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On series of homogeneous polynomials and their partial sums

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Abstract. Let $\sum_{\nu=0}^{\infty} Q_\nu$ be a series of homogeneous polynomials on C^N with $Q_\nu(z) = \sum_{|\alpha|=\nu} c_\alpha z^\alpha$, $\alpha \in Z_+^N$. Let $S_n := Q_0 + \ldots + Q_n$ be the n-th sum of the series, E a subset of C^N, and $\{n_j\}$ an increasing sequence of positive integers. We say that a pair $(E, \{n_j\})$ has A-Property if each series $\sum_0^\infty Q_\nu(z)$ with $\{S_{n_j}(z)\}$ converging for every z in E has a positive radius of convergence. The paper delivers a characterization of pairs $(E, \{n_j\})$ with A-Property in terms of global extremal plurisubharmonic functions and of related capacities as well as of the rate of convergence $\varkappa := \limsup_{j\to\infty} n_{j+1}/n_j$ of the sequence $\{n_j\}$. In particular, it is shown that if $\varkappa$ is finite and $\{S_{n_j}(z)\}$ converges at each point z of a nonpluripolar set E in C^N, then the series converges in the ball $\|z\| < \alpha^\varkappa$, where $\alpha = \alpha(E)$ is a capacity of E.

1. Introduction

Let

$$(1.1)\qquad \sum_{\nu=0}^{\infty} Q_\nu(z) \quad \text{with} \quad Q_\nu(z) = \sum_{|\alpha|=\nu} c_\alpha z^\alpha, \quad z \in C^N$$

be a series of homogeneous polynomials of N complex variables. If $N = 1$ then (1.1) is a power series,

$$(1.2)\qquad \sum_0^\infty c_\nu z^\nu, \quad z \in C.$$

It is the classical result due to Abel that if (1.2) converges at a point $z_0 \neq 0$ then it converges in the disk $|z| < r = |z_0|$. About 60 years ago F. Leja observed that, given any countable set E in C^2, one can construct a series (1.1) convergent on E but having no positive radius of convergence.

DEFINITION 1.1. We say that a subset E of C^N has *Abel's Property* if there is a positive number $R = R(E)$ such that every series (1.1) converging pointwise on E converges in the ball $|z| < R$.

DEFINITION 1.2. We say that a subset E of C^N has *Weak Abel's Property* if every series (1.1) converging on E has a positive radius of convergence.

PROBLEM 1 (due to Leja). Characterize subsets E of C^N with Abel's Property (resp. with Weak Abel's Property).

DEFINITION 1.3. Let E be a subset of C^N and let $\{n_j\}$ be an increasing sequence of positive integers. Let $S_n := Q_0 + \ldots + Q_n$ be the n-th partial sum of series (1.1). We say that a pair $(E, \{n_j\})$ has *A-Property* if each series (1.1) with the subsequence $\{S_{n_j}\}$ converging on E has a positive radius of convergence.

PROBLEM 2. Characterize pairs $(E, \{n_j\})$ with A-Property.

Problem 2 reduces to Problem 1 by taking $n_j = j$, $j \in N$.

Problem 1 for compact sets K in C^2 was solved by Leja ([3], [4]) in terms of his "triangular" transfinite diameter. His method does not work in C^N for $N \geqslant 3$, because it is based on the well-known fact that each homogeneous polynomial of two variables can be written as a product of linear factors. Such a property is no more true for homogeneous polynomials of N variables with $N \geqslant 3$. A partial solution to Problem 1 in C^N ($N \geqslant 2$) was given in [7] (for compact subsets of C^N) and in [10] (for arbitrary subsets of C^N) in terms of the extremal homogeneous function Ψ_E associated with subsets E of C^N (see Chapter 2). Here we give a complete solution to Problem 1 in terms of Ψ_E (or equivalently, in terms of a projective capacity $\varrho(E)$ of E; see next chapter for the definitions). Problem 2 for the plane case has been already studied by Naftalevich [6]. Our solution to Problem 2 given in this paper contains results of [6] as special cases. We characterize pairs $(E, \{n_j\})$ with A-Property in terms of the rate of convergence of $\{n_j\}$,

$$\varkappa := \limsup N_{j+1}/n_j,$$

and of the capacity $\alpha(E)$ of E defined by 2.2 below.

2. Global extremal plurisubharmonic functions and capacities in C^N (Reminder of definitions and main properties)

Let $\mathscr{L}$ be the set of all plurisubharmonic (plsh) functions u in C^N with $\sup\{u(z) - \log(1+|z|);\ z \in C^N\} < +\infty$. Let $\mathscr{H}$ be the set of all functions h plurisubharmonic in C^N such that $h(\lambda z) = |\lambda| h(z)$, $\lambda \in C$, $z \in C^N$. It is well known that $\log h \in \mathscr{L}$ for every $h \in \mathscr{H}$, $h \neq 0$.

2.1. For every subset E of C^N, we define two extremal functions $\Phi_E(z) \equiv \Phi(z, E)$, $\Psi_E(z) \equiv \Psi(z, E)$ in C^N by the formulas

$$\Phi(z, E) := \sup\{\exp u(z);\ u \in \mathscr{L},\ u \leqslant 0 \text{ on } E\},$$
$$\Psi(z, E) := \sup\{h(z);\ h \in \mathscr{H},\ h \leqslant 1 \text{ on } E\}$$

if $E \subset \boldsymbol{C}^N$ is bounded; and

$$\Phi(z, E) := \inf\{\Phi(z, F);\ F \subset E,\ F \text{ is bounded}\},$$
$$\Psi(z, E) := \inf\{\Psi(z, F);\ F \subset E,\ F \text{ is bounded}\}$$

if E is arbitrary.

2.2. Let α and ϱ be set functions defined for all $E \subset \boldsymbol{C}^N$ by the formulas

$$\alpha(E) := 1/\sup\{\Phi(z, E);\ z \in B\}, \qquad \varrho(E) := 1/\sup\{\Psi(z, E);\ z \in B\},$$

where $B := \{z \in \boldsymbol{C}^N;\ \|z\| < 1\}$ is the unit ball with respect to a fixed norm $\|\cdot\|$ in $\boldsymbol{C}^N$.

2.3. Let β be a set function defined for all $E \subset \boldsymbol{C}^N$ by the formula

$$\beta(E) := \varrho(\varphi(E)),$$

where $\varphi(E) := S^{2N+1} \cap \boldsymbol{C}\cdot(\{1\} \times E)$ denotes the intersection of the cone $\boldsymbol{C}\cdot(\{1\} \times E) = \{(t, tz);\ t \in \boldsymbol{C},\ z \in E\}$ with the Euclidean unit sphere S^{2N+1} in $\boldsymbol{C}^{N+1}$.

Now we shall recall some of the properties of Φ, Ψ, α, ϱ and β (the proofs may be found in [1], [7]–[10]).

2.4. If K is a compact subset of $\boldsymbol{C}^N$, then

$$\Phi(z, K) = \sup_{n \geqslant 1} \Phi_n^{1/n}(z) = \lim_{n \to \infty} \Phi_n^{1/n}(z), \qquad \Psi(z, K) = \sup_{n \geqslant 1} \Psi_n^{1/n}(z) = \lim_{n \to \infty} \Psi_n^{1/n}(z),$$
$$\alpha(K) = \inf_{n \geqslant 1} \alpha_n^{1/n} = \lim_{n \to \infty} \alpha_n^{1/n}, \qquad \varrho(K) = \inf_{n \leqslant 1} \varrho_n^{1/n} = \lim_{n \to \infty} \varrho_n^{1/n},$$

where

$$\Phi_n(z) := \sup\{|P(z)|;\ P(z) = \sum_{|\alpha| \leqslant n} c_\alpha z^\alpha,\ \|P\|_K \leqslant 1\},$$
$$\Psi_n(z) := \sup\{|Q(z)|;\ Q(z) = \sum_{|\alpha| = n} c_\alpha z^\alpha,\ \|Q\|_K \leqslant 1\},$$
$$\alpha_n := \inf\{\|P\|_K;\ P(z) = \sum_{|\alpha| \leqslant n} c_\alpha z^\alpha,\ \|P\|_B \geqslant 1\},$$
$$\varrho_n := \inf\{\|Q\|_K;\ Q(z) = \sum_{|\alpha| = n} c_\alpha z^\alpha,\ \|Q\|_B \geqslant 1\}.$$

2.5. If K is a non-pluripolar subset of $\boldsymbol{C}^N$, then $G(z, K) := \log \Phi^*(z, K)$ (where Φ^* denotes the upper semicontinuous regularization of Φ) is the unique function u plurisubharmonic in $\boldsymbol{C}^N$ with the following properties:

(i) $c_1 + \log(1+|z|) \leqslant u(z) \leqslant c_2 + \log(1+|z|)$ in $\boldsymbol{C}^N$, where c_1, c_2 are real constants;

(ii) $(dd^c u)^N = 0$ in $\boldsymbol{C}^N \backslash K$, where $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$;

(iii) $u = 0$ on $K \backslash F$, where F is an F_σ pluripolar subset of K.

In particular, if $N = 1$ and $K \subset \boldsymbol{C}$ is not polar, then $G(z, K)$ is the Green function for $\boldsymbol{C} \backslash \hat{K}$ with pole at infinity.

2.6. Characterization of pluripolar sets in C^N. For a subset E of C^N, the following conditions are equivalent;

(a) E is globally pluripolar (i.e. there is a plsh function u in C^N with $u = -\infty$ on E);

(b) E is $\mathscr{L}$-polar, i.e. there is u in $\mathscr{L}$ with $u = -\infty$ on E;

(c) $\alpha(E) = 0$;

(d) $\Phi^*(z, E) \equiv +\infty$.

2.7. Characterization of circled pluripolar sets in C^N. For a subset E of C^N, the following conditions are equivalent:

(a) the complex cone $C \cdot E$ generated by E is pluripolar;

(b) $\varrho(E) = 0$;

(c) there is h in $\mathscr{H}$ with $h = 0$ on E, $h \neq 0$;

(d) $\Psi^*(z, E) \equiv +\infty$.

2.8. Basic inequalities for polynomials in C^N. If $P(z) = \sum_{|\alpha| \leqslant n} c_\alpha z^\alpha$, $Q(z) = \sum_{|\alpha| = n} c_\alpha z^\alpha$ and K is a compact subset of C^N, then

$$|P(z)| \leqslant \|P\|_K \Phi^n(z, K), \quad |P(z)| \leqslant \|P\|_K (\max\{1, \|z\|\}/\alpha(K))^n,$$
$$|Q(z)| \leqslant \|Q\|_K \Psi^n(z, K), \quad |Q(z)| \leqslant \|Q\|_K (\|z\|/\varrho(K))^n$$

for all z in C^n, where $\|P\|_K := \max_K |P(z)|$.

2.9. Let A, B, E, E_n be arbitrary sets and K, K_n compact sets in C^N. Then

I. $\Phi_A \leqslant \Phi_B$, $\Psi_A \leqslant \Psi_B$ if $B \subset A$,

II. $\Phi_{K_n} \uparrow \Phi_K$, $\Psi_{K_n} \uparrow \Psi_K$ if $K_{n+1} \subset K_n$, $K = \bigcap_1^\infty K_n$,

III. $\Phi^*_{E_n} \downarrow \Phi^*_E$, $\Psi^*_{E_n} \downarrow \Psi^*_E$ if $E_n \subset E_{n+1}$, $E = \bigcup_1^\infty E_n$,

IV. $\Phi^*_{E \cup A} = \Phi^*_E$ if A is pluripolar,
$\Psi^*_{E \cup A} = \Psi^*_E$ if $C \cdot A$ is pluripolar,

V. $\log \Phi^*_E \in \mathscr{L}$ if and only if E is not pluripolar,
$\log \Psi^*_E \in \mathscr{L}$ if and only if $C \cdot E$ is not pluripolar.

2.10. If c denotes any of the set functions α, β, or ϱ, then c is a Choquet capacity, i.e.

(i) $c(A) \leqslant c(B)$ if $A \subset B$,

(ii) $c(K_n) \downarrow c(K)$ if $K_n \downarrow K$,

(iii) $c(E_n) \uparrow c(E)$ if $E_n \uparrow E$,

where A, B, E, E_n are arbitrary and K, K_n are compact sets in C^N.

2.11. Geometric interpretation of the capacities α and ϱ. $1/\alpha(K)$ is equal to the *radius* R of the smallest level domain $\{z \in C^N;\ G(z, K) < \log R\}$ containing the unit ball B.

$\varrho(K) = \sup\{r \geqslant 0;\ r\bar{B} \subset \hat{K}_b\}$, where $\hat{K}_b := \{z \in C^N;\ |Q(z)| \leqslant \|Q\|_K$ for every homogeneous polynomial $Q\}$, i.e. $\varrho(K)$ is the radius of the maximal ball (with respect to the given norm) contained in the convex hull $\hat{K}_b$ of K with respect to homogeneous polynomials.

2.12. $\alpha(E) > 0$ iff $\log \Phi_E^* \in \mathscr{L}$; $\varrho(E) > 0$ iff $\Psi_E^* \in \mathscr{H}$.

2.13. If $\alpha(A) = 0$ (resp. $\varrho(A) = 0$), then $\alpha(E \cup A) = \alpha(E)$ (resp. $\varrho(E \cup A) = \varrho(E)$).

2.14. $\Phi(z, E_b) \equiv \Phi(z, E_c) \equiv \max\{1, \Psi(z, E)\}$, $\alpha(E_b) = \alpha(E_c) = \min\{1, \varrho(E)\}$, where $E_b := \{\lambda z;\ \lambda \in C,\ |\lambda| \leqslant 1,\ z \in E\}$, $E_c := \{\lambda z;\ \lambda \in C,\ |\lambda| = 1,\ z \in E\}$.

2.15. Given a subset E of C^N let $1 \times E$ be the subset of $C \times C^N$ defined by $1 \times E := \{(1, z);\ z \in E\}$. Then

$$\Phi_E(z) = \Psi_{1 \times E}(1, z) \quad \text{in } C^N \quad \text{and} \quad m\varrho(1 \times E) \leqslant \alpha(E) \leqslant M\varrho(1 \times E),$$

m and M being positive constants depending only on the fixed norms in C^N and C^{N+1} but not on E. Remember that $\varrho(1 \times E)$ depends on the norm in C^{N+1} and $\varrho(E)$ on the norm in C^N.

2.16. If β is the set function defined in 2.3, then

$$m\alpha(E) \leqslant m\varrho(1 \times E) \leqslant \beta(E) \leqslant \varrho(1 \times E) \leqslant \alpha(E), \qquad E \subset B(0, R),$$

where

$$m = m(R) := \sup\{\Psi_{\varphi(E)}(1, z);\ \|z\| \leqslant R\}.$$

3. A solution to Problem 1

The aim of this section is to prove the following theorem.

THEOREM 3.1. (i) *If E is a subset of C^N such that the complex cone $C \cdot E$ is not pluripolar, then each series of homogeneous polynomials* (1.1) *convergent ϱ-a.e. on E converges locally uniformly in the domain*

$$\Omega := \{z \in C^N;\ \Psi^*(z, E) < 1\}$$

and consequently in the ball $\|z\| < \varrho(E)$.

(ii) *If E is any F_σ set in C^N, then the following conditions are equivalent:*

(1) *$\varrho(E) > 0$ (i.e. $C \cdot E$ is not plp);*

(2) *E has Abel's Property;*

(3) *E has Weak Abel's Property.*

Conditions (1) *and* (2) *are equivalent for all sets in* C^N.

(iii) *A subset* E *of* C^N *has Weak Abel's Property if and only if there is no* F_σ *complex cone* F *containing* E *with* $\varrho(F) = 0$.

(iv) *If* E *is a subset of* C^N *such that the cone* $C \cdot E$ *is a* G_δ *set dense in an open non-empty subset* Ω *of* C^N, *then* E *has Weak Abel's Property. In particular there are pluripolar subsets* E *of* C^N $(N \geqslant 2)$ *with Weak Abel's Property.*

Proof. (i) Put $E_j := \{z \in E;\ |Q_k(z)| \leqslant j,\ k \geqslant 0,\ |z| \leqslant j\}$. Then $E_j \subset E_{j+1}$ and $\bigcup_1^\infty E_j = E \setminus A$ with $\varrho(A) = 0$. By 2.9, $\Psi^*(z, E_j) \downarrow \Psi^*(z, E)$ in C^N. Let K be any fixed compact subset of Ω. Put $\theta_1 := \max_K \Psi^*(z, E)$. Take θ with $\theta_1 < \theta < 1$. Then by Dini's argument

$$\Psi^*(z, E_j) < \theta, \quad z \in K,\ j > j_0.$$

Hence by 2.8

$$|Q_v(z)| \leqslant j\big(\Psi(z, E_j)\big)^v \leqslant j\theta^v, \quad z \in K,\ j > j_0,\ v \geqslant 0,$$

which shows that the series $\sum Q_v$ is uniformly convergent on K.

(ii) By (i), the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are always true. Suppose now E is an F_σ set with $\varrho(E) = 0$. Then we may assume $E = \bigcup K_j$, where $K_j \subset K_{j+1}$ are compact sets. Moreover, there exists a plsh absolutely homogeneous function h on C^N such that $h = 0$ on E and $h \neq 0$. It is known that h can be written in the form

$$(+) \qquad h = v^* \quad \text{with} \quad v = \limsup_{n \to \infty} |Q_n|^{1/n},$$

Q_n being a homogeneous polynomial of degree n and v^* denoting the upper semicontinuous regularization of v. It is clear that

$$\lim_{n \to \infty} |Q_n(z)|^{1/n} = 0 \quad \text{for all } z \text{ in } E.$$

Let a be a point of C^N with $\limsup_{n \to \infty} |Q_n(a)|^{1/n} \neq 0$. Let $\{n_j\}$ be an increasing sequence of positive integers such that $|Q_{n_j}(a)|^{1/n_j} \geqslant m = \text{const} > 0,\ j \geqslant 1$. By the Hartogs Lemma there exists an increasing sequence of positive integers $\{j_s\}$ such that

$$|Q_{n_{j_s}}(z)|^{1/n_{j_s}} \leqslant s^{-2} \quad \text{on } K_s \text{ for all } s \geqslant 1.$$

Put $d_s := n_{j_s}$. The function

$$P_s(z) := s^{d_s} Q_{d_s}(z)/Q_{d_s}(a), \quad z \in C^N$$

is a homogeneous polynomial of degree d_s. For every positive integer t the series $\sum_n^\infty P_s$ is uniformly convergent on K_t, because

$$\|P_s\|_{K_t}^{1/d_s} \leqslant s^{-1}/|Q_{d_s}(a)|^{1/d_s} \leqslant 1/sm, \quad s \geqslant t.$$

On the other hand, the series $\sum_1^\infty P_s$ is divergent at every point $t^{-1}a$, $t = 1, 2, \ldots$, because $P_s(t^{-1}a) = (s/t)^{d_s}$ for all $s \geqslant 1$. Therefore our series has zero radius of convergence. So we have proved that for F_σ sets E one has $(3) \Rightarrow (1)$.

Suppose now E is a set in C^N with $\varrho(E) = 0$ and let h be a function given by $(+)$ with $h = 0$ on E, $h \neq 0$. For every $r > 0$ there is a point a in C^N such that $|a| < r$ and $h(a) = \lim\sup|Q_n(a)|^{1/n} > 0$. Without loss of generality we may assume $h(a) = 1$. The series $\sum_0^\infty Q_n$ converges at each point of $C \cdot E$ and diverges at each point $z = \lambda a$ with $\lambda \in C$, $|\lambda| > 1$. Therefore E does not satisfy (2). We have thus proved that $(2) \Rightarrow (1)$ for every set E in C^N.

(iii) If there is an F_σ complex cone F with $E \subset F$, $\varrho(F) = 0$, then by (ii) E has not Weak Abel's Property.

Suppose now $\varrho(F) > 0$ for each F_σ complex cone F containing E. Given a series (1.1) converging on E, put $\Psi(z) := \sup_{n \geqslant 1}|Q_n(z)|^{1/n}$ and $F := \{z \in C^N;\ \Psi(z) < +\infty\}$. Then F is an F_σ cone containing E. Hence $\varrho(F) > 0$. The set $F_j := \{z \in C^N;\ \Psi(z) \leqslant j,\ |z| \leqslant j\}$ is compact and $\varrho(F_j) \uparrow \varrho(F)$. Take j so large that $\varrho(F_j) > 0$. Then for all $\nu \geqslant 1$

$$|Q_\nu(z)| \leqslant \big(j\|z\|/\varrho(F_j)\big)^\nu \leqslant 2^{-\nu} \quad \text{if} \quad \|z\| \leqslant (2j)^{-1}\varrho(F_j),$$

which implies that series (1.1) converging on E has a positive radius of convergence. The proof of (iii) is concluded.

(iv) Given any series $f = \sum_{n=0}^\infty Q_n$ of homogeneous polynomials converging on E, the complex cone $F := \{\Psi(z) < +\infty\}$, where $\Psi = \sup_{n \geqslant 1}|Q_n|^{1/n}$ is of type F_σ and $C \cdot E \subset F$. Since $C \cdot E$ is dense in an open non-empty set $\Omega \subset C^N$ with Baire property, it follows that Ψ is bounded in a neighbourhood of 0 in C^N. Hence the series f is convergent in a neighbourhood of the origin.

Example. Let $A = \{a_j\}$ be a countable dense subset of C^N $(N \geqslant 2)$. By 2.10, $\varrho(C \cdot A) = 0$, and by 2.7, there is an absolutely homogeneous plsh function h with $h = 0$ on $C \cdot A$, $h \neq 0$. The cone $E := \{h(z) = 0\}$ is a G_δ dense pluripolar set in C^N. By (iv) E has Weak Abel's Property. By (ii) the set E has not Abel's Property (because $\varrho(E) = 0$). The proof of Theorem 3.1 is finished.

Remark 3.2. Statements (i) and (ii) of Theorem 3.1 are known ([10]). We have inserted them here to get a "round" theorem and because lecture notes [10] are not easily available.

Corollary 3.3 (from (i), (ii) and 2.11). *A compact set K in C^N has Abel's Property if and only if its convex hull $\hat{K}_b$ with respect to homogeneous polynomials of N complex variables has a non-empty interior. If a series of* *homogeneous polynomials converges pointwise on K, then it converges locally uniformly in the interior of* $\hat{K}_b$.

4. Convergence sets of N-tuple power series

We say that an N-tuple power series

$$P(z) = \sum_{\alpha} c_\alpha z^\alpha \quad (\alpha \in \mathbf{Z}_+^N,\ z \in \mathbf{C}^N)$$

is *convergent* if there are two positive real numbers r and M such that $|c_\alpha| z^{|\alpha|} \leqslant M$ ($\alpha \in \mathbf{Z}_+^N$, $|\alpha| = \alpha_1 + \ldots + \alpha_N$), otherwise the series is called *divergent*. It is well known that a power series P is convergent if and only if the series $f = \sum_{n=0}^{\infty} Q_n$ of homogeneous polynomials $Q_n(z) = \sum_{|\alpha|=n} c_\alpha z^\alpha$ converges in a neighbourhood of 0 in $\mathbf{C}^N$.

Let $G(k, N)$ denote the Grassmann manifold of all k-dimensional subspaces V of $\mathbf{C}^N$. We say that a family $E \subset G(k, N)$ of k-dimensional subspaces of $\mathbf{C}^N$ has *Abel's Property* if there is a positive number $R = R(E)$ such that every N-tuple power series P with the property that $P|V$ is a convergent k-tuple power series for every $V \in E$, is absolutely convergent in the polydisk $|z_j| < R$ ($j = 1, \ldots, N$).

We say that $E \subset G(k, N)$ has *Weak Abel's Property* if every N-tuple power series P such that $P|V$ is convergent for every $V \in E$ is convergent. In the last property the radius of the absolute convergence of P may depend on P.

It is clear that Abel's Property implies Weak Abel's Property. We shall see that the inverse implication is not true.

Given $E \subset G(k, N)$, denote by $\tilde{E} = \bigcup_{V \in E} V$ the union of all k-dimensional subspaces V of $\mathbf{C}^N$ belonging to E. The set $\tilde{E}$ is a complex cone in $\mathbf{C}^N$. The set $\tilde{E}$ can be treated in a canonical way as a subset of $G(1, N)$ — the set of complex vector lines in $\mathbf{C}^N$.

As a corollary from Theorem 3.1 one gets the following slight improvement of results due to Levenberg and Molzon [5].

THEOREM 4.1. (i) *A subset E of $G(k, N)$ has Abel's Property if and only if the corresponding complex cone $\tilde{E}$ is not pluripolar.*

(ii) *A subset E of $G(k,N)$ has Weak Abel's Property if and only if $\tilde{E}$ is not contained in any F_σ pluripolar complex cone F.*

(iii) *If $E \subset G(k, N)$ is a family of k-dimensional subspaces of $\mathbf{C}^N$ such that the corresponding cone $\tilde{E}$ is a dense G_δ subset of an open non-empty subset Ω of $\mathbf{C}^N$, then E has Weak Abel's Property. Moreover, if $\tilde{E}$ is pluripolar, then E has Weak Abel's Property but it has not Abel's Property.*

5. A solution to Problem 2

In this chapter we shall prove two theorems.

THEOREM 5.1. *Let $\{n_j\}$ be an increasing sequence of positive integers with finite rate of convergence $\varkappa := \limsup_{j\to\infty} n_{j+1}/n_j$. Let E be a subset of $\boldsymbol{C}^N$ with $\alpha(E) > 0$. Let $S_n := Q_0 + \ldots + Q_n$ be the n-th partial sum of a series of homogeneous polynomials*

$$f = \sum_{\nu=0}^{\infty} Q_\nu \quad \text{with} \quad Q_\nu(z) = \sum_{|\alpha|=\nu} c_\alpha z^\alpha \quad (\alpha \in \boldsymbol{Z}_+^N).$$

Then the following statements are true:

(a) *If the sequence $\{S_{n_{j+1}} - S_{n_j}\}$ is bounded α-a.e. on E, then f converges in the ball*

$$\|z\| < \alpha(E)^\varkappa.$$

(b) *If*

$$M := \limsup_{j\to\infty} \|S_{n_{j+1}} - S_{n_j}\|_E^{1/n_{j+1}}$$

is finite, then the series f converges in the ball

$$\|z\| < \min\{\alpha(E)/M,\ (\alpha(E)/M)^\varkappa\}.$$

(c) *If*

$$\sup_{j\geq 1} |S_{n_{j+1}}(z) - S_{n_j}(z)|^{1/n_{j+1}}$$

is finite α-a.e. on E, then the series f has a positive radius of convergence.

(d) *If*

$$\limsup_{j\to\infty} |S_{n_{j+1}}(z) - S_{n_j}(z)|^{1/n_{j+1}} = 0 \quad \alpha\text{-a.e. on } E,$$

then the series f converges locally uniformly in the whole space $\boldsymbol{C}^N$ (i.e. the series represents an entire function of N complex variables).

Proof. (a) $E_l := \{z \in \boldsymbol{C}^N;\ |S_{n_{j+1}}(z) - S_{n_j}(z)| \leq l,\ j \geq 1,\ |z| \leq l\}$ is a compact set, $E_l \subset E_{l+1}$ and $\alpha(F) \geq \alpha(E)$, where $F := \bigcup_{l=1}^{\infty} E_l$. By 2.8,

$$|S_{n_{j+1}}(z) - S_{n_j}(z)| \leq l\alpha(E_l)^{-n_{j+1}} \quad \text{if} \quad \|z\| \leq 1.$$

Hence by the Cauchy inequalities

$$|Q_l(z) \leq l\|z\|^l/\alpha(E_l)^{n_{j+1}}, \quad n_j + 1 \leq l \leq n_{j+1},\ z \in \boldsymbol{C}^N.$$

Therefore

$$\limsup_{l\to\infty} |Q_l(z)|^{1/l} \leq \|z\|/\alpha(E)^\varkappa \quad \text{in } \boldsymbol{C}^N,$$

which ends the proof of (a).

(b) Given $\varepsilon > 0$ one has $\|S_{n_{j+1}} - S_{n_j}\|_E^{1/n_{j+1}} \leqslant M+\varepsilon$, $j > j_\varepsilon$. Hence, via 2.8 and the Cauchy inequalities,

$$|Q_l(z)| \leqslant \left(\frac{M+\varepsilon}{\alpha(E)}\right)^{n_{j+1}} \|z\|^l, \quad n_j+1 \leqslant l \leqslant n_{j+1}, \quad j > j_\varepsilon,$$

which implies

$$\limsup_{l\to\infty} |Q_l(z)|^{1/l} \leqslant \begin{cases} (M/\alpha(E))^\varkappa \|z\| & \text{if } M \geqslant \alpha(E), \\ (M/\alpha(E)) \|z\| & \text{if } M < \alpha(E). \end{cases}$$

This concludes the proof of (b).

(c) The set $E_l := \{z \in \boldsymbol{C}^N;\ |S_{n_{j+1}}(z) - S_{n_j}(z)|^{1/n_{j+1}} \leqslant l,\ j \geqslant 1,\ \|z\| \leqslant l\}$ is compact, $E_l \subset E_{l+1}$ and $E_l \uparrow F$ with $\alpha(F) \geqslant \alpha(E)$. Take l so large that $\alpha(E_l) > 0$. Then by (b) the series f is convergent at least in the ball

$$\|z\| < \min\{\alpha(E_l)/l,\ (\alpha(E_l)/l)^\varkappa\}.$$

(d) Given $\varepsilon > 0$, the set $E_l(\varepsilon) := \{z \in \boldsymbol{C}^N;\ |S_{n_{j+1}}(z) - S_{n_j}(z)|^{1/n_{j+1}} \leqslant \varepsilon$, $j \geqslant l$, $\|z\| \leqslant l\}$ is compact and $E_l(\varepsilon) \uparrow F$ as $l \to \infty$, where $\alpha(F) \geqslant \alpha(E) > 0$. Take l so large that $\alpha(E_l) > \alpha(E)/2$. Then by (b) the series f is convergent in the ball

$$\|z\| < \min\{\alpha(E)/2\varepsilon,\ (\alpha(E)/2\varepsilon)^\varkappa\}. \ \blacksquare$$

THEOREM 5.2. *Given $\{n_j\}$ and a compact polynomially convex set K in $\boldsymbol{C}^N$, assume that at least one of the following two conditions* (a) *or* (b) *is satisfied*:

(a) $\varkappa > 1$, $\alpha(K) = 0$;

(b) $\varkappa = +\infty$, $0 \notin \operatorname{int} K$.

Then there exists a series f such that the sequence $\{S_{n_j}\}$ is uniformly convergent on K but it has zero radius of convergence.

Proof. Let J be an infinite subset of N such that

(i) $n_{j+1} - n_j < n_{k+1} - n_k$ if $j, k \in J$, $j < k$;

(ii) $\varkappa = \lim n_{j+1}/n_j$, $j \to \infty$, $j \in J$.

Let Φ_K denote the extremal function defined in 2.1. It is known [1] that the set $\{\Phi_K(z) < \Phi_K^*(z)\}$ is pluripolar. Let $A = \{a_j;\ j \in J\}$ be a dense subset of $\boldsymbol{C}^N \backslash K$, where each point a of A is repeated infinitely many times in the family $(a_j)_{j\in J}$ and $\Phi_K^*(a) = \Phi_K(a)$ for every $a \in A$.

For a fixed $j \in J$ let P_j be a polynomial of degree $\leqslant n_{j+1} - n_j - 1$ such that $\|P_j\|_K \leqslant 1$ and $|P_j(a_j)| = \Phi_{n_{j+1}-n_j-1}(a_j)$, where Φ_n is defined by 2.4. Put $M := \sup\{|z|;\ z \in K\}$ and

$$(\%) \qquad S_{n_{j+1}}(z) - S_{n_j}(z) := \begin{cases} j^{-2} P_j(z)(z_1/M)^{n_j+1} & \text{if } j \in J, \\ 0 & \text{if } j \notin J. \end{cases}$$

It is clear that, for every $j \in N$, $S_{n_{j+1}} - S_{n_j} = Q_{n_j+1} + \ldots + Q_{n_{j+1}}$, where Q_{n_j+k} is a homogeneous polynomial of degree $n_j + k$, $k = 1, \ldots, n_{j+1} - n_j$.

Let $\sum_0^\infty Q_n$ be the series of homogeneous polynomials whose partial sums S_{n_j}

are uniquely determined by equations (%). We claim that it is the required series. Indeed, if follows from (%) that

$$|S_{n_{j+1}}(z)-S_{n_j}(z)| \leqslant j^{-2}, \quad j \in \boldsymbol{N},\ z \in K,$$

which implies that the sequence $\{S_{n_j}\}$ is uniformly convergent on K.

Let $w = (w_1, \ldots, w_N)$ be a fixed point in A with $w_1 \neq 0$. Observe that

$$|S_{n_{j+1}}(z)-S_{n_j}(z)| \leqslant j^{-2} \Phi_{n_{j+1}-n_j-1}(z)|z_1/M|^{n_j+1}$$
$$=: g_j(z)^{n_{j+1}-n_j-1}.$$

If (a) is satisfied, then

$$\lim_{j \to \infty, j \in J} g_j(w) = \Phi_K(w)(|w_1|/M)^{1/(\varkappa-1)} = +\infty,$$

because $\Phi_K(w) = +\infty$. Hence $S_{n_{j+1}}(w)-S_{n_j}(w) \to \infty$. So the series $\sum Q_n$ diverges at each point $w \in A$ with $w_1 \neq 0$. Since K is pluripolar it follows that the series diverges in a dense subset of $\boldsymbol{C}^N$.

If (b) is satisfied, then $\lim\limits_{j \to \infty, j \in J} g_j(w) = \Phi_K(w) > 1$. Hence again $S_{n_{j+1}}(w) - S_{n_j}(w) \to \infty$ as $j \to \infty$ $(j \in J)$. So the series $\sum Q_n$ diverges at each point $w \in A$ with $w_1 \neq 0$. Since $0 \notin \operatorname{int} K$, it follows that every ball $B(0, r)$ contains a point w of A with $w_1 \neq 0$, which shows that the series has zero radius of convergence.

Remark. If $N = 1$, then part (b) of Theorem 5.2 and a weaker version of (a) in Theorem 5.1 are due to Naftalevich, whose paper [6] inspired the author to study Problem 2 in $\boldsymbol{C}^N$, $N > 1$.

6. Examples

EXAMPLE 1. Let p be a positive integer. Let S_n denote the n-th partial sum of a power series

$$f(z) = \sum c_n z^n, \quad z \in \boldsymbol{C},$$

on the complex plane and let $\{n_j\}$ be an increasing sequence of positive integers.

(α) If $n_{j+1}-n_j \leqslant p$ $(j \geqslant 1)$ and the sequence $\{S_{n_{j+1}} - S_{n_j}\}$ is bounded at each of the points $z_1, \ldots, z_p$ of $\boldsymbol{C}\setminus\{0\}$ with $z_j \neq z_k$ $(j \neq k)$, then f converges in the disk $|z| < r := \min\{|z_1|, \ldots, |z_p|\}$. In particular, if $\{S_{n_j}\}$ is bounded at each of the points $z_1, \ldots, z_p$, then f converges in the disk $|z| < \min\{|z_1|, \ldots, |z_p|\}$.

(β) If $n_{j+1}-n_j > p$ $(j \geqslant 1)$ and $z_1, \ldots, z_p$ are arbitrary points of $\boldsymbol{C}$, then there is a power series f such that sequence of its partial sums $\{S_{n_j}\}$ converges at each of the points $z_1, \ldots, z_p$ but f has zero radius of convergence.

Statement (α) is a direct consequence of Theorem 3.1 (i). An independent proof in this case is very simple. Namely, observe that

$$S_{n_{j+1}}(z)-S_{n_j}(z) = z^{n_j+1}[c_{n_j+1}+c_{n_j+2}z+ \ldots + c_{n_{j+1}} z^{n_{j+1}-n_j-1}] = z^{n_j+1}P_j(z),$$

where $\deg P_j \leqslant p-1$. Hence by the Lagrange Interpolation Formula with nodes $z_1, \ldots, z_p$ one gets

$$|P_j(z)| \leqslant Mr^{-n_j-1}, \qquad |z| \leqslant 1,\ j \geqslant 1,$$

where $r = \min\{|z_1|, \ldots, |z_p|\}$ and M is a positive constant that depends neither on z nor on j. Hence, by the Cauchy inequalities,

$$|c_k|^{1/k} \leqslant M^{1/k}.\begin{cases} r^{-1} & \text{if } r < 1, \\ r^{-(n_j+1)/n_{j+1}} & \text{if } r \geqslant 1, \end{cases}$$

for all k with $n_j < k \leqslant n_{j+1}$. Theorefore, $\limsup\limits_{k\to\infty} |c_k|^{1/k} \leqslant r^{-1}$.

In order to show (β) take $P(z) := (z-z_1)\ldots(z-z_p)$ and define

$$(+) \qquad S_{n_{j+1}} - S_{n_j} := (jz)^{n_j+1} P(z) = c_{n_{j+1}} z^{n_j+1} + \ldots + c_{n_{j+1}} z^{n_{j+1}}.$$

It is clear that $S_{n_1} + \sum\limits_{j=1}^{\infty} (S_{n_{j+1}} - S_{n_j})$ converges uniformly on $A = \{z_1, \ldots, z_p\}$ but diverges at each point $z \notin (A \cup \{0\})$. Hence the power series f with coefficients given by $(+)$ converges only at 0.

EXAMPLE 2. Let $n_j = j^2$, $E = \{1+1/k;\ k \geqslant 1\} \cup \{1\}$, and $F = \{1+2^{-2^k};\ k \geqslant 1\} \cup \{1\}$.

(i) If the sequence of partial sums $\{S_{j^2}\}$ of a power series f is uniformly convergent on E, then f converges in the disk $|z| < 1$.

(ii) There is a power series f such that the sequence of its partial sums $\{S_{j^2}\}$ is uniformly convergent on F but f converges only at 0.

Proof. (i) Put $P_j := S_{(j+1)^2} - S_{j^2} = z^{j^2+1} Q_j$. Then $\deg Q_j \leqslant 2j$ and $\|Q_j\|_E \leqslant \|P_j\|_E \leqslant M = \text{const}$. Put $L_s(z) = \prod\limits_{k=1, k\neq s}^{2j+1} (z-z_k)/(z_s-z_k)$, where $z_k = 1+1/k$. Then $Q_j(z) = \sum\limits_{s=1}^{2j+1} Q_j(z_s) L_s(z)$. It is clear that $|z_s - z_k| \geqslant 1/2j(2j+1)$, and $|z-z_k| \leqslant 3$ for z in $D := \{|z| < 1\}$. Hence $\|Q_j\|_D \leqslant M_j := (2j+1)M3^{2j}[2j \times (2j+1)]^{-2j}$. Therefore, $\|P_j\|_D \leqslant M_j$. Hence by the Cauchy inequalities one gets

$$|c_s|^{1/s} \leqslant M_j^{1/s} \leqslant M_j^{1/j^2} \quad \text{if} \quad j^2 < s \leqslant (j+1)^2.$$

Therefore, $\limsup\limits_{s\to\infty} |c_s|^{1/s} \leqslant 1$.

(ii) Let $Q_j(z) = (z-x_1)\ldots(z-x_{2j})$ be a polynomial of degree $2j$ such that $x_1, \ldots, x_{2j} \in F$ and $\|Q_j\|_F \leqslant \max\limits_{z\in F} |(z-z_1)\ldots(z-z_{2j})|$ for all $z_1, \ldots, z_{2j}$ in F. Then

$$\|Q_j\|_F \leqslant (1-y_{2j})(1-y_{2j-1})\ldots(1-y_1) = 2^{-(2+2^2+\ldots+2^{2j})},$$

where $y_s = 1+2^{-2^s}$. Therefore

(!) $$\lim_{j\to\infty} \|Q_j\|_F^{1/(1+j)^2} = 0.$$

Put

$$S_{(j+1)^2} - S_{j^2} := \frac{1}{j^2}\left(\frac{z}{2}\right)^{j^2+1} Q_j(z)/\|Q_j\|_F.$$

Then the sequence $\{S_{j^2}\}$ is uniformly convergent on F. But for z in $C\setminus(F \cup \{0\})$ we have

$$|S_{(j+1)^2} - S_{j^2}| \geqslant \frac{1}{j^2}\left(\frac{|z|}{2}\right)^{j^2+1} \frac{r^{2j}}{\|Q_j\|_F} \quad \text{with } r = \operatorname{dist}(z, F),$$

which by (!) implies $\lim\limits_{j\to\infty} |S_{(j+1)^2} - S_{j^2}(z)| = +\infty$.

Theorem 5.2 does not cover the case of $\varkappa = 1$ and $\alpha(K) = 0$. Example 2 shows that in this case the statement of Theorem 5.2 is false.

EXAMPLE 3. Assume $\varkappa = 1$ and $n_{j+1} - n_j \leqslant n_{j+2} - n_{j-1}$ ($j \geqslant 2$) (i.e. $\{n_j\}$ is convex), e.g. $n_j = j^2$. Then for every countable set $E = \{z_p\colon p \geqslant 1\} \subset C$ there is a power series f such that $\{S_{n_j}\}$ converges at each point of E but the series converges only at 0.

Proof. Put $S_{n_1} \equiv 0$ and

(:) $$S_{n_{j+1}}(z) - S_{n_j}(z) := a_j z^{n_j+1} Q_j(z) \equiv c_{n_j+1} z^{n_j+1} + \ldots + c_{n_{j+1}} z^{n_{j+1}},$$

where $Q_j(z) := (z - z_1) \ldots (z - z_{n_{j+1}-n_j-1})$ and $a_j := 1/\max\limits_{|z|=1/j} |z^{n_j+1} Q_j(z)|$.

Then

$$\lim_{j\to\infty} S_{n_j}(z_p) = \sum_{k=1}^{m_p} a_k z_p^{n_k+1} Q_k(z_p) = \sum_{s=0}^{\infty} c_s z_p^s, \qquad p > 1,$$

where c_s are given by (:) and $m_p := \min\{n_{j+1} - n_j - 1;\ n_{j+1} - n_j - 1 > p\}$. If the power series $\sum c_s z^s$ were convergent in a neighbourhood of 0, then the series $\sum\limits_{j=1}^{\infty} \max\limits_{|z|=r} |S_{n_{j+1}}(z) - S_{n_j}(z)|$ would be convergent for all sufficiently small $r > 0$. But this is impossible because

$$\max_{|z|=r} |S_{n_{j+1}}(z) - S_{n_j}(z)| = \max_{|z|=r} |z^{n_j+1} Q_j(z)| / \max_{|z|=1/j} |z^{n_j+1} Q_j(z)| \geqslant 1 \qquad \text{for } j \geqslant 1.$$

EXAMPLE 4. Let $S = \{x \in \boldsymbol{R}^N\colon |x| = 1\}$ be the Euclidean unit sphere in $\boldsymbol{R}^N$. It is known [2] that $\Psi(z, S) = (|x|^2 + |y|^2 + 2\sqrt{|x|^2|y|^2 - \langle x, y\rangle^2})^{1/2}$ for all $z = x + iy$ in $\boldsymbol{C}^N$, so that $\hat{S}_b = \{\Psi(z, S) \leqslant 1\}$ is the unit Lie ball in $\boldsymbol{C}^N$. By Theorem 3.1 (i), if $\sum\limits_s^{\infty} Q_n$ is any series of homogeneous polynomials converging on S, then it converges in the unit Lie ball. In particular, every function u harmonic in the unit Euclidean ball in $\boldsymbol{R}^N$ can be extended to a holomorphic function $\tilde{u}$ of N complex variables in the unit Lie ball in $\boldsymbol{C}^N$.

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