OPERATORS ON SOME FUNCTION SPACES

BY

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1. Introduction. The Gleason-Kahane-Żelazko theorem [2], [4], [10] states that if \( \tau \) is a linear functional on a complex Banach algebra \( A \) such that

\[
\tau(x) \in \sigma(x)
\]

for every \( x \in A \), where \( \sigma(x) \) denotes the spectrum of \( x \) in \( A \), then \( \tau \) is multiplicative on \( A \), i.e.,

\[
\tau(xy) = \tau(x) \tau(y) \quad (x, y \in A).
\]

(See also Żelazko [11] and Rudin [7], p. 233.)

In [9], applying this interesting theorem, we have observed that if \( X \) is a compact Hausdorff space and \( T \) is a linear operator from \( C(X) \) to \( C(X) \), where \( C(X) \) denotes the space of all complex-valued continuous functions on \( X \), such that

\[
|Tf| > 0 \quad \text{on } X
\]

for every \( f \in C(X) \) with \( |f| > 0 \) on \( X \), then \( T \) has the form

\[
Tf(x) = r(x) \cdot f(\varphi(x)) \quad (f \in C(X), x \in X),
\]

where \( r \in C(X) \) and \( \varphi \) is a continuous mapping from \( X \) to \( X \).

In the present paper we intend to apply the Gleason-Kahane-Żelazko theorem to obtain similar results for linear operators on \( H_\infty(D) \), the space of all bounded analytic functions on the open unit disc \( D = \{z : |z| < 1\} \) in the complex plane, and on \( A(C) \), the space of all complex-valued continuous functions on the unit circle \( C = \{z : |z| = 1\} \) in the complex plane that have absolutely convergent Fourier series.

2. Operators on \( H_\infty(D) \). We recall that \( H_\infty(D) \) is a complex Banach space with the norm

\[
\|f\|_\infty = \sup_{z \in D} |f(z)| \quad (f \in H_\infty(D)),
\]
and that it is also a Banach algebra under the usual pointwise multiplication. If \( T \) is a linear operator from \( H_\infty(D) \) to \( H_\infty(D) \), we shall write

\[
Z(T) = \{ z \in D : Tf(z) = 0 \text{ for some invertible } f \in H_\infty(D) \}.
\]

Here the continuity of \( T \) is not assumed. Our first result is the following

**Theorem 1.** Let \( T \) be a linear operator from \( H_\infty(D) \) to \( H_\infty(D) \). Then the following two statements (I) and (II) are equivalent:

(I) \( Z(T) \) has no limit point in \( D \) and \( T \) is one-to-one (i.e., \( Tf = 0 \) implies \( f = 0 \)).

(II) \( T \) has the form

\[
Tf(z) = h(z) \cdot f(a(z)) \quad (f \in H_\infty(D), z \in D),
\]

where \( h, a \in H_\infty(D), h \) is not identically zero on \( D \), \( a \) is not constant on \( D \), and \( \|a\|_\infty \leq 1 \).

**Proof.** (I) implies (II). If we let \( \Omega = D \cap Z(T)^c \), then \( \Omega \) is open and connected. For \( z \in \Omega \) and \( f \in H_\infty(D) \), define

\[
\tau_z(f) = Tf(z)/T1(z).
\]

\( \tau_z \) is a linear functional on \( H_\infty(D) \) satisfying

\[
\tau_z(1) = 1 \quad \text{and} \quad \tau_z(f) \neq 0
\]

for every invertible \( f \in H_\infty(D) \). Therefore we may apply the Gleason-Kahane-Żelazko theorem to infer that \( \tau_z \) is multiplicative on \( H_\infty(D) \).

Let the letter \( j \) denote the identity function: \( j(z) = z \), and define a function \( a \) on \( \Omega \) by the relation

\[
a(z) = \tau_z(j) \quad (z \in \Omega).
\]

It follows that \( |a(z)| \leq \|\tau_z\| \cdot \|j\|_\infty = 1 \) (\( z \in \Omega \)), and \( a \) is an analytic function on \( \Omega \) by (5). Thus \( a \) may and will be regarded as an analytic function on \( D \), because \( Z(T) \) has no limit point in \( D \). We now prove that \( |a(z)| < 1 \) for every \( z \in D \). In fact, if \( |a(z)| = 1 \) for some \( z \in D \), then the maximum modulus principle implies that \( a = c \) on \( D \), where \( c \) is a constant of absolute value 1, and so it follows from (5) that \( Tf(z) = cT1(z) \) (\( z \in D \)), which is impossible, since \( T \) is one-to-one. Hence it follows (cf. Hoffman \[3\], p. 160) that if \( z \in \Omega \) and \( f \in H_\infty(D) \), then \( \tau_z(f) = f(a(z)) \). Thus

\[
Tf(z) = h(z) \cdot f(a(z)) \quad (f \in H_\infty(D), z \in D),
\]

where we let \( h = T1 \). Since \( T \) is one-to-one, \( a \) cannot be constant on \( D \). It is clear that \( a \in H_\infty(D) \).

(II) implies (I). This is immediate from the open mapping theorem and the unicity theorem for analytic functions defined on an open and connected set in the complex plane.
Corollary 2 (cf. Nagasawa [5] and deLeew-Rudin-Wermer [1]). Let \( T \) be a linear operator from \( H_\infty(D) \) to \( H_\infty(D) \). Assume that \( T \) is one-to-one and onto, and that \(|Tf| > 0 \) on \( D \) and \(|T^{-1}f| > 0 \) on \( D \) for every invertible \( f \in H_\infty(D) \). If \( \|T1\|_\infty = 1 \) and \( \|f\|_\infty \leq \|Tf\|_\infty \) for every \( f \in H_\infty(D) \), or if \( T1 = c \) on \( D \), where \( c \) is a constant of absolute value 1, then \( T \) is an isometry.

Proof. By Theorem 1, \( T \) has the form

\[
Tf(z) = h(z) \cdot f(a(z)) \quad (f \in H_\infty(D), z \in D),
\]

where \( h \in H_\infty(D) \) is invertible and \( a \) is a conformal mapping from \( D \) onto \( D \).

If \( \|T1\|_\infty = 1 \) and \( \|f\|_\infty \leq \|Tf\|_\infty \) for every \( f \in H_\infty(D) \), then it follows that

\[
\|1/T1\|_\infty = \|1/h\|_\infty = \|T^{-1}\| \leq 1,
\]

and so \( |T1| = 1 \) on \( D \), thus \( T1 = c \) on \( D \) for some constant \( c \) of absolute value 1. Therefore \( T \) has the form \( Tf(z) = c \cdot f(a(z)) \), and this completes the proof.

3. Operators on \( A(C) \). We recall that \( A(C) \) is a complex Banach space with the norm

\[
\|f\| = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \quad (f \in A(C)),
\]

where \( \hat{f} \) is defined by

\[
\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta.
\]

It is also a Banach algebra under the usual pointwise multiplication.

Theorem 3. Let \( T \) be a linear operator from \( A(C) \) to \( A(C) \). Assume that \( f \in A(C) \) and \( |f| > 0 \) on \( C \) imply \( |Tf| > 0 \) on \( C \). Then \( T \) has the form

\[
Tf(z) = h(z) \cdot f(cz^n) \quad (f \in A(C), z \in C),
\]

where \( h \in A(C) \) is invertible, \( c \) is a constant of absolute value 1, and \( n \) is an integer.

Proof. Write \( h = T1 \). Since \( |h| > 0 \) on \( C \), \( 1/h \in A(C) \) by the inversion theorem of Wiener (see, for example, Rudin [8], p. 399). For \( z \in C \) and \( f \in A(C) \), let us define

\[
\tau_z(f) = Tf(z)/h(z).
\]

Then \( \tau_z \) is a linear functional on \( A(C) \) satisfying \( \tau_z(1) = 1 \) and \( \tau_z(f) \neq 0 \) for every invertible \( f \in A(C) \). Hence \( \tau_z \) is multiplicative on \( A(C) \) by the Gleason-Kahane-Żelazko theorem, and thus there exists a complex number \( \alpha(z) \in C \) satisfying \( \tau_z(f) = f(\alpha(z)) \) for every \( f \in A(C) \). Since

\[
f(\alpha(z)) = Tf(z)/h(z) \in A(C)
\]
for every \( f \in A(C) \), it follows from the Leibenson-Kahane theorem (cf. Rudin \[6\], p. 94) that \( a \) has the form

\[
a(z) = c \cdot z^n \quad (z \in C),
\]

where \(|c| = 1\) and \( n \) is an integer, and the proof is complete.

REFERENCES


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