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Parabolic equations with coefficients depending on t and parameters

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Abstract. The main object of this paper is to study the continuity with respect to both h and t of solutions of the problem

$$\frac{du}{dt}(t)+A_h(t)u(t)=f(h,t), \quad u(0)=u_h^0.$$

Introduction. In this paper, we consider the parabolic problem

$$(*) \qquad \frac{du}{dt}(t)+A_h(t)u(t)=f(h,t), \quad t\in(0,T],\ h\in\Omega,$$

$u(0)=u_h^0$. It is well known (see e.g. [8]) that in some cases the solution is given by the formula

$$(**) \qquad u_h(t)=U_h(t,0)u_h^0+\int_0^t U_h(t,s)f(h,s)ds,$$

where U_h is the fundamental solution of equation (*). Since the sufficient conditions for $u_h(t)$ (given by (**)) to be the solution to problem (*) are well known in the literature, we do not discuss them here. We shall only study the continuity of the mapping

$$(***) \qquad u\colon \Omega\times[0,T]\ni(h,t)\to u_h(t)\in X$$

with $u_h(t)$ given by (**), without specifying assumptions which guarantee that u_h is the solution of (*). Our assumptions enable us to repeat the construction of U_h presented in [8], Chapter 5, and to prove that $u_h(t)$ is continuous with respect to both h and t.

Theorems on the class of regularity of $u_h(t)$ with respect to h and t will be published elsewhere.

Similar problems are considered in [1], [6], [9], [10] for differential equations with coefficients independent of t.

2. Notation and definitions. Let X be a Banach space and let Ω be a locally compact subset of $\boldsymbol{R}^n$.

We denote by $B(X, Y)$ the space of bounded linear operators from X to Y. If $X = Y$, then $B(X, X)$ is denoted by $B(X)$. The space of closed linear operators from X to X will be denoted $\mathscr{C}(X)$. For a given operator $A \in \mathscr{C}(X)$ the resolvent set of A will be denoted by $P(A)$.

We shall consider a family $(A_h(t))_{(h,t)\in\Omega\times[0,T]}$ of closed linear operators from X to X defined for each $h \in \Omega$ on a dense linear subspace $D(A_h(t)) = D$ of X.

ASSUMPTION Z_1. There exists a Banach space Z and a bijective mapping T: $Z \to D$ such that $T \in B(Z, X)$ and the mapping

(1) $$\Omega\times[0, T]\ni(\tau, r)\to A_\tau(r)T\in B(Z, X)$$

is continuous.

ASSUMPTION Z_2. There exist a Banach space Z, a continuous linear bijective mapping T: $Z \to D$ and $\alpha \in (0, 1]$ such that the mapping

(2) $$[0, T]\ni t\to A_\tau(t)T\in B(Z, X) \quad \text{for } \tau\in\Omega$$

is Hölder continuous, i.e. there exists $\tilde{L} > 0$ such that

$$\|A_\tau(t)T - A_\tau(s)T\| \leqslant \tilde{L}|t-s|^\alpha \quad \text{for } \tau\in\Omega,$$

$0 \leqslant s \leqslant T$ and $0 \leqslant t \leqslant T$.

We easily see that Assumptions Z_1, Z_2 are independent of the choice of (Z, T).

ASSUMPTION Z_3. $A_h(t) \in G(C_0)$ for $(h, t) \in \Omega\times[0, T]$, where

(3) $$G(C_0) = \{A\in\mathscr{C}(X)\colon \overline{D(A)} = X,\ [0, +\infty)\subset P(-A),\ \|(A+\xi)^{-k}\| \leqslant C_0\xi^{-k}$$
$$\text{for } \xi > 0,\ k = 1, 2, \ldots \text{ and } \|A\exp(-tA)\| \leqslant C_0t^{-1} \text{ for } t > 0\}.$$

Remark 1. It follows from Assumption Z_1 that if, for $(h, t) \in \Omega\times[0, T]$, $A_h(t)$ is invertible (1-1 and onto), then

(4) $$A_h(t)A_{h_0}^{-1}(s) \to A_{h_0}(t)A_{h_0}^{-1}(s) \quad \text{as } h\to h_0,$$

uniformly on $\Delta_T = \{(t, s)\colon 0 \leqslant s \leqslant t \leqslant T\}$. Hence, if K is a bounded subset of X, then

(5) $$(A_h(t) - A_{h_0}(t))A_{h_0}^{-1}(s)x \to 0 \quad \text{as } h\to h_0$$

uniformly on $\Delta_T\times K$.

LEMMA 1. *If for every $(h, t) \in \Omega\times[0, T]$ the operator $A_h(t)$ is invertible and Assumptions Z_1, Z_2 are satisfied, then for every compact subset K of Ω there exists $L > 0$ such that*

(6) $$\|A_\tau(t)A_h^{-1}(r) - A_\tau(s)A_h^{-1}(r)\| \leqslant L|t-s|^\alpha$$

for $h, \tau \in K$ and $t, s, r \in [0, T]$.

Proof. Let K be a compact subset of Ω. Since the mapping

$$\Omega\times[0, T]\ni(h, r)\to(A_h(r)T)^{-1} = T^{-1}A_h^{-1}(r)\in B(X, Z)$$

is continuous, there exists $\tilde{C}>0$ such that

$$\|(A_h(r)T)^{-1}\|\leqslant\tilde{C}\quad\text{for } (h, r)\in K\times[0, T].$$

Hence

$$\|A_\tau(t)A_h^{-1}(r)-A_\tau(s)A_h^{-1}(r)\|\leqslant\tilde{C}\|A_\tau(t)T-A_h(s)T\|\leqslant\tilde{C}\tilde{L}|t-s|^\alpha = L|t-s|^\alpha.$$

LEMMA 2. *Let U be a compact space, let $\Phi(h, z)\in B(X)$ for $(h, z)\in\Omega\times U$, let D be a dense subset of X and let $h_0\in\Omega$. Suppose that there exists a constant $C_1>0$ such that*

$$\|\Phi(h, z)\|\leqslant C_1\quad\textit{for } (h, z)\in\Omega\times U. \tag{7}$$

If, for any $x\in D$,

$$\Phi(h, z)x\to 0\quad\textit{as } h\to h_0,$$

uniformly in U, then

(i) *for any $x\in X$, $\Phi(h, z)x\to 0$ as $h\to h_0$, uniformly in U;*

(ii) *for any compact subset K of X, $\Phi(h, z)x\to 0$ as $h\to h_0$, uniformly in $U\times K$.*

Proof. (i) is obvious. To prove (ii), let us observe that the family $(\Phi(h, z))$ is equicontinuous. Therefore, for every $x_0\in X$ and $\varepsilon>0$, there exists $\delta=\delta(x_0)>0$ such that

$$\Phi(h, z)(B(x_0, \delta))\subset B(\Phi(h, z)x_0, \tfrac{1}{2}\varepsilon)$$

where $B(x_0, \delta)$ is the ball of radius δ centered at x_0. Since K is a compact subset of X, there exist points $x_j\in K$, $j=1, \ldots, l$, such that

$$K\subset\sum_{j=1}^{l} B(x_j, \delta(x_j)).$$

Therefore, by (i), there exists $\mu>0$ such that

$$\|\Phi(h, z)x_j\|<\tfrac{1}{2}\varepsilon\quad\text{for } j=1, 2, \ldots, l,$$

provided $|h-h_0|<\mu$ and $z\in U$. Thus $\Phi(h, z)(K)\subset B(0, \varepsilon)$ and this ends the proof.

We shall consider the following parabolic problem:

$$\frac{du}{dt}(t)+A_h(t)u(t)=f(h, t),\quad 0<t\leqslant T, \tag{8}$$

$$u(0)=u_h^0 \tag{9}$$

with the parameter $h\in\Omega$, given $u_h^0\in X$ and a continuous function $f\colon\Omega\times[0, T]\to X$.

DEFINITION 1. We say that a function

$$u\colon \Omega \times [0, T] \ni (h, t) \to u_h(t) = u(h, t) \in X$$

is a *solution of problem* (8)–(9) if $u_h \in C([0, T]; X) \cap C^1((0, T]; X)$, $u_h(0) = u_h^0$ and u_h satisfies (9) in $(0, T]$ for $h \in \Omega$.

For convenience we now recall the construction of the fundamental solution $U_h(t, s)$ of (8) (for details see [8], Chapter 5), for $h \in \Omega$. Some inequalities needed in the sequel will also be recalled:

$$R_1^h(t, s) := -(A_h(t) - A_h(s))\exp(-(t-s)A_h(s)), \tag{10}$$

where $\exp(-tA_h(s))$ is the strongly continuous semigroup with the infinitesimal generator $A_h(s)$ for $h \in \Omega$, $s \in [0, T]$.

$$R_m^h(t, s) = \int_s^t R_1^h(t, \tau) R_{m-1}^h(\tau, s)\, d\tau \quad \text{for } m = 2, 3, \ldots, \tag{11}$$

$$R^h(t, s) = \sum_{m=1}^{\infty} R_m^h(t, s), \tag{12}$$

$$W^h(t, s) = \int_s^t \exp(-(t-\tau)A_h(\tau)) R^h(\tau, s)\, d\tau, \tag{13}$$

$$U_h(t, s) = \exp(-(t-s)A_h(s)) + W^h(t, s), \tag{14}$$

$$\|R_1^h(t,s)\| \leqslant LC_0(t-s)^{\alpha-1} \quad \text{for } h \in \Omega,\ s < t \leqslant T, \tag{15}$$

$$\|R_m^h(t, s)\| \leqslant \tilde{M}(t-s)^{m\alpha-1} \quad \text{for } h \in \Omega,\ s < t \leqslant T, \tag{16}$$

where

$$\tilde{M} = (LC_0\Gamma(\alpha))^m / \Gamma(m\alpha),$$

$$\|R^h(t, s)\| \leqslant C(t-s)^{\alpha-1} \quad \text{for } s < t, \tag{17}$$

where

$$C = \sum_{m=1}^{\infty} (LC_0\Gamma(\alpha))^m T^{(m-1)\alpha} (\Gamma(m\alpha))^{-1},$$

$$\|W^h(t, s)\| \leqslant C(t-s)^{\alpha}, \tag{18}$$

$$\|U_h(t, s)\| \leqslant C. \tag{19}$$

3. Preparatory lemmas. In this section we consider the case of a one-point set $\Omega = \{h_0\}$. For simplicity we omit the parameter h_0.

LEMMA 3. *If Assumptions* Z_1, Z_2, Z_3 *are fulfilled, then the mapping* $\Phi\colon [0, T] \times [0, T] \times X \to D \subset X$ *given by*

$$\Phi(\tau, s, x) = \exp(-\tau A(s))x \tag{20}$$

is continuous.

Proof. This is a simple consequence of Proposition 1 from [9].

LEMMA 4. *If Assumptions Z_1, Z_2, Z_3 are fulfilled, then for any $x \in D$ the mapping*

$$\Delta_T \ni (t, s) \to R_1(t, s)x \in X$$

is continuous.

Proof. For $x \in D$ we have (see [3])

$$A(s)\exp(-(t-s)A(s))x = \exp(-(t-s)A(s))A(s)x,$$

$$R_1(r, s)x = (A(t)-A(s))A^{-1}(s)\exp(-(t-s)A(s))A(s)x.$$

Thus, by assumptions and Lemma 3, $R_1(t, s)x$ is a continuous function of (t, s, x).

Remark 2. Combining Lemma 4 and Lemma 2, we have:

(a) for every $x \in X$ the mapping

$$\Delta_T^0 = \{(t, s) \in \Delta_T\colon\ t > s\} \ni (t, s) \to R_1(t, s)x$$

is continuous;

(b) for any compact subset K of X and any $(t_0, s_0) \in \Delta_T^0$ the convergence

$$R_1(t, s)x \to R_1(t_0, s_0)x \quad \text{when } (t, s) \to (t_0, s_0)$$

is uniform on K.

LEMMA 5. *If Assumptions Z_1, Z_2, Z_3 are fulfilled, then:*

(a) *the mappings*

$$\Delta_T^0 \times X \ni (t, s, x) \to R_m(t, s)x \in X, \quad m = 1, 2, \ldots \tag{21}$$

are continuous;

(b) *for any $x \in D$ the mappings*

$$\Delta_T \ni (t, s) \to R_m(t, s)x, \quad m = 1, 2, \ldots \tag{22}$$

are continuous.

Proof (induction on m). We start with $m = 1$. Let $(t_\nu, s_\nu) \in \Delta_T^0$, $x_\nu \in X$, $\nu = 1, 2, \ldots$, and let $(t_\nu, s_\nu) \to (t_0, s_0) \in \Delta_T^0$, $x_\nu \to x_0$. We have, by Remark 2(b),

$$\|R_1(t_\nu, s_\nu)x_\nu - R_1(t_0, s_0)x_0\|$$
$$\leqslant \|R_1(t_\nu, s_\nu)x_\nu - R_1(t_0, s_0)x_\nu\| + \|R_1(t_0, s_0)(x_\nu - x_0)\| \to 0 \quad \text{as } \nu \to \infty.$$

Now suppose that (a) is true for $k = 1, \ldots, m-1$ and fix $(t_0, s_0) \in \Delta_T^0$, $x_0 \in X$, $\varepsilon > 0$ and $0 < \delta < \frac{1}{4}(t_0 - s_0)$. For $(t, s, x) \in \Delta_T^0 \times X$ such that $|s - s_0| < \delta$, $|t - t_0| < \delta$, $\|x\| \leqslant \|x_0\| + 1$ we have

$$\Big\|\int_s^t R_1(t,\tau)R_{m-1}(\tau,s)x\,d\tau-\int_{s_0}^{t_0} R_1(t_0,\tau)R_{m-1}(\tau,s_0)x_0\,d\tau\Big\|$$

$$=\Big\|\int_s^{s_0+\delta} R_1(t,\tau)R_{m-1}(\tau,s)x\,d\tau\Big\|+\Big\|\int_{t_0-\delta}^{t} R_1(t,\tau)R_{m-1}(\tau,s)x\,d\tau\Big\|$$

$$+\Big\|\int_{s_0}^{s_0+\delta} R_1(t_0,\tau)R_{m-1}(\tau,s_0)x_0\,d\tau\Big\|+\Big\|\int_{t_0-\delta}^{t_0} R_1(t_0,\tau)R_{m-1}(\tau,s_0)x_0\,d\tau\Big\|$$

$$+\Big\|\int_{s_0+\delta}^{t_0-\delta}\big(R_1(t,\tau)R_{m-1}(\tau,s)x-R_1(t_0,\tau)R_{m-1}(\tau,s_0)x_0\big)d\tau\Big\|.$$

Thus, by inequalities (15), (16),

$$\Big\|\int_s^{s_0+\delta} R_1(t,\tau)R_{m-1}(\tau,s)x\,d\tau\Big\|\leqslant\int_s^{s_0+\delta} C_0L(t-\tau)^{\alpha-1}M(\tau-s)^{(m-1)\alpha-1}\|x\|\,d\tau$$

$$\leqslant(\|x_0\|+1)C_0L\tilde{M}\big(\tfrac{1}{2}(t_0-s_0)\big)^{\alpha-1}\int_s^{s_0+\delta}(\tau-s)^{(m-1)\alpha-1}d\tau$$

$$=M_1(s_0+\delta-s)^{(m-1)\alpha}$$

and so, for small enough δ,

$$\Big\|\int_s^{s_0+\delta} R_1(t,\tau)R_{m-1}(\tau,s)x\,d\tau\Big\|<\tfrac{1}{8}\varepsilon\quad\text{for } s\in(s_0-\delta,\,s_0+\delta),\ t\in(t_0-\delta,\,t_0+\delta).$$

Similarly,

$$\Big\|\int_{t_0-\delta}^{t} R_1(t,\tau)R_{m-1}(\tau,s)x\,d\tau\Big\|<\tfrac{1}{8}\varepsilon,$$

$$\Big\|\int_{s_0}^{s_0+\delta} R_1(t_0,\tau)R_{m-1}(\tau,s_0)x_0\,d\tau\Big\|<\tfrac{1}{8}\varepsilon$$

and

$$\Big\|\int_{t_0-\delta}^{t_0} R_1(t_0,\tau)R_{m-1}(\tau,s_0)x_0\,d\tau\Big\|<\tfrac{1}{8}\varepsilon\quad\text{for } s\in(s_0-\delta,\,s_0+\delta),\ t\in(t_0-\delta,\,t_0+\delta).$$

To finish the proof of (a), it is enough to prove that there exists $0<\delta_1<\frac{1}{2}\delta$ such that

$$\Big\|\int_{s_0+\delta}^{t_0-\delta}\big(R_1(t,\tau)R_{m-1}(\tau,s)x-R_1(t_0,\tau)R_{m-1}(\tau,s_0)x_0\big)d\tau\Big\|<\tfrac{1}{2}\varepsilon,$$

whenever $|t-t_0|<\delta_1$, $|s-s_0|<\delta_1$, $\|x-x_0\|<\delta_1$.

Setting

$$\Psi(t,\tau,s)x=R_1(t,\tau)R_{m-1}(\tau,s)x,$$

we have

$$\|\Psi(t, \tau, s)x-\Psi(t_0, \tau, s_0)x_0\| \leqslant \|\Psi(t, \tau, s)(x-x_0)\|+\|(\Psi(t, \tau, s)-\Psi(t_0, \tau, s_0))x_0\|$$

and, by (15) and (16),

$$\|\Psi(t, \tau, s)\| \leqslant \begin{cases} LC_0\tilde{M}(\tfrac{1}{2}\delta)^{\alpha-1}(\tfrac{1}{2}\delta)^{(m-1)\alpha-1} & \text{if } (m-1)\alpha-1<0, \\ LC_0\tilde{M}(\tfrac{1}{2}\delta)^{\alpha-1}T^{(m-1)\alpha-1} & \text{if } (m-1)\alpha-1 \geqslant 0, \end{cases}$$

when $|t-t_0| \leqslant \frac{1}{2}\delta$, $|s-s_0| \leqslant \frac{1}{2}\delta$, $\tau\in[s_0+\delta, t_0-\delta]$.

Since Ψ is continuous on the compact set $K\times\{x_0\}$, where

$$K = \{(t, \tau, s)\colon |t-t_0| \leqslant \tfrac{1}{2}\delta,\ |s-s_0| \leqslant \tfrac{1}{2}\delta,\ \tau\in[s_0+\delta, t_0-\delta]\},$$

there exists $0<\delta_1<\frac{1}{2}\delta$ such that

$$\|\Psi(t, \tau, s)x_0-\Psi(t_0, \tau, s_0)x_0\| < \varepsilon/4T$$

and

$$\|\Psi(t, \tau, s)\|\,\|x-x_0\| < \varepsilon/4T$$

for $|t-t_0|<\delta_1$, $|s-s_0|<\delta_1$, $\|x-x_0\|<\delta_1$ and $\tau\in[s_0+\delta, t_0-\delta]$.

To prove assertion (b) it is enough to show that the mapping (22) is continuous at any point (t_0, s_0) with $t_0 = s_0$. By Lemma 4 this is true for $m = 1$. Suppose that (22) is continuous for $k = 1, \ldots, m-1$. Since $R_m(t_0, t_0) = 0$,

$$R_m(t, s)x-R_m(t_0, t_0)x = \int_s^t R_1(t, \tau)R_{m-1}(\tau, s)x\,d\tau.$$

By the inductive assumption there exists $M>0$ such that $\|R_{m-1}(\tau, s)x\| \leqslant M$. Thus,

$$\|R_m(t, s)x\| \leqslant \text{const}\cdot(t-s)^\alpha \to 0 \quad \text{as } t-s\to 0.$$

PROPOSITION 1. *If Assumptions* Z_1, Z_2, Z_3 *are fulfilled, then*

(a) *the mapping*

$$\Delta_T^0\times X \ni (t, s, x) \to R(t, s)x$$

is continuous;

(b) *for every* $x\in D$ *the mapping*

$$\Delta_T \ni (t, s) \to R(t, s)x$$

is continuous.

Proof. Let $x\in D$, $m_0 = [1/\alpha]+1$ and let

$$\Phi(m) = \big(LC_0\Gamma(\alpha)\big)^m/\Gamma(m\alpha).$$

By (16)

$$\|R_m(t, s)\| \leqslant \Phi(m)(t-s)^{m\alpha-1}.$$

Since (cf. [4]).

(23) $$\Gamma(x) = \sqrt{2\pi}x^{x-1/2}e^{-x}(1+r(x))$$

with $|r(x)| \leqslant e^{1/12x}-1$, we have

$$\lim_{m\to\infty} \sqrt[m]{\Phi(m)} = 0.$$

Since

(24) $$\|R_m(t, s)\| \leqslant \Phi(m)(t-s)^{m\alpha-1} \leqslant \Phi(m)T^{m\alpha-1}$$

for $m \geqslant m_0$, $(t, s) \in \Delta_T$, the series $\sum_{m=1}^{\infty} R_m(t, s)$ is uniformly convergent on Δ_T. Hence and by Lemma 5, Proposition 1 follows.

4. Continuity with respect to parameters. In this section we study the continuity of the function $u_h(t)$ given by the formula

(25) $$u_h(t) = U_h(t, 0)u_h^0 + \int_s^t U_h(t, s)f(h, s)ds$$

with respect to both h and t, where U_h is given by (14).

Since conditions (14) for U_h to be the fundamental solution of (8) are well known in the literature (see e.g. [8], Chapter 5), we do not discuss them here. We shall only inspect the continuity of the mapping

$$u\colon \Omega\times[0, T]\ni(h, t)\to u_h(t)\in X$$

with $u_h(t)$ given by (25), without specifying assumptions which would guarantee that $u_h(t)$ is the solution of problem (8)–(9). The continuity of u will be proved here under Assumptions Z_1, Z_2, Z_3, which are weaker than the conditions given in [8], sufficient for the function (25) to be the solution of problem (8)–(9).

Remark 3. Suppose that Assumptions Z_1, Z_2, Z_3 are fulfilled; then (following [9], Proposition 1) the function

$$[0, T]\times\Omega\times[0, T]\times X\ni(r, h, \tau, x)\to\exp(-rA_h(\tau))x$$

is continuous. This fact will be used in the sequel.

LEMMA 6. *If Assumptions* Z_1, Z_2, Z_3 *are fulfilled, then*

(a) *for any* $x\in D$ *and* $h_0\in\Omega$

$$\lim_{h\to h_0} R_1^h(t, s)x = R_1^{h_0}(t, s)x$$

uniformly on Δ_T;

(b) *for any compact sets* $K_1\subset\Delta_T^0$, $K_2\subset X$

$$\lim_{h\to h_0} R_1^h(t, s)x = R_1^{h_0}(t, s)x$$

uniformly on $K_1\times K_2$.

Proof. To prove (a) let us fix $x \in D$ and $h_0 \in \Omega$. Then

$$\|R_1^h(t,s)x - R_1^{h_0}(t,s)x\| \leqslant \|(A_h(t)-A_h(s))\exp(-(t-s)A_h(s))x - (A_{h_0}(t)-A_{h_0}(s))\exp(-(t-s)A_{h_0}(s))x\|$$
$$\leqslant \|(A_h(t)-A_{h_0}(t))\exp(-(t-s)A_h(s))x\| + \|(A_h(s)-A_{h_0}(s))\exp(-(t-s)A_h(s))x\|$$
$$+ \|(A_{h_0}(t)-A_{h_0}(s))[\exp(-(t-s)A_h(s))-\exp(-(t-s)A_{h_0}(s))]x\|.$$

Setting $\exp(-(t-s)A_h(s)) = T(h,t,s)$ we have

$$\|T(h,t,s)\| \leqslant \tilde{C} \quad \text{for } (h,t,s) \in \Omega \times \Delta_T$$

with a positive constant $\tilde{C}$. Thus

$$\|(A_h(t)-A_{h_0}(t))T(h,t,s)x\| \leqslant \|(A_h(t)-A_{h_0}(t))A_{h_0}^{-1}(s)A_{h_0}(s)T(h,t,s)x\|.$$

By (4),

$$\lim_{h\to h_0} \|(A_h(t)-A_{h_0}(t))A_{h_0}^{-1}(s)\| = 0$$

uniformly on Δ_T. Next, we have

$$A_{h_0}(s)T(h,t,s)x = A_{h_0}(s)A_h^{-1}(s)A_h(s)T(h,t,s)x.$$

But

$$A_{h_0}(s)A_h^{-1}(s) = (A_h(s)A_{h_0}^{-1}(s))^{-1}.$$

By (4) applied to $t = s$, $s \in [0, T]$, we have

$$\lim_{h\to h_0} \|(A_h(s)-A_{h_0}(s))A_{h_0}^{-1}(s)\| = 0 \tag{26}$$

uniformly on $[0, T]$.

Observe that the mappings

$$(h,s) \to A_{h_0}(0)A_{h_0}^{-1}(s), \quad (h,s) \to A_h(s)A_{h_0}^{-1}(s),$$
$$(h,s) \to A_{h_0}(s)A_h^{-1}(s), \quad (h,s) \to A_{h_0}(s)x$$

and

$$\varphi\colon (h,s) \to A_h(s)A_{h_0}^{-1}(s)A_{h_0}(s)x$$

are continuous in $\Omega \times [0, T]$ and

$$A_h(s)T(h,t,s)x = T(h,t,s)A_h(s)x = T(h,t,s)A_h(s)A_{h_0}^{-1}(s)A_h(s)x.$$

Let $V \subset \Omega$ be a compact neighbourhood of h_0. Since φ is continuous, there exists $K > 0$ such that

$$\|A_h(s)T(h,t,s)\| \leqslant K \quad \text{for } (h,s) \in V \times [0, T]. \tag{27}$$

Now, it is clear that

(28) $\|(A_h(t)-A_{h_0}(t))T(h,t,s)x\|$

$$\leqslant \|(A_h(t)-A_{h_0}(t))A_{h_0}^{-1}(s)\| \; \|A_{h_0}(s)A_h^{-1}(s)\| \; \|A_h(s)T(h,t,s)x\| \to 0$$

as $h \to h_0$, uniformly on Δ_T.

Similarly,

(29) $$\lim_{h\to h_0} (A_h(s)-A_{h_0}(s))T(h,t,s)x = 0$$

uniformly on Δ_T.

We finally prove that

$$\lim_{h\to h_0} \|(A_{h_0}(t)-A_{h_0}(s))(T(h,t,s)-T(h_0,t,s))x\| = 0$$

uniformly on Δ_T. Since by Lemma 1, $\|(A_{h_0}(t)-A_{h_0}(s)A_{h_0}^{-1}(s))\|$ is bounded in Δ_T, we have to prove that

$$\lim_{h\to h_0} \|A_{h_0}(s)(T(h,t,s)-T(h_0,t,s))x\| = 0$$

uniformly on Δ_T. Now,

$$\|A_{h_0}(s)(T(h,t,s)-T(h_0,t,s))x\|$$
$$\leqslant \|A_{h_0}(s)A_h^{-1}(s)-A_{h_0}(s)A_{h_0}^{-1}(s)\| \, \|T(h,t,s)A_h(s)x\|$$
$$+\|T(h,t,s)A_h(s)x-T(h_0,t,s)A_{h_0}(s)x\|.$$

By (4) and (27) we see that the first term of the right-hand side of this inequality tends to 0 as $h \to h_0$. Since

$$\|T(h,t,s)A_h(s)x-T(h_0,t,s)A_{h_0}(s)x\|$$
$$\leqslant \|T(h,t,s)(A_h(s)x-A_{h_0}(s)x)\|+\|(T(h,t,s)-T(h_0,t,s))A_{h_0}(s)x\|,$$

we see, by our assumptions and Remark 3, that

$$\lim_{h\to h_0} \|T(h,t,s)A_h(s)x-T(h_0,t,s)A_{h_0}(s)x\| = 0$$

uniformly in Δ_T.

To prove (b) it is enough to use inequality (15) and Lemma 2.

PROPOSITION 2. *If Assumptions Z_1, Z_2, Z_3 are fulfilled and $x \in D$, then*

$$\lim_{h\to h_0} R_m^h(t,s)x = R_m^{h_0}(t,s)x, \quad m = 1, 2, \ldots,$$

uniformly on Δ_T.

Proof. Fix $x \in D$, $h_0 \in \Omega$. If $m = 1$, then the statement is true, by Lemma 6.

Suppose that the claim is true for $k = 1, \ldots, m-1$. Then

(30) $$\|R_m^h(t, s)x - R_m^{h_0}(t, s)x\| \leqslant \int_s^t \|(R_1^h(t, \tau) - R_1^{h_0}(t, \tau)) R_{m-1}^{h_0}(\tau, s)x\| d\tau + \int_s^t \|R_1^h(t, \tau)(R_{m-1}^h(\tau, s)x - R_{m-1}^{h_0}(\tau, s)x)\| d\tau.$$

Let us fix $\varepsilon > 0$. Setting

$$\mu(h) = \sup\{\|R_{m-1}^h(\tau, s)x - R_{m-1}^{h_0}(\tau, s)x\| : (\tau, s) \in \Delta_T\}$$

we have, by the inductive assumption

$$\lim_{h \to h_0} \mu(h) = 0.$$

Next, we see that

$$\int_s^t \|R_1^h(t, \tau)(R_{m-1}^h(\tau, s)x - R_{m-1}^{h_0}(\tau, s))x\| d\tau$$

$$\leqslant \mu(h) \int_s^t LC_0 (t-\tau)^{\alpha-1} d\tau = LC_0 \mu(h)(t-s)^\alpha/\alpha \leqslant \frac{LC_0}{\alpha} \mu(h) T^\alpha \quad \text{for } (t, s) \in \Delta_T.$$

Hence, there exists $\delta_1 > 0$ such that

(31) $$\int_s^t \|R_1^h(t, \tau)(R_{m-1}^h(\tau, s)x - R_{m-1}^{h_0}(\tau, s))x\| d\tau < \tfrac{1}{4}\varepsilon$$

for $(t, s) \in \Delta_T$, whenever $|h - h_0| < \delta_1$.

To prove that the first term of the right-hand side of (30) tends uniformly to 0, let us observe that there exists $\mu > 0$ such that

If $0 \leqslant c \leqslant d \leqslant T$, $d - c \leqslant \mu$, $d \leqslant t \leqslant T$, $0 \leqslant s \leqslant c$ *and* $h \in \Omega$, *then*

(32) $$\int_c^d \|(R_1^h(t, \tau) - R_1^{h_0}(t, \tau)) R_{m-1}^{h_0}(\tau, s)x\| d\tau < \tfrac{1}{4}\varepsilon.$$

Indeed, this follows from (15) and Lemma 5b).

We now write

(33) $$\Delta_T = K^\mu \cup (\Delta_T \setminus K^\mu),$$

where

(34) $$K^\mu = \{(t, s) \in \Delta_T : t - s \geqslant \mu\}.$$

Then, by (32), if $(t, s) \in \Delta_T \setminus K^\mu$ then

$$\int_s^t \|(R_1^h(t, \tau) - R_1^{h_0}(t, \tau)) R_{m-1}^{h_0}(\tau, s)x\| d\tau < \tfrac{1}{4}\varepsilon$$

for all $h \in \Omega$.

If $(t, s) \in K^{\mu}$, then

$$\int_s^t \|(R_1^h(t,\tau) - R_1^{h_0}(t,\tau)) R_{m-1}^{h_0}(\tau, s)x\| \, d\tau$$

$$= \int_s^{s+\mu/4} \|(R_1^h(t,\tau) - R_1^{h_0}(t,\tau)) R_{m-1}^{h_0}(\tau, s)x\| \, d\tau$$

$$+ \int_{s+\mu/4}^{t-\mu/4} \|(R_1^h(t,\tau) - R_1^{h_0}(t,\tau)) R_{m-1}^{h_0}(\tau, s)x\| \, d\tau$$

$$+ \int_{t-\mu/4}^{t} \|(R_1^h(t,\tau) - R_1^{h_0}(t,\tau)) R_{m-1}^{h_0}(\tau, s)x\| \, d\tau$$

and, by Lemma 5(b) and (32), there exists $\delta_2 > 0$ such that

$$\int_s^t \|(R_1^h(t,\tau) - R_1^{h_0}(t,\tau)) R_{m-1}^{h_0}(\tau, s)x\| \, d\tau < \tfrac{3}{4}\varepsilon$$

whenever $|h - h_0| < \delta_2$. Thus

$$\|R_m^h(t, s)x - R_m^{h_0}(t, s)x\| < \varepsilon \quad \text{for } (t, s) \in \Delta_T,$$

whenever $|h - h_0| < \delta = \min(\delta_1, \delta_2)$.

PROPOSITION 3. *If Assumptions* Z_1, Z_2, Z_3 *are fulfilled and* $x \in D$, *then*

$$\lim_{h \to h_0} R^h(t, s)x = R^{h_0}(t, s)x$$

uniformly on Δ_T.

Proof. Fix $x \in D \setminus \{0\}$, $h_0 \in \Omega$ and $\varepsilon > 0$. Since (see the proof of Proposition 1) the series is uniformly convergent on $\Omega \times \Delta_T$, there exists ν_0 such that

$$\|R^h(t, s) - \sum_{m=1}^{\nu_0} R_m^h(t, s)\| \leqslant \varepsilon/4\|x\| \quad \text{for } (h, t, s) \in \Omega \times \Delta_T. \tag{35}$$

Thus

$$\|R^h(t, s)x - \sum_{m=1}^{\nu_0} R_m^h(t, s)x\| < \tfrac{1}{4}\varepsilon \quad \text{for } (h, t, s) \in \Omega \times \Delta_T. \tag{36}$$

Moreover, in view of Proposition 2, there exist $\delta_j > 0$, $j = 1, 2, \ldots$, such that

$$\|R_j^h(t, s)x - R_j^{h_0}(t, s)x\| < \varepsilon/2\nu_0 \quad \text{for } (t, s) \in \Delta_T \tag{37}$$

and $|h - h_0| < \delta_j$. Hence and from (36) and (37) we get

$$\|R^h(t, s)x - R^{h_0}(t, s)x\| \leqslant \|R^h(t, s)x - \sum_{m=1}^{\nu_0} R_m^h(t, s)x\|$$

$$+ \|R^{h_0}(t, s)x - \sum_{m=1}^{\nu_0} R_m^{h_0}(t, s)x\| + \|\sum_{m=1}^{\nu_0} R_m^h(t, s)x - \sum_{m=1}^{\nu_0} R_m^{h_0}(t, s)x\| < \varepsilon$$

for $(t, s) \in \Delta_T$, $|h - h_0| < \delta = \min(\delta_1, \ldots, \delta_{\nu_0})$.

PROPOSITION 4. *If Asumptions* Z_1, Z_2, Z_3 *are fulfilled and* $x \in D$, *then*

$$\lim_{h \to h_0} W^h(t, s)x = W^{h_0}(t, s)x$$

uniformly on Δ_T.

Proof. Fix $x \in D$ and $h_0 \in \Omega$. We have

(38) $$\|W^h(t, s)x - W^{h_0}(t, s)x\|$$
$$\leqslant \Big\|\int_s^t [\exp(-(t-\tau)A_h(\tau)) - \exp(-(t-\tau)A_{h_0}(\tau))]\, R^{h_0}(\tau, s)x\, d\tau\Big\|$$
$$+ \Big\|\int_s^t \exp(-(t-\tau)A_h(\tau))[R^h(\tau, s)x - R^{h_0}(\tau, s)x]\, d\tau\Big\|.$$

Let us write, for $\delta > 0$,

$$\mu(\delta) = \sup\Big\{ \big\|\big(\exp(-rA_h(\tau)) - \exp(-rA_{h_0}(\tau))\big) R^{h_0}(p, s)x\big\| :$$
$$r \in [0, T],\ |h - h_0| \leqslant \delta,\ \tau \in [0, T],\ (p, s) \in \Delta_T\Big\}$$

and

$$\mu_1(\delta) = \sup\{\|R^h(\tau, s)x - R^{h_0}(\tau, s)x\| :\ (\tau, s) \in \Delta_T\}.$$

Since, by (38),

$$\|W^h(t, s)x - W^{h_0}(t, s)x\| \leqslant T\big(\mu(\delta) + \tilde{C}\mu_1(\delta)\big)$$

and $\lim_{\delta \to 0} \mu(\delta) = \lim_{\delta \to 0} \mu_1(\delta) = 0$, we get

$$\lim_{h \to h_0} W^h(t, s)x = W^{h_0}(t, s)x$$

uniformly on Δ_T.

PROPOSITION 5. *If Assumptions* Z_1, Z_2, Z_3 *are fulfilled, then the mapping*

$$\Omega \times \Delta_T \times X \ni (h, t, s, x) \to U_h(t, s)x$$

is continuous.

Proof. Let $h_\nu \to h_0$, $t_\nu \to t_0$, $s_\nu \to s_0$, $x_\nu \to x_0$. Then

$$U_{h_\nu}(t_\nu, s_\nu)x_\nu - U_{h_0}(t_0, s_0)x_\nu = U_{h_0}(t_\nu, s_\nu)x_\nu - U_{h_0}(t_\nu, s_\nu)x_\nu$$
$$+ \big(U_{h_0}(t_\nu, s_\nu) - U_{h_0}(t_0, s_0)\big)x_\nu + U_{h_0}(t_0, s_0)(x_\nu - x_0).$$

If $x \in D$ then, by Proposition 4 and Remark 4,

$$\lim_{h \to h_0} U_h(t, s)x = U_{h_0}(t, s)x$$

uniformly on Δ_T. Hence, by (19) and Lemma 2, for any compact $K \subset X$

$$\lim_{h\to h_0} U_h(t,s)x = U_{h_0}(t, s)x$$

uniformly on $\Delta_T \times K$. Thus,

$$\lim_{\nu\to\infty} U_{h_\nu}(t_\nu, s_\nu)x_\nu = U_{h_0}(t_0, s_0)x_0$$

ends the proof.

As a simple consequence of Proposition 5 we obtain the main result of this paper.

THEOREM 1. *If Assumptions* Z_1, Z_2, Z_3 *are fulfilled and the mappings*

$$\Omega \ni h \to u_h^0 \in X \tag{39}$$

and

$$\Omega \times [0, T] \ni (h, t) \to f(h, t) \tag{40}$$

are continuous, then the mapping

$$\Omega \times [0, T] \ni (h, t) \to u_h(t) \in X$$

with $u_h(t)$ *given by* (25) *is continuous.*

Proof. By (25) and Proposition 5, we only have to prove that the mapping

$$\Omega \times [0, T] \ni (h, t) \to \int_0^t U_h(t, s) f(h, s) ds \tag{41}$$

is continuous. If $h_0 \in \Omega$ and $t_0 \in [0, T]$ are fixed, then

$$\int_0^t U_h(t, s) f(h, s) ds - \int_0^{t_0} U_{h_0}(t_0, s) f(h_0, s) ds$$
$$= \int_0^T [\chi_{[0,t]}(s) U_h(t, s) f(h, s) - \chi_{[0,t_0]}(s) U_{h_0}(t_0, s) f(h_0, s)] ds,$$

where $\chi_{[a,b]}$ denotes the characteristic function of $[a, b]$. Thus, by Lebesgue's theorem, the mapping (41) is continuous.

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