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EQUATIONAL COMPACTNESS IN INFINITARY ALGEBRAS

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Taylor [6] recently proved, among other results, that an equational class of finitary algebras contains enough equationally compact algebras iff the subdirectly irreducible algebras in the class constitute, up to isomorphism, a set. This note* provides a negative answer to the natural question whether the same equivalence holds for equational classes of *infinitary* algebras by exhibiting examples in which there are, up to isomorphism, *only one* subdirectly irreducible algebra in the class but *no* non-trivial equationally compact algebras at all. The same examples, as a by-product, also provide cases where Birkhoff's Subdirect Representation Theorem, the well-known fundamental result concerning equational classes of finitary algebras, is no longer valid.

We note that even in the infinitary case the existence of enough equationally compact algebras, i.e. the fact that every algebra in the given equational class can be embedded in an equationally compact algebra belonging to the class, implies a bound on the cardinality of the subdirectly irreducible algebras in the class, and hence the stated smallness condition (cf. Banaschewski and Nelson [1]). In the finitary case, the reverse implication is relatively trivial. The restriction on the subdirectly irreducible algebras of the class provides for maximal essential extensions of these which, in turn, are readily seen to be equationally compact; Birkhoff's Subdirect Representation Theorem and the equational compactness of products of equationally compact algebras (cf. Mycielski [4]) then provides enough equationally compact algebras. The crucial steps of this argument depend, on the surface at least, heavily on the assumption of finitariness, and the work presented here grew from an attempt to understand how essential this assumption really is; our examples, which are in a sense extreme even though only at most $\aleph_0$-ary operations are needed, provide some insight into this point.

To begin with, recall the familiar notions and facts. For any algebra A (in an arbitrary equational class), an extension B of A is called a *free*

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extension of A by the set X iff $X \subseteq B$ and B is generated by A and X such that, for any extension C of A, any set map $X \to C$ extends (necessarily uniquely) to a homomorphism $B \to C$ over A, i.e. mapping A identically. Given any A and set X it is clear that such extensions exist; in the following $A[X]$ will always be an extension of A of this type. Now, an extension B of A is called *pure* iff for any pairs $(u_1, v_1), \ldots, (u_n, v_n)$ in $A[X]^2$ for which one has a homomorphism f: $A[X] \to B$ over A which equalizes these pairs, i.e. $f(u_k) = f(v_k)$ for all $k = 1, \ldots, n$, there also exists a homomorphism g: $A[X] \to A$ over A equalizing these pairs. An algebra A is called *equationally compact* iff a set S of pairs in $A[X]$ can be equalized by a homomorphism $A[X] \to A$ over A whenever this is the case for *each of its finite* subsets. This condition implies, for arbitrary algebras, that A is a retract of every pure extension $B \supseteq A$, i.e. there exists a homomorphism $B \to A$ over A, and for *finitary* algebras the converse also holds (cf. Węglorz [7]). An algebra which is a retract of *every* extension is called an *absolute retract.*

1. βN-algebras. Our first example is given by what we shall call *βN-algebras*, these being the algebras $A = (X, (f_{\mathfrak{U}}))$, $\mathfrak{U}$ extending over the set βN of all ultrafilters on the set N of natural numbers, each $f_{\mathfrak{U}}$ being a map $X^N \to X$, satisfying the following conditions, where $\mathfrak{U}\sigma = f_{\mathfrak{U}}(\sigma)$ for any $\mathfrak{U}$ and any sequence σ:

(A1) If σ: $N \to X$ is constant with a value a, then $\mathfrak{U}\sigma = a$ for all $\mathfrak{U}$.

(A2) For any $\mathfrak{U}$, if σ, τ: $N \to X$ coincide on some member of $\mathfrak{U}$, then $\mathfrak{U}\sigma = \mathfrak{U}\tau$.

(A3) For all $\mathfrak{U}$, ξ: $N \to \beta N$, and σ: $N \to X$, $(\xi * \sigma) = \lim \xi(\mathfrak{U})\sigma$, where $\xi * \sigma$: $N \to X$ is defined by $(\xi * \sigma)(k) = \xi(k)\sigma$, $k \in N$, and the limit is taken in the usual topology of βN.

We note that these conditions (which can evidently be described by equations) have a topological origin as follows:

If $\mathfrak{O}$ is any regular Hausdorff topology on a set X such that every ultrafilter $\mathfrak{U}$ on X of *countable type*, i.e. containing a countable subset of X, converges (with respect to $\mathfrak{O}$), then, for any $\mathfrak{U} \in \beta N$ and σ: $N \to X$, the image ultrafilter basis $\sigma(\mathfrak{U})$ on X converges, so that we can put $\mathfrak{U}\sigma = \lim \sigma(\mathfrak{U}) = \bar{\sigma}(\mathfrak{U})$, where $\bar{\sigma}$: $\beta N \to X$ is the unique continuous map such that $\sigma = \bar{\sigma}\eta$, η: $N \to \beta N$ the natural map which assigns to each $k \in N$ the ultrafilter fixed at k, the existence of $\bar{\sigma}$ resulting from the regularity of $\mathfrak{O}$ by a fundamental proposition on the continuous extension of continuous maps (cf. Bourbaki [2], p. 81). The $\aleph_0$-ary operations thus obtained on X clearly satisfy conditions (A1) and (A2); regarding (A3) one has

$$\mathfrak{U}(\xi * \sigma) = \lim(\xi * \sigma)(\mathfrak{U}) = \lim \{\{\xi(k)\sigma \mid k \in V\} \mid V \in \mathfrak{U}\}$$

$$= \lim \bar{\sigma}\{(\xi(V)) \mid V \in \mathfrak{U}\} = \lim \bar{\sigma}\xi(\mathfrak{U}) = \bar{\sigma}(\lim \xi(\mathfrak{U})) = \lim \xi(\mathfrak{U})\sigma.$$

The algebra obtained in this manner will be called the *βN-algebra of the given space.*

A particular instance of such algebras is obtained as follows. For any set X, let $\beta_0 X$ be the subspace of βX consisting of all ultrafilters on X of countable type. Then, for any countable $\Sigma \subseteq \beta_0 X$ and any ultrafilter Φ on Σ, one has a countable $S_{\mathfrak{V}} \subseteq X$ in $\mathfrak{V}$ for each $\mathfrak{V} \in \Sigma$ so that the countable set $S = \bigcup S_{\mathfrak{V}}$ belongs to all $\mathfrak{V} \in \Sigma$. Now, for $\mathfrak{W} = \lim \Phi$ (in βX), one knows from the properties of the topology of βX that $\mathfrak{W} \supseteq \bigcap \mathfrak{V}$ ($\mathfrak{V} \in \Sigma$); thus one has $S \in \mathfrak{W}$, and hence $\mathfrak{W} \in \beta_0 X$. As a result we have the βN-algebra of the space $\beta_0 X$. The significance of these algebras is described in

LEMMA 1. *The βN-algebra of $\beta_0 X$ is a free βN-algebra with basis $\eta(X)$.*

Proof. First, we prove this for $\beta_0 N = \beta N$. For any $\sigma\colon N \to A$, set map into (the underlying set of) a βN-algebra A, let $h\colon \beta N \to A$ be defined by $h(\mathfrak{U}) = \mathfrak{U}\sigma$. Then, for any $k \in N$, $h(\eta(k)) = \eta(k)\sigma = \sigma(k)$ by (A2) and (A1), and hence $h\eta = \sigma$, as required. Now, for any $\xi\colon N \to \beta N$ and $\mathfrak{U}$, one has $h(\lim \xi(\mathfrak{U})) = \lim \xi(\mathfrak{U})\sigma = \mathfrak{U}(\xi * \sigma)$ by (A3), and here $(\xi * \sigma)(k) = \xi(k)\sigma = h(\xi(k))$ so that $\xi * \sigma = h\xi$. For the algebra in the question this reads $h(\mathfrak{U}\xi) = \mathfrak{U}(h\xi)$, i.e. h is a homomorphism such that $h\eta = \sigma$. The uniqueness of h follows from the fact that $\eta(N)$ is a set of βN-algebra generators.

Now, consider an arbitrary $\beta_0 X$, taking X to be infinite since there is nothing to prove in the finite case. From general principles we know there exists a free βN-algebra $FX \supseteq X$ with basis X, and thus also a homomorphism $h\colon FX \to \beta_0 X$ with $h(x) = \eta(x)$. Then, for any distinct $a, b \in FX$ there exists a countable subset $Y \subseteq X$ such that $a, b \in A$, the subalgebra of FX generated by Y, since the operations all have arity $\aleph_0$. A is free with basis Y, and h induces an onto homomorphism $A \to \beta Y$; however, βY is also free with basis $\eta(Y)$, and this homomorphism induces a one-one onto map between the bases of these free algebras, hence is an isomorphism. In particular, $h(a) \neq h(b)$, and, therefore, h is an isomorphism.

Any algebra, of any type, determines on its underlying set X the *closure system* of (the underlying sets of) its subalgebras and the *closure operator* Γ associated with this. For the βN-algebras, one has $\Gamma\emptyset = \emptyset$ since there are no 0-ary operations; beyond that, though, the operators Γ are closure operators of T_1-topologies, as is expressed in

LEMMA 2. *In any βN-algebra, the singletons and the finite unions of subalgebras are subalgebras.*

Proof. The first part follows directly from (A1). As to the second, let B and C be subalgebras of a βN-algebra A and consider any $\sigma\colon N \to B \cup C$; then $N = \sigma^{-1}(B) \cup \sigma^{-1}(C)$, and any $\mathfrak{U} \in \beta N$ contains one of these two sets, say $\sigma^{-1}(B) \in \mathfrak{U}$. Then, let $\tau\colon N \to B$ be such that it coincides with σ on

$\sigma^{-1}(B)$; from (A2) one then has $\mathfrak{U}\sigma = \mathfrak{U}\tau$ and $\mathfrak{U}\tau \in B$ since B is a subalgebra. In general, this says that $\mathfrak{U}\sigma \in B \cup C$, i.e. $B \cup C$ is a subalgebra.

COROLLARY 1. *For βN-algebras, finite coproducts are disjoint unions.*

Proof. We know the coproduct $A \amalg B$, with canonical maps $i\colon A \to A \amalg B$ and $j\colon B \to A \amalg B$, exists for any A and B. Then, by Lemma 2, $A \amalg B = i(A) \cup j(B)$. Moreover, since any non-trivial βN-algebra contains disjoint subalgebras by Lemma 2 (e.g. different singletons), one has $i(A) \cap j(B) = \emptyset$. Finally, since there exist algebras C into which both A and B can be embedded, e.g. $C = A \times B$ with $A \times \{b\}$, $\{a\} \times B \subseteq C$, i and j are embeddings. This shows $A \amalg B$ is the disjoint union of subalgebras isomorphic to A and B, respectively.

COROLLARY 2. *Any two finite βN-algebras of the same cardinality are isomorphic.*

Proof. They are coproducts of the same number of singleton algebras.

COROLLARY 3. *For βN-algebras, pushouts preserve monomorphisms.*

Proof. For any equational class of algebras, given homomorphisms $f\colon A \to B$ and $g\colon A \to C$, the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "v"] \\ C \arrow[r, "u"'] & D \end{tikzcd}$$

is always constructed by taking $B \amalg C$ modulo a suitable congruence θ, and u and v are the homomorphisms

$$C \xrightarrow{i} B \amalg C \xrightarrow{\nu} D = (B \amalg C)/\theta \quad \text{and} \quad B \xrightarrow{j} B \amalg C \xrightarrow{\nu} D,$$

respectively, where i and j are the canonical maps and ν is the quotient map. Here, one has by the lemma that

$$\Delta \cup \{(f(x), g(x)) \mid x \in A\} \cup \{(g(x), f(x)) \mid x \in A\},$$

Δ the diagonal of $B \amalg C$, is already a congruence: each of the three sets occurring is a subalgebra of $(B \amalg C)^2$, hence the union is, and is otherwise an equivalence. It follows that this is the required relation θ. Now, the restriction of θ to C is clearly the identity relation, and hence u is a monomorphism — no matter, actually, whether f is one or not.

We now turn to the discussion of equational compactness for βN-algebras.

Let A be any βN-algebra and S a set of pairs in $A[X]$. Note that, actually, $A[X] = A \amalg FX$, FX the βN-algebra of $\beta_0 X$; hence, these pairs are of the form (a, b), $(a, \mathfrak{U})$ or $(\mathfrak{V}, \mathfrak{W})$, where $a, b \in A$ and $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in FX$.

From the point of view of equalizing such pairs by homomorphisms over A, moreover, the pairs (a, b) with $a, b \in A$ can be disregarded; hence, we consider S not to contain any of these. Let Σ_a be the set of those $\mathfrak{U}$ for which there exists a sequence $\mathfrak{U}_1, \mathfrak{U}_2, \ldots, \mathfrak{U}_n = \mathfrak{U}$ for which $(a, \mathfrak{U}_1) \in S$ and $(\mathfrak{U}_i, \mathfrak{U}_{i+1})$ or $(\mathfrak{U}_{i+1}, \mathfrak{U}_i)$ belongs to S for all $i = 1, \ldots, n-1$. For a finite S one then has

LEMMA 3. *S can be equalized by a homomorphism h: $A \amalg FX \to B$ over A, B any extension of A, iff all Σ_a are disjoint.*

Proof. If there exists such an h, then $a = h(\mathfrak{U})$ for each $\mathfrak{U} \in \Sigma_a$, so the condition then holds. Conversely, given this condition, there exists a decomposition $X = E \cup \bigcup E_a$ of X such that $E_a \in \mathfrak{U}$ for all $\mathfrak{U} \in \Sigma_a$ and $E \in \mathfrak{B}$ for the remaining $\mathfrak{B} \in FX$ occurring in S. This results from the properties of the topology of βX and the fact that there is only a finite number of finite, hence closed, subsets of βX involved. Now, one can define a homomorphism g: $FX \to B$ by putting $g(x) = a$ for $x \in E_a$ and $g(x) = c$ arbitrary for $x \in E$. Then, for each $\mathfrak{U} \in \Sigma_a$, take ξ: $N \to E_a$ and $\mathfrak{B} \in \beta N$ such that $\mathfrak{U} = \xi(\mathfrak{B}) = \mathfrak{B}\xi$; this implies $g(\mathfrak{U}) = \mathfrak{B}(g\xi) = a$, and, similarly, $g(\mathfrak{W}) = c$ for all $\mathfrak{W}$ occurring in S not belonging to Σ_a. It follows from this that g together with the identity map on A provides the desired homomorphism h.

COROLLARY 4. *Every extension of βN-algebras is pure.*

Proof. The condition for the existence of h does not depend on B.

PROPOSITION 1. *There are only trivial equationally compact βN-algebras.*

Proof. Let A be such a βN-algebra and take any elements $a, b \in A$. Then, consider the set S of pairs in $A \amalg \beta N$, where $(a, \mathfrak{U}) \in S$ for all fixed ultrafilters $\mathfrak{U} \in \beta N$ and $(b, \mathfrak{B}) \in S$ for all $\mathfrak{B} \in \beta N$. By Lemma 3 one has that every finite subset of S can be equalized by a homomorphism $A \amalg \beta N \to A$ over A, and thus there exists a homomorphism h: $A \amalg \beta N \to A$ over A equalizing all pairs in S. This, however, means that h is constant, with value a, on the generating set $\eta(N)$ of βN (as βN-algebra), hence h is constant, and, therefore, $a = b$.

On the other hand, concerning subdirect irreducibility, we have

PROPOSITION 2. *The subdirectly irreducible βN-algebras are exactly the two-element algebras, and thus, up to isomorphism, there is only one subdirectly irreducible βN-algebra.*

Proof. Consider any βN-algebra A with three distinct elements a, b and c. It follows from Lemma 2 that $\Theta = \Delta \cup \{(a, b), (b, a)\}$ and $\Lambda = \Delta \cup \{(b, c), (c, b)\}$, Δ the diagonal of $A \times A$, are congruences on A; and evidently $\Delta = \Theta \cap \Lambda$ and $\Theta, \Lambda \supset \Delta$. This shows A is not subdirectly irreducible, and the converse is trivial.

COROLLARY 5. *Birkhoff's Subdirect Representation Theorem fails in the class of βN-algebras.*

Proof. It has to be shown that there are βN-algebras which cannot be embedded into a product of two-element βN-algebras. Let $A = \prod A_\alpha$ with projections $p_\alpha: A \to A_\alpha$ be such a product. Now, for the topologies given by the subalgebra closure operators, the p_α are, as homomorphisms between βN-algebras, continuous maps; thus the topology in question on A is finer than the product topology given by the discrete topologies on the two-element algebras A_α. It follows that any subalgebra $B \subseteq A$, whose subalgebra topology is compact, is actually a subspace, in that topology, with respect to the product topology on A, since compact Hausdorff topologies are minimal Hausdorff. Consequently, a βN-algebra which is compact Hausdorff in its subalgebra topology and embeddable into a product of two-element algebras must be zero-dimensional in that topology. It follows that the βN-algebra of any connected second countable compact Hausdorff space cannot be embedded into any such product.

Remark 1. The class of *all* compact Hausdorff spaces, though it does not constitute an equational class of (infinitary) algebras in the usual sense (which assumes only a *set* of operations) is still rather like an equational class (cf. Linton [3], p. 84-94), and the considered questions can be raised for it; the answers are the same: there are no non-trivial equationally-compact compact Hausdorff spaces, and the only subdirectly irreducible ones are the two-element spaces. Again, the same holds if one considers the counterparts of βN-algebras for any cardinal greater than $\aleph_0$; these will be equational classes, the equations being the analogues of conditions (A1)-(A3).

Remark 2. For βN-algebras, Corollary 3 of Lemma 2 implies that injectivity is the same as being an absolute retract, and the same holds for compact Hausdorff spaces. For the latter, of course, one has that there are enough injectives since the unit interval is injective (Tietze's Extension Theorem), and every compact Hausdorff space is embeddable into some power thereof. We know nothing about the existence of injective βN-algebras; if there are non-trivial such algebras, these would show that, for infinitary algebras, being a retract of every pure extension (which here means: every extension, by the Corollary 4) does not imply equational compactness, in contrast with the finitary case.

Remark 3. Since all extensions of βN-algebras are pure, the two-element algebras are the only pure-irreducible ones, and hence there is only a set of these, up to isomorphism, whereas no non-trivial βN-algebra has an equationally compact pure extension. This shows that Theorem 3.12 of Taylor [6] also fails in the infinitary case.

Remark 4. It is easy to single out the basic features of βN-algebras which enter into the proofs of Propositions 1 and 2.

Let $\boldsymbol{A}$ be any equational class of infinitary algebras satisfying the following conditions:

(i) For any $A \in \boldsymbol{A}$, the singletons in A are subalgebras of A.

(ii) For any $A \in \boldsymbol{A}$, the union of any two subalgebras of A is a subalgebra of A.

(iii) If $A \in \boldsymbol{A}$ is a free algebra in $\boldsymbol{A}$ with basis X, then, for any finitely many elements $a_1, \ldots, a_n \in A$, there exists a decomposition $X = X_1 \cup \ldots \cup X_n$ of X such that a_i belongs to the subalgebra of A generated by X_i for each i.

Then, $\boldsymbol{A}$ has only trivial equationally compact algebras, and its subdirectly irreducible algebras are just the (mutually isomorphic) two-element algebras.

2. Boolean m-algebras. The second example is given by the equational class of m-complete Boolean algebras for a fixed infinite cardinal m, the operations being the usual finitary Boolean operations (including 0 and 1 as 0-ary operations) together with the two m-ary operations which assign to each $\sigma \in A^m$ the meet $\bigwedge\sigma$ and the join $\bigvee\sigma$, respectively, and the equations being the obvious ones. We refer to these algebras as *Boolean m-algebras*.

Recall that an algebra A is *weakly* equationally compact iff a set of pairs in a free algebra (in the class $\boldsymbol{A}$ is considered in) can be equalized by a homomorphism into A, whenever this is the case for each of its finite subsets. Evidently, weak equational compactness is implied by equational compactness, though the converse does not hold, as is shown by the class of Boolean algebras where every algebra is weakly equationally compact but the equationally compact algebras are exactly the complete ones (cf. Węglorz [7]).

The situation is quite different for Boolean m-algebras:

PROPOSITION 3. *There are no non-trivial weakly equationally compact Boolean m-algebras.*

Proof. Let A be such an algebra and F a free Boolean m-algebra with infinite basis X whose cardinal is greater than the number of elements of A. Then, let S be the set of all pairs in F of the form $(0, \bigwedge\sigma)$ and $(1, \bigvee\sigma)$ for those $\sigma \in X^m$ whose images contain infinitely many elements. Now, given any such $\sigma_1, \ldots, \sigma_n$ with images $X_1, \ldots, X_n$, choose $x_i \in X_i$ for each i, and define a homomorphism $h\colon F \to A$ by putting $h(x_i) = 1$ for each i, and $h(x) = 0$ for the other $x \in X$. Since each $X_i - \{x_1, \ldots, x_n\}$ is non-void, it follows that $h(\bigwedge\sigma_i) = \bigwedge h\sigma_i = 0$, whereas $h(\bigvee\sigma_i) = 1$ by the choice of the x_i, and hence every finite subset of S can be equalized by a homomorphism $F \to A$. Consequently, there exists a homomorphism $g\colon F \to A$ equalizing all pairs in S. Now, by the choice of the size of X, g must be constant on some infinite subset X_0 of X, and thus one has pairs

$(0, \bigwedge\sigma)$ and $(1, \bigvee\sigma)$ in S for which the image of σ is contained in X_0; this implies $\bigvee g\sigma = \bigwedge g\sigma$, and hence $0 = 1$ in A, i.e. A is trivial.

PROPOSITION 4. *The subdirectly irreducible Boolean m-algebras are exactly the two-element algebras.*

Proof. Clearly, the two-element Boolean m-algebras are subdirectly irreducible. Conversely, let A be any Boolean m-algebra which has an element $a \in A$ distinct from 0 and 1. Then, the intervals $[0, a]$ and $[a, 1]$ in A are again Boolean m-algebras, and the maps $f: A \to [0, a]$ and $g: A \to [a, 1]$, given by $f(x) = a \wedge x$ and $g(x) = a \vee x$, are Boolean m-algebra homomorphisms by the distributivity laws which hold in any Boolean algebra with respect to any existing joins and meets (cf. Sikorski [5], p. 55). Now, the kernels of f and g (i.e. the ideals mapped to zero) are given by the conditions $a \wedge x = 0$ and $a \vee x = a$, respectively, which shows their intersection is zero. Since neither kernel itself is zero by the choice of a, it follows that A is not subdirectly irreducible.

COROLLARY 6. *Birkhoff's Subdirect Representation Theorem fails in the class of Boolean m-algebras.*

Proof. To be a Boolean m-subalgebra of a product of two-element algebras amounts to being isomorphic to an m-field of sets, and, for any infinite cardinal m, there are m-complete Boolean algebras which do not have this property (cf. Sikorski [5], p. 79).

These arguments also work for suitable distributive lattices. We call a *distributive m-lattice*, for a given infinite cardinal m, any m-complete distributive lattice with the binary and m-ary operations of meet and join, subject to the usual equations together with the m-ary distribution laws $x \wedge \bigvee \sigma = \bigvee x \wedge \sigma(i)$ and $x \vee \bigwedge \sigma = \bigwedge x \vee \sigma(i)$. The proof of Proposition 3 can then be used, with two arbitrary elements a and b such that $a \leqslant b$ in place of 0 and 1, to show that equational compactness implies $a = b$. Further, as in the proof of Proposition 4, one considers the maps $A \to [\leftarrow, a]$ and $A \to [a, \to]$ as given there, these being m-lattice homomorphisms in view of the m-ary distribution laws. Now, the intersection of the congruences on A determined, in the usual way, by these homomorphisms is the identity relation since $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ imply $x = y$ in every distributive lattice. Moreover, if A has more than two elements, then there also exist $a, b, c \in A$ with $b < a < c$; in this case, since $a \wedge c = a \wedge a$ and $a \vee b = a \vee a$, both these congruences are non-trivial, and hence A is not subdirectly irreducible.

This together with the fact that the underlying distributive m-lattice of a Boolean m-algebra which is not isomorphic to an m-field of sets is itself not isomorphic to an m-ring of sets proves

PROPOSITION 5. *There are no non-trivial equationally compact distributive m-lattices, the subdirectly irreducible distributive m-lattices are exactly* *the two-element lattices, and Birkhoff's Subdirect Representation Theorem fails in the class of distributive m-lattices.*

In conclusion we note that, in contrast with the situation for Boolean m-algebras, all distributive m-lattices are *weakly* equationally compact — an obvious consequence of the fact that they all have one-element subalgebras.

REFERENCES

[1] B. Banaschewski and Evelyn Nelson, unpublished notes.

[2] N. Bourbaki, *General topology I*, Paris-Reading 1966.

[3] F. E. J. Linton, *Some aspects of equational theories*, Proceedings of the Conference on Categorical Algebra, La Jolla 1965, New York 1966.

[4] J. Mycielski, *Some compactifications of general algebras*, Colloquium Mathematicum 13 (1964), p. 1-9.

[5] R. Sikorski, *Boolean algebras*, Berlin-Göttingen-Heidelberg 1960.

[6] W. Taylor, *Residually small varieties*, Algebra Universalis 2 (1972), p. 33-53.

[7] B. Węglorz, *Equationally compact algebras (I)*, Fundamenta Mathematicae 59 (1968), p. 289-298.

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