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A relation between Laplace and Stieltjes transforms of two variables

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1. Introduction. The Laplace transform of two variables of a function $f(x, y)$ is given ([1], p. 657) by the equation

$$F(p, q) = \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\, dx\, dy\,, \qquad R(p, q) > 0 \text{ (}^1\text{)}; \tag{1.1}$$

denote it symbolically as

$$F(p, q) \doteqdot f(x, y)\,.$$

We define the generalized Stieltjes transform of order n_1 and n_2 of a function $f(x, y)$ by the integral equation

$$\mathfrak{S}_{n_1,n_2}\{f(x, y)\colon\ p, q\} = \int_0^\infty \int_0^\infty \frac{f(x, y)}{(x+p)^{n_1}(y+q)^{n_2}}\, dx\, dy\,, \tag{1.2}$$

where $|\arg p| < \pi$, $|\arg q| < \pi$, $R(n_1, n_2) > 0$ and $R(p, q) > 0$.

If $n_1 = n_2 = 1$ in (1.2), we obtain

$$\mathfrak{S}_{1,1}\{f(x, y)\colon\ p, q\} = \int_0^\infty \int_0^\infty \frac{f(x, y)}{(x+p)(y+q)}\, dx\, dy\,, \tag{1.3}$$

where $|\arg p| < \pi$, $|\arg q| < \pi$ and $R(p, q) > 0$.

We shall call (1.3) an *ordinary Stieltjes transform of two variables of* $f(x, y)$ and denote it symbolically as $\mathfrak{S}\{f(x, y)\colon\ p, q\}$.

The object of this note is to obtain a relation between Laplace and Stieltjes transforms of two variables. The main result is stated in the form of a theorem and is then illustrated by an example.

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(1) For the sake of brevity the symbol $R(p, q) > 0$ has been used to denote $R(p) > 0$ and $R(q) > 0$, throughout this paper.

2. THEOREM. *If*

$$g(p,q) \doteqdot f(x,y) \tag{2.1}$$

and

$$F(p,q) \doteqdot x^{n_1-1}y^{n_2-1}g(x,y)\,, \tag{2.2}$$

then

$$\frac{1}{\Gamma(n_1)\Gamma(n_2)}F(p,q) = \mathfrak{S}_{n_1,n_2}\{f(x,y):\ p,q\}\,, \tag{2.3}$$

provided a Laplace transform of $|f(x,y)|$ *and* $|x^{n_1-1}y^{n_2-1}g(x,y)|$ *and a Stieltjes transform of* $|f(x,y)|$ *exist,* $|\arg p| < \pi$, $|\arg q| < \pi$, $R(p,q) > 0$ *and* $R(n_1,n_2) > 0$.

Proof. We have

$$g(p,q) = \int_0^\infty\int_0^\infty e^{-px-qy}f(x,y)\,dx\,dy\,, \tag{2.4}$$

and

$$F(p,q) = \int_0^\infty\int_0^\infty e^{-px-qy}x^{n_1-1}y^{n_2-1}g(x,y)\,dx\,dy\,. \tag{2.5}$$

Substituting the value of $g(x,y)$ from (2.4) in (2.5), we obtain

$$\begin{aligned} F(p,q) &= \int_0^\infty\int_0^\infty e^{-px-qy}x^{n_1-1}y^{n_2-1}\Big[\int_0^\infty\int_0^\infty e^{-xs-yt}f(s,t)\,ds\,dt\Big]dx\,dy \\ &= \int_0^\infty\int_0^\infty f(s,t)\Big[\int_0^\infty\int_0^\infty e^{-x(p+s)-y(q+t)}x^{n_1-1}y^{n_2-1}dx\,dy\Big]ds\,dt\,. \end{aligned} \tag{2.6}$$

The change in the order of integration is justified under the conditions stated in the theorem. Evaluating the inner double integral with the use of a known result [3], we obtain

$$\frac{\Gamma(n_1)\Gamma(n_2)}{(p+s)^{n_1}\Gamma(q+t)^{n_2}} \doteqdot x^{n_1-1}y^{n_2-1}e^{-sx-ty}\,,$$

where $R(p+s,q+t) > 0$ and $R(n_1,n_2) > 0$.

The result (2.3) follows.

COROLLARY. *Taking* $n_1 = n_2 = 1$ *in the theorem, we obtain the following result:*

If

$$g(p,q) \doteqdot f(x,y)\,,$$

and

$$F(p,q) \doteqdot g(x,y)\,,$$

then

$$F(p,q) = \mathfrak{S}\{f(x,y):\ p,q\}\,,$$

provided Laplace transforms of $|f(x, y)|$, $|g(x, y)|$ *and a Stieltjes transform of* $|f(x, y)|$ *exist,* $R(p, q) > 0$, $|\arg p| < \pi$ *and* $|\arg q| < \pi$.

3. Application. Let

$$f(x, y) = x^{m_1} y^{m_2} G_{l,r}^{h,k}\left(xy \middle| \begin{matrix} a_1, \ldots, a_l \\ b_1, \ldots, b_r \end{matrix}\right), \tag{3.1}$$

where $(l+r) < 2(h+k)$ and $|\arg(xy)| < (h+k-\frac{1}{2}l-\frac{1}{2}r)\pi$.

On using [2], p. 419, (5), we get

$$g(p, q) = \frac{1}{p^{m_1+1} q^{m_2+1}} G_{l+2,r}^{h,k+2}\left(\frac{1}{pq} \middle| \begin{matrix} -m_1, -m_2, a_1, \ldots, a_l \\ b_1, \ldots, b_r \end{matrix}\right), \tag{3.2}$$

where

$$|\arg p| < \pi/2, \quad |\arg q| < \pi/2, \quad R(m_i + b_j) > -1, \quad R(p, q) > 0 \quad (i = 1, 2;\ j = 1, 2, \ldots, h);$$

and

$$F(p, q) = p^{m_1-n_1+1} q^{m_2-n_2+1} G_{l+2,r+2}^{h+2,k+2}\left(pq \middle| \begin{matrix} -m_1, -m_2, a_1, \ldots, a_l \\ n_1-m_1-1, n_2-m_2-1, b_1, \ldots, b_r \end{matrix}\right), \tag{3.3}$$

provided $R(n_1, n_2) > 0$, $|\arg p| < \pi/2$, $|\arg q| < \pi/2$, $R(n_t - m_j - a_i) > 0$, $R(p, q) > 0$ $(t = 1, 2;\ i = 1, 2, \ldots, k;\ j = 1, 2)$.

Hence on using (3.1) and (3.3) in (2.3), we obtain

$$\frac{p^{m_1-n_1+1} q^{m_2-n_2+1}}{\Gamma(n_1)\Gamma(n_2)} G_{l+2,r+2}^{h+2,k+2}\left(pq \middle| \begin{matrix} -m_1, -m_2, a_1, \ldots, a_l \\ n_1-m_1-1, n_2-m_2-1, b_1, \ldots, b_r \end{matrix}\right) = \mathfrak{S}_{n_1,n_2}\left\{x^{m_1} y^{m_2} G_{l,r}^{h,k}\left(xy \middle| \begin{matrix} a_1, \ldots, a_l \\ b_1, \ldots, b_r \end{matrix}\right) : p, q\right\} \tag{3.4}$$

valid under the conditions given in (3.1) to (3.3).

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References

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[2] Bateman M. Project, *Tables of integral transforms*, Vol. 2, New York 1954.

[3] N. C. Jain, *On Laplace transform of two variables and self reciprocal functions*, Journ. Math. Pures et Appl. 47 (1968), pp. 47-55.

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