iii) is concluded.

(iv) Given any series $f = \sum_{n=0}^{\infty} Q_n$ of homogeneous polynomials converging on $E$, the complex cone $F = \{\Psi(z) < +\infty\}$, where $\Psi = \sup_{n \geq 1} |Q_n|^{1/n}$ is of type $F_\sigma$ and $C \cdot E \subset F$. Since $C \cdot E$ is dense in an open non-empty set $\Omega \subset C^N$ with Baire property, it follows that $\Psi$ is bounded in a neighbourhood of 0 in $C^N$. Hence the series $f$ is convergent in a neighbourhood of the origin.

Example. Let $A = \{a_j\}$ be a countable dense subset of $C^N$ ($N \geq 2$). By 2.10, $\varrho(C \cdot A) = 0$, and by 2.7, there is an absolutely homogeneous plsh function $h$ with $h = 0$ on $C \cdot A$, $h \neq 0$. The cone $E = \{h(z) = 0\}$ is a $G_\delta$ dense pluripolar set in $C^N$. By (iv) $E$ has Weak Abel's Property. By (ii) the set $E$ has not Abel's Property (because $\varrho(E) = 0$). The proof of Theorem 3.1 is finished.

Remark 3.2. Statements (i) and (ii) of Theorem 3.1 are known ([10]). We have inserted them here to get a "round" theorem and because lecture notes [10] are not easily available.

Corollary 3.3 (from (i), (ii) and 2.11). A compact set $K$ in $C^N$ has Abel's Property if and only if its convex hull $K_\delta$ with respect to homogeneous polynomials of $N$ complex variables has a non-empty interior. If a series of
homogeneous polynomials converges pointwise on \( K \), then it converges locally uniformly in the interior of \( \tilde{K}_b \).

4. Convergence sets of \( N \)-tuple power series

We say that an \( N \)-tuple power series

\[
P(z) = \sum_{\mathbf{z}} c_{\mathbf{z}} z^\mathbf{z} \quad (\mathbf{z} \in \mathbb{Z}_+^N, \ z \in \mathbb{C}^N)
\]

is convergent if there are two positive real numbers \( r \) and \( M \) such that \( |c_{\mathbf{z}}| |z|^{\alpha} \leq M \) \( (\mathbf{z} \in \mathbb{Z}_+^N, \ |\alpha| = \alpha_1 + \ldots + \alpha_N) \), otherwise the series is called divergent. It is well known that a power series \( P \) is convergent if and only if the series \( f = \sum_{n=0}^{\infty} Q_n \) of homogeneous polynomials \( Q_n(z) = \sum_{|\alpha| = n} c_{\alpha} z^\alpha \) converges in a neighbourhood of 0 in \( \mathbb{C}^N \).

Let \( G(k, N) \) denote the Grassmann manifold of all \( k \)-dimensional subspaces \( V \) of \( \mathbb{C}^N \). We say that a family \( E \subset G(k, N) \) of \( k \)-dimensional subspaces of \( \mathbb{C}^N \) has Abel's Property if there is a positive number \( R = R(E) \) such that every \( N \)-tuple power series \( P \) with the property that \( P|V \) is a convergent \( k \)-tuple power series for every \( V \in E \), is absolutely convergent in the polydisk \( |z_j| < R \) (\( j = 1, \ldots, N \)).

We say that \( E \subset G(k, N) \) has Weak Abel's Property if every \( N \)-tuple power series \( P \) such that \( P|V \) is convergent for every \( V \in E \) is convergent. In the last property the radius of the absolute convergence of \( P \) may depend on \( P \).

It is clear that Abel's Property implies Weak Abel's Property. We shall see that the inverse implication is not true.

Given \( E \subset G(k, N) \), denote by \( \tilde{E} = \bigcup_{V \in E} V \) the union of all \( k \)-dimensional subspaces \( V \) of \( \mathbb{C}^N \) belonging to \( E \). The set \( \tilde{E} \) is a complex cone in \( \mathbb{C}^N \). The set \( \tilde{E} \) can be treated in a canonical way as a subset of \( G(1, N) \) — the set of complex vector lines in \( \mathbb{C}^N \).

As a corollary from Theorem 3.1 one gets the following slight improvement of results due to Levenberg and Molzon [5].

**Theorem 4.1.** (i) A subset \( E \) of \( G(k, N) \) has Abel's Property if and only if the corresponding complex cone \( \tilde{E} \) is not pluripolar.

(ii) A subset \( E \) of \( G(k, N) \) has Weak Abel's Property if and only if \( \tilde{E} \) is not contained in any \( F_\delta \) pluripolar complex cone \( F \).

(iii) If \( E \subset G(k, N) \) is a family of \( k \)-dimensional subspaces of \( \mathbb{C}^N \) such that the corresponding cone \( \tilde{E} \) is a dense \( G_\delta \) subset of an open non-empty subset \( \Omega \) of \( \mathbb{C}^N \), then \( E \) has Weak Abel's Property. Moreover, if \( \tilde{E} \) is pluripolar, then \( E \) has Weak Abel's Property but it has not Abel's Property.

5. A solution to Problem 2

In this chapter we shall prove two theorems.
Theorem 5.1. Let \( \{n_j\} \) be an increasing sequence of positive integers with finite rate of convergence \( \chi := \limsup_{j \to \infty} n_{j+1}/n_j \). Let \( E \) be a subset of \( \mathbb{C}^N \) with \( \alpha(E) > 0 \). Let \( S_n := Q_0 + \ldots + Q_n \) be the \( n \)-th partial sum of a series of homogeneous polynomials

\[
f = \sum_{\nu=0}^{\infty} Q_\nu \quad \text{with} \quad Q_\nu(z) = \sum_{|x| = \nu} c_\nu z^x \quad (x \in \mathbb{Z}_+^N).
\]

Then the following statements are true:

(a) If the sequence \( \{S_{n_j+1} - S_{n_j}\} \) is bounded \( \alpha \)-a.e. on \( E \), then \( f \) converges in the ball

\[
\|z\| < \alpha(E)^{\chi}.
\]

(b) If

\[
M := \limsup_{j \to \infty} \|S_{n_{j+1}} - S_{n_j}\|_E^{1/n_{j+1}}
\]

is finite, then the series \( f \) converges in the ball

\[
\|z\| < \min \{\alpha(E)/M, \ (\alpha(E)/M)^\chi\}.
\]

(c) If

\[
\sup_{j \geq 1} |S_{n_{j+1}}(z) - S_{n_j}(z)|^{1/n_{j+1}}
\]

is finite \( \alpha \)-a.e. on \( E \), then the series \( f \) has a positive radius of convergence.

(d) If

\[
\limsup_{j \to \infty} |S_{n_{j+1}}(z) - S_{n_j}(z)|^{1/n_{j+1}} = 0 \quad \alpha \text{-a.e. on } E,
\]

then the series \( f \) converges locally uniformly in the whole space \( \mathbb{C}^N \) (i.e. the series represents an entire function of \( N \) complex variables).

Proof. (a) \( E_l := \{z \in \mathbb{C}^N; |S_{n_{j+1}}(z) - S_{n_j}(z)| \leq l, j \geq 1, |z| \leq l\} \) is a compact set, \( E_l \subset E_{l+1} \) and \( \alpha(F) \geq \alpha(E) \), where \( F := \bigcup_{l=1}^{\infty} E_l \). By 2.8,

\[
|S_{n_{j+1}}(z) - S_{n_j}(z)| \leq l \alpha(E_l)^{-n_{j+1}} \quad \text{if} \quad \|z\| \leq 1.
\]

Hence by the Cauchy inequalities

\[
|Q_l(z)| \leq l \|z\|^l/\alpha(E_l)^{n_{j+1}}, \quad n_{j+1} + 1 \leq l \leq n_{j+1}, \ z \in \mathbb{C}^N.
\]

Therefore

\[
\limsup_{l \to \infty} |Q_l(z)|^{1/l} \leq \|z\|/\alpha(E)^{\chi} \quad \text{in } \mathbb{C}^N,
\]

which ends the proof of (a).
(b) Given \( \varepsilon > 0 \) one has \( \|S_{n_j+1} - S_{n_j}\|^{1/n_j+1} \leq M + \varepsilon, \ j > j_\varepsilon \). Hence, via 2.8 and the Cauchy inequalities,

\[
|Q_l(z)| \leq \left( \frac{M + \varepsilon}{\alpha(E)} \right)^{n_j+1} \|z\|^l, \quad n_j+1 \leq l \leq n_{j+1}, \quad j > j_\varepsilon,
\]

which implies

\[
\limsup_{l \to \infty} |Q_l(z)|^{1/l} \leq \begin{cases} 
\left( \frac{M}{\alpha(E)} \right)^{n_j+1} \|z\| & \text{if } M \geq \alpha(E), \\
\left( \frac{M}{\alpha(E)} \right) \|z\| & \text{if } M < \alpha(E).
\end{cases}
\]

This concludes the proof of (b).

(c) The set \( E_{i,j} = \{ z \in \mathbb{C}^N; \|S_{n_{i,j}}(z) - S_{n_j}(z)\|^{1/n_j+1} \leq l, \ i \geq 1, \ \|z\| \leq l \} \) is compact, \( E_i \subset E_{i+1} \) and \( E_i \uparrow F \) with \( \alpha(F) \geq \alpha(E) \). Take \( l \) so large that \( \alpha(E_l) > \varepsilon \). Then by (b) the series \( f \) is convergent at least in the ball

\[
\|z\| < \min \{ \alpha(E_l)/l, \ (\alpha(E_l)/l)^* \}.
\]

(d) Given \( \varepsilon > 0 \), the set \( E_{i,\varepsilon} = \{ z \in \mathbb{C}^N; \|S_{n_{i,\varepsilon}}(z) - S_{n_j}(z)\|^{1/n_j+1} \leq \varepsilon, \ i \geq l, \ \|z\| \leq l \} \) is compact and \( E_{i,\varepsilon} \uparrow F \) as \( l \to \infty \), where \( \alpha(F) \geq \alpha(E) > 0 \). Take \( l \) so large that \( \alpha(E_l) > \alpha(E)/2 \). Then by (b) the series \( f \) is convergent in the ball

\[
\|z\| < \min \{ \alpha(E)/2\varepsilon, \ (\alpha(E)/2\varepsilon)^* \}.
\]

**Theorem 5.2.** Given \( \{n_j\} \) and a compact polynomially convex set \( K \) in \( \mathbb{C}^N \), assume that at least one of the following two conditions (a) or (b) is satisfied:

- (a) \( \lambda > 1, \ \alpha(K) = 0 \);
- (b) \( \lambda = +\infty, \ 0 \notin \text{int} K \).

Then there exists a series \( f \) such that the sequence \( \{S_{n_j}\} \) is uniformly convergent on \( K \) but it has zero radius of convergence.

**Proof.** Let \( J \) be an infinite subset of \( N \) such that

(i) \( n_{j+1} - n_j < n_{k+1} - n_k \) if \( j, k \in J, \ j < k \);

(ii) \( \lambda = \lim n_{j+1}/n_j, \ j \to \infty, \ j \in J \).

Let \( \Phi_K \) denote the extremal function defined in 2.1. It is known [1] that the set \( \{ \Phi_K(z) < \Phi_K^*(z) \} \) is pluripolar. Let \( A = \{ a_j; j \in J \} \) be a dense subset of \( \mathbb{C}^N \setminus K \), where each point \( a \) of \( A \) is repeated infinitely many times in the family \( (a_j)_{j \in J} \) and \( \Phi_K^*(a) = \Phi_K(a) \) for every \( a \in A \).

For a fixed \( j \in J \) let \( P_j \) be a polynomial of degree \( \leq n_{j+1} - n_j - 1 \) such that \( \|P_j\|_K \leq 1 \) and \( |P_j(a)| = \Phi_{n_{j+1} - n_j}(a), \) where \( \Phi_n \) is defined by 2.4. Put \( M := \sup \{ |z|; \ z \in K \} \) and

\[
S_{n_{j+1}}(z) - S_{n_j}(z) = \begin{cases} 
\frac{1}{1-j} P_j(z)(z/M)^{n_j+1} & \text{if } j \in J, \\
0 & \text{if } j \notin J.
\end{cases}
\]

It is clear that, for every \( j \in N \), \( S_{n_{j+1}} - S_{n_j} = Q_{n_{j+1}} + \ldots + Q_{n_j} \), where \( Q_{n_{j+k}} \) is a homogeneous polynomial of degree \( n_j + k, \ k = 1, \ldots, n_{j+1} - n_j \).

Let \( \sum_{n=0}^\infty Q_n \) be the series of homogeneous polynomials whose partial sums \( S_{n_j} \).
are uniquely determined by equations (9). We claim that it is the required series. Indeed, if follows from (9) that

$$|S_{n_j+1}(z) - S_{n_j}(z)| \leq j^{-2}, \quad j \in \mathbb{N}, \quad z \in K,$$

which implies that the sequence $\{S_{n_j}\}$ is uniformly convergent on $K$.

Let $w = (w_1, \ldots, w_N)$ be a fixed point in $A$ with $w_1 \neq 0$. Observe that

$$|S_{n_j+1}(z) - S_{n_j}(z)| \leq j^{-2} \Phi_{n_j+1-n_j-1}(z)|z_1/M|^{n_j+1} =: g_j(z)^{n_j+1-n_j-1}.$$

If (a) is satisfied, then

$$\lim_{j \to \infty, j \in J} g_j(w) = \Phi_K(w)(|w_1|/M)^{1/(\kappa-1)} = +\infty,$$

because $\Phi_K(w) = +\infty$. Hence $S_{n_j+1}(w) - S_{n_j}(w) \to \infty$. So the series $\sum Q_n$ diverges at each point $w \in A$ with $w_1 \neq 0$. Since $K$ is pluripolar it follows that the series diverges in a dense subset of $C^N$.

If (b) is satisfied, then $\lim_{j \to \infty, j \in J} g_j(w) = \Phi_K(w) > 1$. Hence again $S_{n_j+1}(w) - S_{n_j}(w) \to \infty$ as $j \to \infty$ $(j \in J)$. So the series $\sum Q_n$ diverges at each point $w \in A$ with $w_1 \neq 0$. Since $0 \notin \text{int} \ K$, it follows that every ball $B(0, r)$ contains a point $w$ of $A$ with $w_1 \neq 0$, which shows that the series has zero radius of convergence.

Remark. If $N = 1$, then part (b) of Theorem 5.2 and a weaker version of (a) in Theorem 5.1 are due to Naftalewich, whose paper [6] inspired the author to study Problem 2 in $C^N, N > 1$.

6. Examples

Example 1. Let $p$ be a positive integer. Let $S_n$ denote the $n$-th partial sum of a power series

$$f(z) = \sum c_n z^n, \quad z \in C,$$

on the complex plane and let $\{n_j\}$ be an increasing sequence of positive integers.

(a) If $n_{j+1} - n_j \leq p$ $(j \geq 1)$ and the sequence $\{S_{n_j+1} - S_{n_j}\}$ is bounded at each of the points $z_1, \ldots, z_p$ of $C \setminus \{0\}$ with $z_j \neq z_k$ $(j \neq k)$, then $f$ converges in the disk $|z| < r := \min \{|z_1|, \ldots, |z_p|\}$. In particular, if $\{S_{n_j}\}$ is bounded at each of the points $z_1, \ldots, z_p$, then $f$ converges in the disk $|z| < \min \{|z_1|, \ldots, |z_p|\}$.

(b) If $n_{j+1} - n_j > p$ $(j \geq 1)$ and $z_1, \ldots, z_p$ are arbitrary points of $C$, then there is a power series $f$ such that sequence of its partial sums $\{S_{n_j}\}$ converges at each of the points $z_1, \ldots, z_p$ but $f$ has zero radius of convergence.

Statement (a) is a direct consequence of Theorem 3.1 (i). An independent proof in this case is very simple. Namely, observe that

$$S_{n_j+1}(z) - S_{n_j}(z) = z^{n_j+1} \left[ c_{n_j+1} z^{-1} + c_{n_j+2} z^{-2} + \cdots + c_{n_j+1} z^{n_j+1-n_j-1} \right] = z^{n_j+1} P_j(z),$$
where \( \deg P_j \leq p - 1 \). Hence by the Lagrange Interpolation Formula with nodes \( z_1, \ldots, z_p \) one gets

\[
|P_j(z)| \leq Mr^{-n_j-1}, \quad |z| \leq 1, \quad j \geq 1,
\]

where \( r = \min \{|z_1|, \ldots, |z_p|\} \) and \( M \) is a positive constant that depends neither on \( z \) nor on \( j \). Hence, by the Cauchy inequalities,

\[
|c_k|^{1/k} \leq M^{1/k} \begin{cases} r^{-1} & \text{if } r < 1, \\ r^{-(n_j+1)/n_j+1} & \text{if } r \geq 1, \end{cases}
\]

for all \( k \) with \( n_j < k < n_{j+1} \). Therefore, \( \limsup_{k \to \infty} |c_k|^{1/k} \leq r^{-1} \).

In order to show (\( \beta \)) take \( P(z) = (z-z_1) \ldots (z-z_p) \) and define

\[
+(+) \quad S_{n_j+1}-S_{n_j} = (xz_j + 1)P(z) = c_{n_j+1}z^{n_j+1} + \ldots + c_{n_j+1}z^{n_j+1}.
\]

It is clear that \( S_{n_j+1} + \sum_{j=1}^{\infty} (S_{n_{j+1}} - S_{n_j}) \) converges uniformly on \( A = \{z_1, \ldots, z_p\} \) but diverges at each point \( z \notin (A \cup \{0\}) \). Hence the power series \( f \) with coefficients given by (\( + \)) converges only at 0.

**Example 2.** Let \( n_j = j^2, \quad E = \{1+1/k; \quad k \geq 1\} \cup \{1\}, \) and \( F = \{1+2^{-2k}; \quad k \geq 1\} \cup \{1\} \).

(i) If the sequence of partial sums \( \{S_{n_j}\} \) of a power series \( f \) is uniformly convergent on \( E \), then \( f \) converges in the disk \(|z| < 1\).

(ii) There is a power series \( f \) such that the sequence of its partial sums \( \{S_{n_j}\} \) is uniformly convergent on \( F \) but \( f \) converges only at 0.

**Proof.** (i) Put \( P_j = S_{(j+1)^2} - S_j = z^{j^2+1}Q_j \). Then \( \deg Q_j \leq 2j \) and

\[
\|Q_j\|_E \leq \|P_j\|_E \leq M = \text{const}. \quad \text{Put} \quad L_s(z) = \prod_{k=1, k \neq s}^{2j+1} (z-z_k)/(z_s-z_k), \quad \text{where} \quad z_k = 1+1/k. \quad \text{Then} \quad Q_j(z) = \sum_{s=1}^{2j+1} Q_j(z_s)L_s(z). \quad \text{It is clear that} \quad |z_s-z_k| \geq 1/2j(2j+1), \quad \text{and} \quad |z-z_k| \leq 3 \text{ for } z \text{ in } D = \{|z| < 1\}. \quad \text{Hence} \quad \|Q_j\|_D \leq M_j = (2j+1)M3^{2j}[2j \times (2j+1)]^{-2j}. \quad \text{Therefore,} \quad \|P_j\|_D \leq M_j. \quad \text{Hence by the Cauchy inequalities one gets}
\]

\[
|c_s|^{1/s} \leq M_j^{1/s} \leq M_j^{1/2^j} \quad \text{if} \quad j^2 < s < (j+1)^2.
\]

Therefore, \( \limsup_{s \to \infty} |c_s|^{1/s} \leq 1 \).

(ii) Let \( Q_j(z) = (z-x_1) \ldots (z-x_{2j}) \) be a polynomial of degree \( 2j \) such that \( x_1, \ldots, x_{2j} \in F \) and \( \|Q_j\|_F \leq \max_{x \in F} |(z-z_1) \ldots (z-z_{2j})| \) for all \( z_1, \ldots, z_{2j} \in F \). Then

\[
\|Q_j\|_F \leq (1-y_{2j})(1-y_{2j-1}) \ldots (1-y_1) = 2^{-(2+2^2+\ldots+2^{2j})},
\]

where \( y_s = 1+2^{-2s} \). Therefore
\[ \lim_{j \to \infty} \|Q_j\|_F^{1/(1+j)^2} = 0. \]

Put

\[ S_{j+1} - S_j := \frac{1}{j^3} \left( \frac{z}{2} \right)^{j^2+1} Q_j(z)/\|Q_j\|_F. \]

Then the sequence \( \{S_j\} \) is uniformly convergent on \( F \). But for \( z \) in \( C \cap (F \cup \{0\}) \) we have

\[ |S_{j+1} - S_j| \leq \frac{1}{j^3} \left( \frac{r}{2} \right)^{j^2+1} \frac{r^{2j}}{\|Q_j\|_F} \quad \text{with} \quad r = \text{dist}(z, F), \]

which by (1) implies \( \lim_{j \to \infty} |S_{j+1} - S_j(z)| = +\infty \).

Theorem 5.2 does not cover the case of \( \kappa = 1 \) and \( \kappa(K) = 0 \). Example 2 shows that in this case the statement of Theorem 5.2 is false.

**Example 3.** Assume \( \kappa = 1 \) and \( n_{j+1} - n_j \leq n_{j+2} - n_{j-1} \) (i.e. \( \{n_j\} \) is convex), e.g. \( n_j = j^2 \). Then for every countable set \( E = \{z_p; p \geq 1\} \subset C \) there is a power series \( f \) such that \( \{S_{n_j}\} \) converges at each point of \( E \) but the series converges only at 0.

**Proof.** Put \( S_{n_1} \equiv 0 \) and

\[ S_{n_{j+1}}(z) - S_{n_j}(z) := a_j z^{n_{j+1}+1}Q_j(z) = c_{n_{j+1}} z^{n_{j+1}} + \ldots + c_{n_j} z^{n_j}, \]

where \( Q_j(z) := (z-z_1) \ldots (z-z_{n_{j+1}}-n_{j-1}) \) and \( a_j := 1/\max_{|z|=1/j} |z^{n_j+1}Q_j(z)|. \)

Then

\[ \lim_{j \to \infty} S_{n_j}(z_p) = \sum_{k=1}^{m_p} a_k z_p^{n_k+1}Q_k(z_p) = \sum_{s=0}^{\infty} c_s z_p^s, \quad p > 1, \]

where \( c_s \) are given by (1) and \( m_p := \min \{n_{j+1} - n_j - 1; n_{j+1} - n_j - 1 > p\} \). If the power series \( \sum c_s z^s \) were convergent in a neighbourhood of 0, then the series

\[ \sum_{j=1}^{\infty} \max_{|z|=r} |S_{n_{j+1}}(z) - S_{n_j}(z)| \]

would be convergent for all sufficiently small \( r > 0 \). But this is impossible because

\[ \max_{|z|=r} |S_{n_{j+1}}(z) - S_{n_j}(z)| = \max_{|z|=r} |z^{n_{j+1}+1}Q_j(z)|/\max_{|z|=r} |z^{n_j+1}Q_j(z)| \geq 1 \quad \text{for} \quad j > 1. \]

**Example 4.** Let \( S = \{x \in \mathbb{R}^N; |x| = 1\} \) be the Euclidean unit sphere in \( \mathbb{R}^N \). It is known [2] that \( \Psi(z, S) = (|x|^2 + |y|^2 + 2\sqrt{|x|^2|y|^2 - \langle x, y \rangle^2})^{1/2} \) for all \( z = x + iy \) in \( C^N \), so that \( \hat{S}_b = \{\Psi(z, S) \leq 1\} \) is the unit Lie ball in \( C^N \). By Theorem 3.1 (i), if \( \sum \Phi_n \) is any series of homogeneous polynomials converging
on $S$, then it converges in the unit Lie ball. In particular, every function $u$ harmonic in the unit Euclidean ball in $\mathbb{R}^N$ can be extended to a holomorphic function $\tilde{u}$ of $N$ complex variables in the unit Lie ball in $\mathbb{C}^N$.

References


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