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Local analysis of non-standard C^∞ functions of pre-distributional type

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Abstract. The authors define pre-distributions as equivalence classes of non-standard $*C^\infty$ functions, and analyze the classes of the so-called K-pre-distributions. Discrete oscillation, essential oscillation and other properties are defined in the monad of a standard point. Distributions of Schwartz are classified with respect to the essential behaviour in the monads of standard points. Two different definitions of order are shown to be equivalent. This analysis deliberately avoids the L_2 approach suggested by A. Robinson, instead pursuing the basic L_1 setting, which is lifted to a non-standard model.

0. Introductory comments regarding terminology for the non-standard model

The basic "non-standard" concepts of this paper are all derived from the classic work of Abraham Robinson, cf. [12]. We make no pretense of enlarging upon the theory of non-standard models *per se*; all the results in the paper are "applied". Various technical lemmas of Robinson [12], Luxemburg [10], or the general folklore will be used, often *implicitly*, to link together the steps in our arguments. A certain very simple formal language $\mathscr{L}$ will suffice for our "logical" purposes. The key properties of $\mathscr{L}$, in interaction with its non-standard model $*\boldsymbol{R}$ are given in Appendix II, to which we shall frequently refer. ($*\boldsymbol{R}$ is defined in Appendix I.) In this introduction, we shall morely specify our main notational and terminological conventions and allude to some of the most fundamental properties of $*\boldsymbol{R}$.

For simplicity's sake, we shall take as our base (or "standard") structure the real line $\boldsymbol{R}$. (All our arguments are valid for base structure = arbitrary $\boldsymbol{R}^n$, $n \geqslant 1$, and several of them extend without significant modification to an arbitrary metric space as base.) The finite-rank "universe" $\hat{R}$, based on $\boldsymbol{R}$, is described in Appendix I; from there one passes to the non-standard extension $*\boldsymbol{R}$ in which the analysis is to be done. The most immediate property of $\hat{R}$ is that all functions and relations in which we are interested, for the purposes of ordinary real analysis, are *elements of* $\hat{R}$. The most immediate property of $\tilde{\boldsymbol{R}}$ is that $\hat{R}$ is embedded in $\tilde{\boldsymbol{R}}$ as an elementary submodel (see Appendix II) relative to the language $\mathscr{L}$. $\tilde{\boldsymbol{R}}$ is much bigger than $\hat{R}$: it contains "real numbers" which are larger in absolute value than any positive element of $\boldsymbol{R}$; the reciprocals of such giant reals are called *infinitesimals*. The set of all infinitesimal reals in the $\tilde{\boldsymbol{R}}$-version, $^*\boldsymbol{R}$, of $\boldsymbol{R}$, together with zero, constitute the so-called "monad of zero", and will be denoted by $\mu(0)$. $\mu(0)_+$ will denote the positive elements of $\mu(0)$. More generally, if $r \in \boldsymbol{R}$, then $\mu(r)$ denotes $\{x \in {}^*\boldsymbol{R} \mid x - r \in \mu(0)\}$. (We shall avoid the pedantry of using different symbols to denote subtraction in $\boldsymbol{R}$ and its extension to $^*\boldsymbol{R}$; similarly with other operations and relations of analysis.) $^*\boldsymbol{R}^\infty_+$ and $^*\boldsymbol{R}^\infty_-$ will denote, respectively, the sets of positive infinite and negative infinite elements of $^*\boldsymbol{R}$, and $^*\boldsymbol{R}^\infty = {}^*\boldsymbol{R}^\infty_- \cup {}^*\boldsymbol{R}^\infty_+$. $^*\boldsymbol{R} - {}^*\boldsymbol{R}^\infty$ will be denoted by $^*\boldsymbol{R}_{\mathrm{Bd}}$ (the *bounded part* of the non-standard real line). For $x, y \in {}^*\boldsymbol{R}$, we write $x \approx y$ to mean that $x - y \in \mu(0)$. If $x \in {}^*\boldsymbol{R}_{\mathrm{Bd}}$, we denote by $\mathrm{st}(x)$ that unique element y of $\boldsymbol{R}$ (see [12]) such that $y \approx x$.

Finally, analogous with our use of the notation $^*\boldsymbol{R}$, we shall denote by $^*S$ the $\tilde{\boldsymbol{R}}$-version (i.e., the non-standard extension) of any set S belonging to $\hat{R}$. The reader may wish, at this point, to examine the appendices.

1. Distributions, monads, and elements of $^*C^\infty$

The classical Schwarzian theory of distributions, in a nutshell, says: "look at the bounded-on-compacta linear functionals from the space C_0^∞ (of C^∞ functions with compact support) into the reals; you will see the Dirac Delta and a host of other useful gadgets". An alternative approach, of a somewhat more "constructive" character, was subsequently developed by Mikusiński ([1], [11]), using the notion of a *fundamental sequence* of continuous functions. (For a non-standard discussion of the Mikusiński point of view, see [9].) In the work which follows, we are *motivated* by the Mikusiński approach; however, we shall in general give definitions and arguments which bear the mark of the Schwarzian point of view. The point here is, that non-standard analysis leaps to mind when we are confronted with a fundamental sequence of C^∞ functions: just pick off a term having infinite index in the $\hat{R}$-version of the sequence, and a distribution in $\hat{R}$ is thereby converted into a C^∞ function in $\hat{\boldsymbol{R}}$. On the other hand, in formulating our proofs it has been convenient to manipulate inner product integrals rather than work with extensions of fundamental sequences.

In the present paper, we are not concerned with "generalizing" distribution theory via non-standard analysis, nor even with reproducing it in its present state of generality by that method. Rather, we wish to *apply* distributional ideas in studying the infinitesimally local behaviour of a significant class of $^*C^\infty$ functions (naturally enough, the significant class in question is then the class of "pre-distributions"); hopefully, such a study will provide us with some non-standard means whereby to double back and launch a fresh attack on some old problems, e.g., the multiplication problem. Accordingly, we shall for now restrict ourselves to the following framework: elements of $*C^\infty$ defined on the whole of $*\mathbf{R}$, and (standard) C_0^∞ as the "test space". Somewhat greather generality, if desired, can be purchased at the usual price of increased irksomeness in the book-keeping, although it is not clear whether certain compactness restrictions can be avoided.

We proceed now to basic definitions and preliminary results. The first step is to define "pre-distribution"; for the classical distribution-theoretic genesis of our definition, the reader is referred to Chapter 2 of [3].

1.1. DEFINITION. Let (D_K) denote (as in [3]) the set of all those elements of C_0^∞ with support $\subseteq K$, K a compact subset of $\mathbf{R}$. Let a functional $F\colon (D_K) \to *\mathbf{R}$ be called *strongly bounded* in case there is a positive number $M \in *\mathbf{R}_{\mathrm{Bd}}$ such that $|F(g)| < M \cdot \sup_{x \in *K} |g(x)|$ holds for all $g \in (D_K)$. A function $f(x) \in *C^\infty$, defined on all of $*\mathbf{R}$, is a *pre-distribution* iff the (extended Lebesgue) integral $\langle f(x), \varphi(x)\rangle_{*K} = \int_{*K} f(x)\varphi(x)\,dx$ is a strongly bounded linear functional on (D_K), for *all* compact $K \subseteq \mathbf{R}$.

Since the foregoing definition involves a markedly *external* object, namely, (D_K), the reader is fully justified in immediately demanding examples of pre-distributions; in particular, it is incumbent upon us to present a $*C^\infty$ version of Dirac's delta in the form of a pre-distribution. It turns out that this is not hard. Let $\langle a_n\rangle$ be a decreasing (standard) sequence of positive reals such that $\lim_{n\to\infty} a_n = 0$; and fix a number $c \in \mathbf{R}$, c positive. Let a sequence $\langle f_n(x)\rangle$ of functions be defined thus:

$$f_n(x) = \begin{cases} \dfrac{1}{ca_n}\exp\left(\dfrac{-a_n^2}{a_n^2 - x^2}\right) & \text{for } |x| < a_n; \\ 0 & \text{for } |x| \geqslant a_n. \end{cases}$$

Each $f_n(x)$ can be verified to be C_0^∞; moreover, we readily see that

$$\int_{-\infty}^{+\infty} f_n(x)\,dx = \frac{1}{ca_n}\int_{-a_n}^{a_n} \exp\left(\frac{-a_n^2}{a_n^2 - x^2}\right)dx = \frac{1}{c}\int_{-1}^{1} \exp\left(\frac{-1}{1-\xi^2}\right)d\xi.$$

The last integral, however, is independent of a_n; so, if we simply choose

$$c = \int_{-1}^{1} \exp\left(\frac{-1}{1-\xi^2}\right)d\xi,$$

we then have that $\int_{-\infty}^{+\infty} f_n(x)\,dx = 1$ for all n. Now, consider the extension,

$*\langle f_n(x)\rangle$, of the sequence $\langle f_n(x)\rangle$ in $\hat{\boldsymbol{R}}$, and let $n = \omega =$ an *infinite* positive integer. Using integration by parts (and recalling that $\varphi(x) \in C^\infty \Rightarrow \varphi(0) \approx \varphi(t)$ for all $t \in \mu(0)$), we readily calculate $\int_{-\infty}^{\infty} f_\omega(x)\varphi(x)\,dx \approx \varphi(0)$ for each $\varphi(x) \in C_0^\infty$. Thus, $f_\omega(x)$ is a pre-distribution representing Dirac's delta. (Upon taking *standard parts*, i.e., forming $\mathrm{St}(\int_{-\infty}^{+\infty} f_\omega(x)\varphi(x)\,dx)$, $f_\omega(x)$ *literally* behaves as δ.)

Having exhibited δ in terms of a pre-distribution, we can exhibit much more. The following simple proposition shows that the usual notion of distributional derivative is precisely the one which applies to our definition of pre-distribution:

1.2. Proposition. *Let $f(x)$ be any pre-distribution, and let $\varphi(x) \in C_0^\infty$, support $(\varphi(x)) \subseteq I$, where $I =$ a standard closed finite interval of R. Then $\int_I f'(x)\varphi(x)\,dx = -\int_I f(x)\varphi'(x)\,dx$. (Thus, in terms of Definition 1.1, we have: $\langle f', \varphi\rangle_I = -\langle f, \varphi'\rangle_I$, a formula which is immediately recognized from standard distribution theory.)*

Proof. The proof is just the usual elementary application of "integration by parts": $\int_I f'(x)\varphi(x)dx = f(x)\varphi(x)|_a^b - \int_I f(x)\varphi'(x)dx$, where $I = [a, b]$. But $f(x)\varphi(x)|_a^b = 0$, since support $(\varphi(x)) \subseteq$ the interior of I. That is all.

Since $\int_I f(x)\varphi(x)\,dx$ is a strongly bounded linear operator on (D_I), so also must be $-\int_I f(x)\varphi'(x)\,dx$; thus $\delta, \delta', \delta'', \ldots$ are *all* pre-distributional in our sense; moreover, they have *uniform* pre-distributional representatives $f_\omega(x), f_\omega'(x), f_\omega''(x), \ldots$

(Note. The foregoing trivial proof of Proposition 1.2 readily generalizes to arbitrary compact $K \subseteq \boldsymbol{R}$: just enclose K in an *interval* I, and observe that no support has been added, for $\varphi(x)$.)

Since, clearly, all finite (in the standard sense) sums of pre-distributions and likewise all scalar multiples of pre-distributions *by elements of* $*\boldsymbol{R}_{\mathrm{Bd}}$ are again pre-distributions, the class of pre-distributions forms a vector space over $*\boldsymbol{R}_{\mathrm{Bd}}$, closed under differentiation. It follows (see [3], Chapter 4) that, in particular, every classical Schwarz distribution with point support is represented by a $*C^\infty$ function which induces a strongly bounded integral functional on every (D_K). This is by no means *everything* that is "pre-distributional" according to Definition 1.1; but it should be enough to convince the reader that we are not necessarily wasting our time.

We wish next to exhibit the criterion according to which we shall lump pre-distributions together into equivalence classes which we shall call *K-distributions*.

1.3. Definition. Let $f(x)$, $g(x)$ be two pre-distributions in $^*C^\infty$; and let Ω be a bounded open subset of $\boldsymbol{R}$. We say that $f(x)$ and $g(x)$ are *Ω-equivalent*, and write $f(x) \equiv_\Omega g(x)$, in case $\int_\Omega (f(x)-g(x))\varphi(x)\,dx \approx 0$ holds for all $\varphi(x) \in C_0^\infty$ such that support$(\varphi(x)) \subseteq \bar{\Omega}$.

For convenience, let us henceforth write $C^\infty(\Omega)$ to mean the set of C^∞ functions f: $\boldsymbol{R} \to \boldsymbol{R}$ such that (support$(f)) \subseteq \bar{\Omega}$.

1.4. Definition. By a *$K(\Omega)$-distribution* (Notation: $\alpha \in K(\Omega)$) we shall mean an equivalence class α of pre-distributions ("$K(\Omega)$-pre-distributions", as we shall call them) under the relation $\equiv_\Omega$. (It is obvious that $\equiv_\Omega$ is an equivalence relation.)

A $K(\Omega)$-distribution will be called *trivial* (or, the *zero distribution*) if $\int_\Omega f(x)\varphi(x)\,dx \approx 0$ holds for all $f \in \alpha$ and all $\varphi \in C^\infty(\Omega)$. It will be denoted by $\emptyset$.

1.5. Definition. Let $p \in \Omega$, Ω a bounded open subset of $\boldsymbol{R}$; and let $\alpha \in K(\Omega)$. p is called an *essential point of support* of α if $(\forall f(x) \in \alpha)$ $[\int_J f(x)\,dx \not\approx 0$ holds for some open interval $J \subseteq \mu(p)]$.

1.6. Definition. Let Ω, p, and α be as in Definition 1.5. p is called a *point of infinite support* of α if $(\forall f(x) \in \alpha)$ $[\int_J f(x)\,dx \in {}^*\boldsymbol{R}^\infty$ holds for some open interval $J \subseteq \mu(p)]$.

We pause to note that no $K(\Omega)$-distribution can have in it a pre-distribution witnessing an entire (standard) interval's worth of points of infinite support. (Later on, we shall get a much stronger result; the proposition at hand is merely by way of preliminary reassurance.)

1.7. Proposition. $(\forall \alpha \in K(\Omega))(\forall f(x) \in \alpha)$ $(\forall S = a$ *non-empty open subinterval of* $\Omega)$ $(\exists x \in S)[f(x) \in {}^*\boldsymbol{R}_{\mathrm{Bd}}]$.

Proof. Assume the contrary. Let $\varphi(x)$ be a $C^\infty(\Omega)$ function satisfying $\varphi(x) = 0$ for $x \in \Omega - S$ and $\varphi(x) > 0$ for $x \in S$, where $S, f(x)$, and α form a counterexample to the proposition. We may assume, w.l.o.g., that $f(x)$ does not change sign on S. But then $\int_{\bar{\Omega}} f(x)\varphi(x)\,dx \in {}^*\boldsymbol{R}^\infty$, which is a contradiction.

We observe that if $p \in \Omega$ is a point of infinite support of $f(x) \in \alpha$, then $f(x)$ must have a zero within any open set $S \subseteq \Omega$ for which $p \in S$. It is easily seen from "countable saturation" of $\boldsymbol{R}$ (see Appendix II) that this latter statement concerning $f(x)$ is equivalent to the assertion that $f(x)$ has a zero in $\mu(p)$. (We shall formally verify this assertion later on.)

We now introduce a notion of "oscillation at a (standard) point" which is closely related to the concept of infinite support at a point.

1.8. Definition. Let $\alpha \in K(\Omega)$; and let k be a (not necessarily standard) positive integer. α is said to *oscillate at least k times at p*, $p \in \Omega$, if $(\forall f(x) \in \alpha)[f(x)$ has at least k zeros in $\mu(p)]$. α is said to *oscillate k times at p* if α oscillates at least k times at p and $(\exists f(x) \in \alpha)\,[f(x)$ has *exactly* k zeros in $\mu(p)]$. Finally, we say that α is of *discrete oscillation at p* if there is an integer $k \in {}^*N$, $N = \{0, 1, 2, \ldots\}$, and a function $f(x) \in \alpha$ such that $f(x)$ has fewer than $k+1$ zeros in $\mu(p)$. (In this last case, we also say that the predistribution $f(x)$ is *of discrete oscillation at p*.)

Remarks. (a) We shall demonstrate further along, without much effort, that *all $K(\Omega)$-distributions are of discrete oscillation at p*. With somewhat greather effort, we shall then derive in section 4 one of our central results: they are in fact all of *standard* finite oscillation at p. (b) As the classic nontrivial example of "exact oscillation", consider the *k-th derivative of the Dirac Delta*; this distribution oscillates k times at 0.

1.8a. Definition 1.8 implies the definition of *essential oscillation* of a predistribution. $f(x)$ essentially oscillates k-times at p if and only if every predistribution belonging to the same distribution $\alpha \supset f(x)$ has at least k zeros in the monad of p, and some $g(x) \in \alpha$ has exactly k zeros in $\mu(p)$.

2. Theorems lifted from R to $^*R$

There are various facts concerning standard C^∞ functions which we need to "lift" (via condition 2 of Appendix II) to the non-standard real line $^*\boldsymbol{R}$. Most of these liftings require no comment and the lifted versions will be stated directly; but in a few cases involving "smoothing" (or "mollification") we shall explicitly state the *standard* theorems, which are then subsequently to be interpreted in $^*\boldsymbol{R}$. The following list of theorems is a representative sample, rather a *complete* list, of the "upstairs-downstairs" results needed in subsequent sections of this paper.

2.1. THEOREM. *Every non-empty open subset of $^*\boldsymbol{R}$ (in the interval topology) is of the form $\bigcup_n I_n$, where n ranges over $^*N$, $n \neq m \Rightarrow I_n \cap I_m = \emptyset$ and $\langle I_n \rangle$ is a strongly internal sequence* (see Appendix II) *of non-empty open intervals of $^*\boldsymbol{R}$.*

2.2. THEOREM. *Let $f(x) \in {}^*C^\infty$. Then the zero set $\{x \in {}^*\boldsymbol{R} |\ f(x) = 0\}$, of $f(x)$ is closed in $^*\boldsymbol{R}$ (i.e., has open complement in the interval topology on $^*\boldsymbol{R}$ determined by the extension of the standard relation $\leqslant$).*

2.3. THEOREM. *Let $f(x) \in {}^*\boldsymbol{C}$; and let $[a, b]$, $a < b$, be a finite closed interval in $^*\boldsymbol{R}$. Then there is a number $M \in {}^*\boldsymbol{R}$, $M > 0$, such that $|f(x)| < M$ for $a \leqslant x \leqslant b$ $(x \in {}^*\boldsymbol{R})$.*

2.4. THEOREM. ("Weierstrass C^∞ Approximation".) *Let $[a, b]$ be a finite closed subinterval of $^*\boldsymbol{R}$; and let $f(x) \in {}^*C^\infty(\Omega)$, Ω some open subset of $^*\boldsymbol{R}$ such* *that* $[a, b] \subseteq \Omega$. *Then there exists a (non-standard) sequence* $\langle B_n(x)\rangle$ *of (non-standard) polynomials over* $*\boldsymbol{R}$ *such that for all non-negative* $k \in *\boldsymbol{N}$ *we have:* $\lim_{n\to\infty} B_n^{(k)}(x) = f_n^{(k)}(x)$ *holds uniformly (in the sense of* $*\boldsymbol{R}$*) on* $[a, b]$. (Here, of course, "$B_n^{(k)}(x)$" denotes the k-th derivative of $B_n(x)$ with $B_n^{(0)}(x) = B_n(x)$, and similarly for "$f^{(k)}(x)$".)

We are ready to prove some basic theorems concerning properties of $*C^\infty$ functions.

2.5. THEOREM. *Let* $f(x) \in *C^\infty$ *have zeros in* $[a, b]$, *where* $[a, b]$ *is a finite closed subinterval of* $*\boldsymbol{R}$ *with* $a < b$; *and suppose* $\operatorname{supp}_0(f(x); [a, b]) = \bigcup_{n\in\boldsymbol{N}} I_n$, *where* $\operatorname{supp}_0(f(x); [a, b])$ *is defined as* $\{x \in [a, b] | f(x) \neq 0\}$ *and where* $\langle I_n | n \in *\boldsymbol{N}\rangle$ *is a disjoint (strongly internal) sequence of open subintervals of* $[a, b]$ *not all of which are empty. Finally, let* τ *be a positive element of* $*\boldsymbol{R}$. *Then there is a positive integer* $n_0 \in *\boldsymbol{N}$ *such that* $\left|\int_a^b |f(x)|\,dx - \sum_{n\leqslant n_0} \int_{I_n} |f(x)|\,dx\right| < \tau$. *Moreover, if E is the set of endpoints of the intervals* I_n, $n \leqslant n_0$ (all of which we may assume are *non-degenerate* intervals), *then we can arrange E into a non-decreasing sequence* $\{x_i | i \leqslant 2n_0 - 1\}$ *such that:*

(i) $x_{2i} < x_{2i+1} \leqslant x_{2i+2} < x_{2i+3}$ *holds for* $i \leqslant 2n_0 - 4$ *and* $x_0 < x_1$ *holds in case* $n_0 = 1$;

(ii) *each* x_i, *if not* $= a$ *or* b, *is a zero of* $f(x)$;

(iii) *no zero of* $f(x)$ *lies strictly between* x_{2i} *and* x_{2i+1}, $i \leqslant n_0 - 1$; *and*

(iv) *there is a positive element* δ *of* $*\boldsymbol{R}$ *such that* $\min\{x_{2i+1} - x_{2i} | i \leqslant n_0 - 1\} > \delta$ (whence, no point of $[a, b]$ is an accumulation point of the set $\{x_i | i \leqslant 2n_0 - 1\}$).

The remaining results to be cited in this section are standard theorems on "smoothing". We shall state them in standard form and let the reader carry out the (routine) "liftings" to $*\boldsymbol{R}$.

2.6. THEOREM. *Let* $g\colon \boldsymbol{R} \to \boldsymbol{R}$ *be* C^∞, *and let* a, b *be elements of* $\boldsymbol{R}$ *with* $a < b$. *Let* ε_1 *be any positive element of* $\boldsymbol{R}$. *Assume that* c, d *are elements of* $\boldsymbol{R}$ *such that* $a < c < d < b$ *and* $(\forall x)\,[c \leqslant x \leqslant d \Rightarrow g(x) = 0]$. *Assume, further, that* $g(x)$ *is positive in a left neighbourhood of c and in a right neighbourhood of d. Then there exist numbers e and f, with* $a < e < c$ *and* $d < f < b$, *a positive element* ε_2 *of* $\boldsymbol{R}$ *with* $|\varepsilon_2| < \varepsilon_1$, *and a* C^∞ *function* $h\colon \boldsymbol{R} \to \boldsymbol{R}$ *such that*

(i) $h(x) = g(x)$ *for* $x \notin [e, f]$;

(ii) $h(x) = \varepsilon_2$ *for* $x \in [c, d]$;

(iii) $h(x) > 0$ *for* $e < x < c$ *and for* $d < x < f$ *and*

(iv) $\max\left\{\int_e^c |h(x) - g(x)|\,dx,\ \int_d^f |h(x) - g(x)|\,dx\right\} < \varepsilon_1$.

If $g(x)$ is negative in a left neighbourhood of c and in a right neighbourhood of d, the same conclusion holds except that we must choose ε_2 to be negative and allow $h(x) < 0$ for $e < x < c$ and $d < x < f$. If $g(x)$ is negative in a left (right) neighbourhood of c and positive in a right (left) neighbourhood of d, then again the conclusion holds except that we must allow $h(x)$ to change sign, *once*, in the interval (e, c) (in the interval (d, f)).

2.7. THEOREM. *Let f: $\boldsymbol{R} \to \boldsymbol{R}$ be a C^∞ function; and let a, b be elements of $\boldsymbol{R}$ with $a < b$. Let ε be an arbitrarily given positive element $\boldsymbol{R}$. Then there exist numbers $c(\varepsilon)$ and $d(\varepsilon)$, and a C^∞ function h: $\boldsymbol{R} \to \boldsymbol{R}$, such that the following conditions are satisfied:*

(i) $a < c(\varepsilon) < d(\varepsilon) < b$;

(ii) $h(x) = 0$ *holds for* $x \notin [a, b]$;

(iii) $h(x) = f(x)$ *holds for* $x \in [c(\varepsilon), d(\varepsilon)]$ *and*

(iv) $\max\{\int_a^u |f(x)-h(x)|\,dx, \int_w^b |f(x)-h(x)|\,dx\} \leqslant \varepsilon$ *holds whenever* $a < u \leqslant c(\varepsilon)$ *and* $d(\varepsilon) \leqslant w < b$.

The proof of 2.7, as a theorem of standard analysis, consists in a straightforward application of "exponential mollification" arguments to the elementary fact that f is bounded on $[a, b]$.

2.8. THEOREM. *Let $f(x) \in C^\infty$ be given. Let $a, b \in \boldsymbol{R}$ with $a < b$; and let ε_1 be a positive real number. Let $\{x_i |\ i \leqslant 2m_1 - 1\}$, $m_1 > 0$, be a finite sequence of elements of $[a, b]$ such that:*

(1) $x_{2i} < x_{2i+1} \leqslant x_{2i+2} < x_{2i+3}$ *for all relevant values of i (i.e., for $i \leqslant 2m_i - 4$, or with $i = 0$ and the last two inequalities omitted in case $m_1 = 1$);*

(2) *each x_i is either a, b or a zero of $f(x)$;*

(3) *no zeros of $f(x)$ occur strictly between x_{2i} and x_{2i+1};*

(4) *there is a fixed positive number m such that* $\min\{x_{2i+1} - x_{2i} |\ i \leqslant m_1 - 1\} > m$.

Then there is a function $h(x) \in C^\infty$ such that:

(a) $h(x) = f(x)$ *for* $x \in \boldsymbol{R} - [a, b]$;

(b) *the zeros of $h(x)$ in $[a, b]$ occur exactly at the members of $\{x_i |\ i \leqslant 2m_1 - 1$ and $f(a) \neq 0 \Rightarrow x_i \neq a$ and $f(b) \neq 0 \Rightarrow x_i \neq b\}$; and*

(c)
$$e < d \Rightarrow \int_e^d |h(x) - f(x)|\,dx \leqslant \varepsilon_1 + \int_{[a,b]-H} |f(x)|\,dx,$$
where $H = \bigcup_{i \leqslant m_1 - 1} [x_{2i}, x_{2i+1}]$.

(Statement (4) in the hypotheses of Theorem 2.8 is, to be sure, an automatic consequence of the assumption that the given sequence is non-decreasing and finite; we mention it explicitly because in the "lifted" version it is necessary to allow $m \notin \mu(0)$).

2.9. THEOREM. *Let $f(x)$: $\boldsymbol{R} \to \boldsymbol{R}$ be a (standard) C^∞ function. Let a, b, c be elements of $\boldsymbol{R}$ with $a < b < c$. Let ε_1, ε_2 be any two positive elements of $\boldsymbol{R}$.*

If $f(x)$ is negative in a left neighbourhood of a, positive for $a < x < b$, zero at a, b and c, negative for $b < x < c$, and positive in a right neighbourhood of c, then there are C^∞ functions $g(x)$: $\boldsymbol{R} \to \boldsymbol{R}$ and $h(x)$: $\boldsymbol{R} \to \boldsymbol{R}$ and numbers $x_1, \ldots, x_6$ such that: $x_1 < a < x_2$ and $x_2 < b < x_3 < c$ and $g(x) = f(x)$ on $\boldsymbol{R} - [x_1, c]$ and $g(x) = -\varepsilon_2$ on $[x_2, x_3]$ and $\max\{\int_{x_1}^{x_2} |f(x) - g(x)|\,dx, \int_{x_3}^{c} |f(x) - g(x)|\,dx\} < \varepsilon_1$; $a < x_4 < b$ and $b < x_5 < c < x_6$ and $h(x) = f(x)$ on $\boldsymbol{R} - [a, x_6]$ and $h(x) = \varepsilon_2$ on $[x_4, x_5]$ and $\max\{\int_{a}^{x_4} |h(x) - f(x)|\,dx, \int_{x_5}^{x_6} |h(x) - f(x)|\,dx\} < \varepsilon_1$.

A corresponding statement (the exact formulation of which we shall leave here to the reader) holds in case $f(x)$ is positive in a left neighbourhood of a, negative for $a < x < b$, zero at a, b, and c, positive for $b < x < c$, and negative in a right neighbourhood of c.

2.10. THEOREM. *Let $f(x)$: $\boldsymbol{R} \to \boldsymbol{R}$ be a (standard) C^∞ function having finitely many zeros in the (standard) interval $[a, b]$. Let $\varepsilon > 0$ be given. Then, there is a (standard) C^∞ function $g(x)$: $\boldsymbol{R} \to \boldsymbol{R}$ and a (standard) interval $[c, d]$, with $[a, b] \subset [c, d]$, such that*

(i) $g(x) \geqslant 0$ *for all* $x \in [a, b]$;

(ii) $g(x) = f(x)$ *for* $x \in [c, d]$;

(iii) $\max\{a - c, d - b\} < \varepsilon$ *and*

(iv) $\big||f(x)| - g(x)\big| < \varepsilon$ *for all* $x \in [a, b]$.

Theorems 2.9, 2.10 can be lifted to $^*C^\infty$ functions f: $^*\boldsymbol{R} \to {}^*\boldsymbol{R}$ without changes.

We are now in a position to derive some initial results on the behaviour of a pre-distribution within the monad, $\mu(p)$ of a standard real number p; we shall make frequent use of the "countable saturation" property of $\boldsymbol{R}$ (see Appendix II).

3. Pre-distributions inside a monad: preliminary results

Throughout this section, Ω denotes some fixed bounded open subset of $\boldsymbol{R}$. Our first proposition is one which will be strengthened greatly in a later section.

3.1 PROPOSITION. *Let α be a $K(\Omega)$-distribution; and let $p \in \Omega$. Then α is of discrete oscillation at p.*

Proof. Let $f(x) \in \alpha$, with $f(\varepsilon) = 0$ holding for at least one $\varepsilon \in \mu(p)$; and let $[a, b] \subseteq \Omega$ be a standard interval such that $\mu(p) \subseteq {}^*[a, b]$. Let τ be a positive infinitesimal. By Theorems 2.1 and 2.5, together with the obvious fact that the finite additivity of the integral on $\boldsymbol{R}$ (relative to *disjoint* subsets of $\boldsymbol{R}$) "lifts" to the non-standard setting, we have the existence of a *-finite (i.e., *finite in the sense of* $^*\boldsymbol{N}$) disjoint (strongly internal) set S of closed subintervals of $[a, b]$ such that $\left|\int_S |f(x)|\,dx - \int_a^b |f(x)|\,dx\right| < \tau$. Let E denote the set of *end-points* of elements of S; then, since S is a *-finite collection, E contains a *largest* number x_{m_1} and a *smallest* number x_0. (We are assuming here, with no essential loss of generality, that $\mathrm{Supp}_0(f(x);\ [a, b]) \neq \emptyset$; in the contrary case, we need only apply the lifted version of Theorem 2.6, to an interval which slightly extends $[a, b]$ at each end.) Further, in view of Theorem 2.5, we can assume that no point of $[a, b]$ is a limit point of elements of E (note that each element of E other that a or b is a zero of $f(x)$). Applying the lifted form of Theorem 2.8, we obtain a function $g(x) \in \alpha$ such that $g(x)$ has no more than $m_1 + 1$ zeros in $[a, b]$ and hence no more than that in $\mu(p)$. (The function $g(x)$ provided by the lifted version of 2.8 is, in fact, *indistinguishable from* $f(x)$, in the sense that $c < d \Rightarrow \int_c^d |g(x) - f(x)|\,dx \approx 0$; this property obviously places $g(x)$ in α along with $f(x)$.) Proposition 3.1 clearly follows. There is another way to prove 3.1, and it involves the simplest case of a notion which plays an important role in the developments of Section 4. Letting $[a, b]$ be any standard interval, and $f(x)$ a function defined on $[a, b]$, we call $f(x)$ (as is customary) a *spline* (on $[a, b]$) in case $f(x)$ is *piecewise polynomial* on the interval $[a, b]$, i.e., in case there is an integer $n \geqslant 1$ and a partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ such that $f(x)$ restricted to $[x_{i-1}, x_i]$ is a polynomial, for $1 \leqslant i \leqslant n$. By the *degree* of the spline $f(x)$ on $[a, b]$ we mean $\max\{\mathrm{degree}(f|_{[x_{i-1}, x_i]})\}$, $1 \leqslant i \leqslant n$. Now, by application of the "smoothing" theorems of Section 2, we see that splines can be approximated by C^∞ functions in the following sharp sense: if $f(x)$ is a spline on $[a, b]$, and if two (standard) real numbers $\varepsilon > 0$ and $\tau > 0$ are given, then there is a C^∞ function $g(x)$ such that

(i) $|f(x) - g(x)| < \varepsilon$ for $x \in [a, b] \cap \bigcup_{i=1}^{n-1} [x_i - \tau, x_i + \tau]$;

(ii) $f(x) = g(x)$ for $x \in [a, b] - \bigcup_{i=1}^{n-1} [x_i - \tau, x_i + \tau]$ and

(iii) the zeros of $g(x)$ on $[a, b]$ are the same as the zeros of $f(x)$ on $[a, b]$.

Given a spline $f(x)$ on $[a, b]$ and two positive reals ε and τ, we define as follows the class $M(f, a, b, \varepsilon, \tau)$ of "(ε, τ)-mollified splines": $M(f, a, b, \varepsilon, \tau) = \{g(x) \in C^\infty \mid g(x)$ satisfies conditions (i), (ii), and (iii) with respect to $f(x)\}$. Keeping a and b standard but allowing ε and τ to be non-standard (i.e., elements of $^*\boldsymbol{R}_{\mathrm{Bd}}$), and permitting $f(x)$ to be an element of $^*S$, where S is the set of splines over $[a, b]$, we define $M_{\boldsymbol{R}}(f, a, b, \varepsilon, \tau) = \{g(x) \in {}^*C^\infty \mid g(x)$

satisfies (i), (ii), and (iii), in $\boldsymbol{R}$, with respect to $f(x)\}$. $\bigcup_{f\in{}^*S} M_{\boldsymbol{R}}(f, a, b, \varepsilon, \tau)$ is an internal set (i.e., an element of $\hat{\boldsymbol{R}}$) which we shall refer to as the set of (ε, τ)-*mollified splines over* $[a, b]$ *with respect to* $[a, b]$. The alternative approach to Proposition 3.1 which we mentioned above in now simply this: choose $\varepsilon = \tau$ = an element of $\mu(0)_+$, and apply Theorem 2.4 to obtain a (non-standard) polynomial $P(x)$ such that $|f(x) - P(x)| < \varepsilon$ holds uniformly on $[a, b]$, where $p \in [a, b]$ and $f(x)$ is a given predistribution (or, for that matter, any ${}^*C^\infty$ function); then define $g(x) = f(x)$ on ${}^*\boldsymbol{R} - [a, b]$, $g(x) = P(x)$ on $[a, b]$. Finally, by suitably applying a "smoothing" theorem (as in Section 2) to $g(x)$ *at the points a and b*, we obtain a ${}^*C^\infty$ function $h(x)$ such that

(1) $h(x)$ is a pre-distribution belonging to the same distribution as $f(x)$ and

(2) $h(x)$ is polynomial on some interval $[c, d]$ such that $a < c < p < d < b$. Since a (non-standard) polynomial is of discrete oscillation at every standard real, we are done. This argument, of course, does not require the general definition of (ε, τ)-mollified splines; polynomials would suffice. Later on, however, mollified splines become important, and this seems to be a reasonable point for their introduction.

3.2. PROPOSITION. *Let $\alpha \in K(\Omega)$, and let $p \in \Omega$. If p is a point of infinite support of α, then $f(x) \in \alpha \Rightarrow f(x)$ has a zero in $\mu(p)$.*

(Note. This generalizes the obvious fact that Dirac's delta has no point of infinite support.)

Proof. Suppose p is a point of infinite support of α; and let $f(x) \in \alpha$. As was noted relative to our proof of Proposition 1.7, $f(x)$ must then have a zero in any standard finite open interval containing p (otherwise, $f(x)$ would *fail to change sign* within some standard open interval (a, b) containing p, and the resultant fact that $\int_{(a,b)} f(x)\,dx \in {}^*\boldsymbol{R}^\infty$ would belie the pre-distributional character of $f(x)$). For each standard positive rational number q, let $F_q(v_0)$ be a formula which asserts of v_0 that $v_0 \in (p - 1/q, p + 1/q)$ and $f(v_0) = 0$. (These formulae, of course, contain an "added constant", namely, a denotation for the ${}^*C^\infty$ function $f(x)$.) Now, any *finite* subset $\{F_{q_1}(v_0), \ldots, F_{q_n}(v_0)\}$ of this collection of formulae is simultaneously satisfied by a suitably chosen element of ${}^*\boldsymbol{R}$. Therefore, by the countable saturation of $\boldsymbol{R}$, all $F_q(v_0)$ are simultaneously satisfied by some $x_0 \in {}^*\boldsymbol{R}$. Clearly, x_0 is a zero of $f(x)$ lying in $\mu(p)$.

Further on, we shall examine fully the question of a converse for Proposition 3.2 (i.e., does "essential oscillation" in $\mu(p)$ imply infinite support at p?) For the time being, we content ourselves with a fairly easy partial observation:

3.3 PROPOSITION. *Let $\alpha \in K(\Omega)$, let $p \in \Omega$, and assume that α oscillates k*

times at p, $k \geqslant 3$. Then $(\exists f(x) \in \alpha)\ [\int_J f(x)\,dx \in {}^*\boldsymbol{R}^{\infty}$ *holds for some open interval* $J \subseteq \mu(p)]$.

Note. In fact, if $f(x) \in \alpha$ witnesses exact oscillation k at p, then $f(x)$ is the required function.

Proof (sketch). Let $f(x) \in \alpha$ have exactly k zeros in $\mu(p)$ (¹). Then, since $f(x)$ witnesses discrete oscillation at p, we can choose three consecutive zeros x_1, x_2, x_3 in $\mu(p)$: $x_1 < x_2 < x_3$ and $f(x_1) = f(x_2) = f(x_3) = 0$ and $(\forall y)\ [(x_1 < y < x_2$ or $x_2 < y < x_3) \Rightarrow f(y) \neq 0]$. If either $\int_{x_1}^{x_2} f(x)\,dx \in {}^*\boldsymbol{R}^{\infty}$ or $\int_{x_2}^{x_3} f(x)\,dx \in {}^*\boldsymbol{R}^{\infty}$, we are done; hence, assume for a proof by contradiction, that both $\int_{x_1}^{x_2} f(x)\,dx$ and $\int_{x_2}^{x_3} f(x)\,dx$ are in ${}^*\boldsymbol{R}_{\mathrm{Bd}}$. Now, since all of x_1, x_2, x_3 are *essential* zeros of $f(x)$ in $\mu(p)$, one of $\int_{x_1}^{x_2} f(x)\,dx$, $\int_{x_2}^{x_3} f(x)\,dx$ is positive and the other negative. Let $\varepsilon_2 \approx \int_{x_1}^{x_3} f(x)\,dx$, and apply the appropriate instance of (the lifted version of) Theorem 2.9 to produce an element $g(x)$ of α such that $g(x)$ has two fewer zeros in $\mu(p)$ than does $f(x)$: contradiction. (Note that $x \in \mu(p) \Rightarrow \varphi(x) \approx \varphi(p)$ for each standard C_0^{∞} function φ defined at p; hence,

$$\begin{aligned}\int_{\Omega} g(x)\,\varphi(x)\,dx &= \int_{\Omega-[x_1,x_3]} g(x)\,\varphi(x)\,dx + \int_{[x_1,x_3]} g(x)\,\varphi(x)\,dx \\ &\approx \int_{\Omega-[x_1,x_3]} f(x)\,\varphi(x)\,dx + \varphi(p) \int_{[x_1,x_3]} g(x)\,dx \\ &\approx \int_{\Omega-[x_1,x_3]} f(x)\,\varphi(x)\,dx + \varphi(p) \int_{[x_1,x_3]} f(x)\,dx \\ &= \int_{\Omega} f(x)\,\varphi(x)\,dx.\end{aligned}$$

Thus, $g(x) \in \alpha$.)

Remark. Clearly, *one* of the things we must do in order to improve significantly on Proposition 3.3 is to show that $(\exists f(x) \in \alpha)[\int_J f(x)\,dx \in {}^*\boldsymbol{R}^{\infty}$ for some open interval $J \subseteq \mu(p)] \Rightarrow p$ is a point of infinite support of α. Or, what is the same thing, to show that the trivial $K(\Omega)$-distribution admits no representative $f(x)$ witnessing infinite support. This is one of the questions we address ourselves to in Section 4. (Also, to be sure, we wish to look into the "missing cases": $k = 1$, $k = 2$.)

(¹) It is not hard to see, by a countable saturation argument, that this implies that $f(x)$ has exactly k zeros in some standard interval $[a, b] \supseteq \mu(p)$. Cf. our proof of Proposition 3.4 below.

The final proposition of this section is one which will play a major role in the developments of Section 4.

3.4. PROPOSITION. *Let $f(x)$ be a $K(\Omega)$-pre-distribution which is of discrete oscillation at p, $p \in \Omega$. Then:*

(a) *there exist a number $\varepsilon \in \mu(0)_+$ and a standard positive integer n such that all zeros of $f(x)$ in $[p-1/n, p+1/n]$ lie inside $[p-\varepsilon, p+\varepsilon]$;*

(b) *there exist a number $\eta \in \mu(0)_+$ and a standard positive real M s.t. for any $\beta \in \mu(0)_+$ it is true that $|\int_{p-\eta-\beta}^{p+\eta+\beta} f(x)dx| < M$.*

(*Part* (b) *holds even if $f(x)$ is not of discrete oscillation.*)

Proof. (a) Suppose $f(x)$ has exactly ω zeros in $\mu(p)$, where $\omega \in {}^*N$ = the set of non-negative integers in $^*\boldsymbol{R}$. We claim, first, that there must exist a *standard* positive integer, n_0, such that $f(x)$ has exactly ω zeros in the interval $(p-1/n_0, p+1/n_0)$. For, if not, consider formulae $F_n(v_0)$, n any standard natural number > 0, such that:

(i) $F_n(v_0)$ contains v_0 as its only free variable;

(ii) $F_n(v_0)$ contains special constants to denote $f(x)$ and ω; and

(iii) $F_n(v_0)$ asserts that $0 < v_0 < 1/n$ and $[p-v_0, p+v_0]$ contains at least $\omega+1$ zeros of $f(x)$. Clearly any finite subcollection of these formulae is simultaneously satisfiable (by our assumption that n_0 does not exist); hence, by countable saturation, *all* $F_n(v_0)$ are satisfiable by a single (infinitesimal) $z \in {}^*\boldsymbol{R}$. But, obviously, the interval $[p-z, p+z]$ is then a subinterval of $\mu(p)$ containing at least $\omega+1$ zeros of $f(x)$: contradiction. Hence, we may suppose that $f(x)$ has exactly ω zeros in $(p-1/n_0, p+1/n_0)$, where n_0 is some standard positive integer. We are now in a position to apply a second saturation argument, as follows. Since $f(x)$ has exactly ω zeros in $(p-1/n_0, p+1/n_0)$, those zeros must in fact all lie in $\mu(p)$. Hence, we can simultaneously satisfy any finite subset of the following countably infinite list of formulae:

$$\hat{F}_{n_0+1}(v_0),\ \hat{F}_{n_0+2}(v_0), \ldots, \hat{F}_{n_0+j}(v_0), \ldots,$$

where $\hat{F}_{n_0+k}(v_0)$ has v_0 has its only free variable, contains constants to denote $f(x)$ and ω, and aserts of v_0 that $0 < v_0 < 1/(n_0+k)$ and $[p-1/n_0, p-1/v_0] \cup [p+1/v_0, p+1/n_0]$ contains no zeros of $f(x)$. Thus, by countable saturation, we conclude the existence of a number $\varepsilon \in \mu(0)_+$ such that all ω zeros of $f(x)$ in $[p-1/(n_0+1), p+1/(n_0+1]$ actually lie in the interval $[p-\varepsilon, p+\varepsilon]$.

(b) Let n_0 be a standard positive integer such that $[p-1/n_0, p+ +1/n_0] \subseteq \Omega$. For each positive integer $n \in {}^*N$, $n > n_0$, define: $\psi(n) = \max_{-1/n \leqslant x \leqslant 1/n} \{|\int_{p-x}^{p+x} f(x)dx|\}$. Then ψ is an *internal sequence* of real numbers; accordingly, by a well-known lemma of Robinson ([12]), if there is a standard positive real number M_0 such that $\psi(n) \leqslant M_0$ holds for all *standard* $n > n_0$, there is an *infinite* positive integer ω such that $n_0 < n \leqslant \omega$ implies that $\psi(n) \leqslant M_0$. But indeed, by the *strong boundedness* property of the functional $\int_{{}^*K} f(x)\varphi(x)\,dx$ we see (taking $\varphi(x)$ identically equal to 1 on $[p-1/n_0,\ p+1/n_0]$) that such an M_0, and hence such an ω, does indeed exist. But this implies that $\left|\int_{p-1/\omega-\beta}^{p+1/\omega+\beta} f(x)\,dx\right| < M_0+1$ holds for all $\beta \in \mu(0)_+$, and the proof is complete.

4. Pre-distributions inside a monad: the main results

We shall now launch a more serious attack on the monadal behaviour of pre-distributions. For convenience, we shall from now on assume that, $0 \in \Omega$ and work in $\mu(0)$ instead of in an arbitrary monad $\mu(p)$; this involves no actual loss of generality. In order to obtain the result mentioned at the end of Section 1, it will prove expedient to fix upon a singly, discretely oscillating representative $f(x)$ of a $K(\Omega)$-distribution α, and then consider the set consisting of those discretely oscillating elements $g(x)$ of α which are "almost indistinguishable" from $f(x)$.

The reason for this sort of procedure is to get our hands on a sufficiently comprehensive *internal* subset of α; α itself appears to be hopelessly external, relative to the class of constructions and arguments that we wish to employ. Here is the key definition:

4.1. Definition. Let α be a $K(\Omega)$-distribution, and let $f(x) \in \alpha$. Let $[a, b]$ be a standard interval such that $a < 0 < b$ and $[a, b] \subseteq \Omega$; and let $c \in \mu(0)_+$. By $N[f\colon a, b, c]$ we mean $\{n \in {}^*N \mid (\exists g(x))[g(x) \in {}^*C^\infty$ and $g(x)$ has exactly n zeros in $[a, b](\forall e)(\forall h)[(a \leqslant e \leqslant h \leqslant -c \Rightarrow \int_e^h |f(x)-g(x)|\,dx \approx 0)$ and $(c \leqslant e \leqslant h \leqslant b \Rightarrow \int_e^h |f(x)-g(x)|\,dx \approx 0)]$ and $\int_{-c}^{c} f(x)\,dx \approx \int_{-c}^{c} g(x)\,dx]\}$.

4.2. Lemma. *$N[f\colon a, b, c] \neq 0 \Rightarrow N[f\colon a, b, c]$ contains a smallest (non-standard) integer.*

Proof. $N[f\colon a, b, c]$ is an *internal* set of non-negative integers and hence, if non-empty, has a least element.

Naturally, Lemma 4.2 remains true if $N[f\colon a, b, c]$ is replaced by any one of its *internal* subsets.

4.3. Theorem. *The discrete oscillation property implies continuity at any point in the support of a distribution.*

Assuming the discrete oscillation property we intend to show that given a sequence of standard C^∞functions φ_n converging uniformly on some interval containing a (standard) point p (say $p = 0$) to a C^∞ function φ, and given any pre-distribution $f(x)$, there exists a positive number $\varepsilon_0 \approx 0$, such that for any $\bar{\varepsilon} > \varepsilon_0$, $\bar{\varepsilon} \approx 0$, for any infinite integer n, $\{\int_{-\bar{\varepsilon}}^{+\bar{\varepsilon}} f(x)\varphi_n(x)\,dx - \int_{-\bar{\varepsilon}}^{+\bar{\varepsilon}} f(x)\varphi(x)\,dx\}$. Equivalently given a standard number $\varepsilon > 0$ there exists a $\delta > 0$ such that $\{\int_{-\delta}^{+\delta} f\cdot\varphi_n\,dx - \int_{-\delta}^{+\delta} f\varphi\,dx\} \leqslant \varepsilon$.

Proof. Utilizing the result of 3.4, we can choose $\varepsilon_0 > 0$ inside the monad of zero, so that all zeros of $f(x)$ are contained in $[-\varepsilon_0, +\varepsilon_0]$, and $\int_{\varepsilon_0}^{\eta} f(x)\,dx \approx \int_{-\varepsilon_2-\eta_2}^{\varepsilon_0} f(x)\,dx \approx 0$, $\eta_2 \geqslant \eta_1 > \eta_0 > 0$ for $\eta_1 \approx 0 \approx \eta_2$. We make use of the uniform convergence of $\varphi_n(x)$ to $\varphi(x)$ on standard interval containing $[-\varepsilon_0, +\varepsilon_0]$. Using the definition and lifting the appropriate statement to $^*\boldsymbol{R}$, we choose n such that for all $x \in [-\varepsilon, +\varepsilon]\,|\varphi_n(x) - \varphi(x)| < 1/M$, where $M = \max_{x\in[-\varepsilon_0, +\varepsilon_0]} f(x)$. Now the conclusion follows.

4.4. The order of a distribution. A (K, Ω)-distribution α is called a *(K, Ω)-distribution of order zero at* x (or, it is called *finite*) if there exists a function $f(x) \in \alpha$, such that $f(x) \in {}^*\hat{\boldsymbol{R}}_{bd}$ for all $x \approx \bar{x}$.

We now introduce an inductive definition of a distribution of order $n \geqslant 1$, where n is a standard positive integer. A (K, Ω)-distribution α is said to be *of order n at $\bar{x}$* if it is not of order $n-1$ and if there exists $f(x) \in \alpha$ such that $(x-\bar{x})^n f(x) \in {}^*\hat{\boldsymbol{R}}_{bd}$ for all x in the monad of $\bar{x}$.

A distribution α is said to be *of order n in Ω* if it is of order n at some (standard) point of Ω.

The following property of distributions of Schwartz, $D'(\Omega)$ (obtained by taking $K(\Omega) = D(\Omega) = C_0^\infty(\Omega)$ with Ω contained in some compact subset of $\boldsymbol{R}$), is well known; we restate it in non-standard terms: *Let α be any distribution of Schwartz* (in $D'(\Omega)$); *then α is of some finite order.*

LEMMA. *The order of the distribution α, over a test space $K(\Omega)$ with $K \subseteq C^m(\Omega)$ for some $m \geqslant 0$, is well defined.* (Trivial.)

EXAMPLES. Let $K(\Omega)$ be the test space of Schwartz (of C^∞ functions with support in the precompact set Ω. Then the Dirac-delta equivalence class of distributions over $K(\Omega)$ is of order one. We also observe that $\delta^{(n)}(x)$ is of order n. An example of a distribution of infinite order (over some suitable space K of standard C^∞ functions having an essential zero at zero) is the formal solution of the axially symmetric problem in $\boldsymbol{R}^2$ posed by the differential equation

$$\frac{dY(r)}{dr} = -\frac{1}{2\pi} Y \frac{d}{dr}\left(\nabla^2 \log\frac{1}{r}\right), \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad (r = \sqrt{x^2+y^2}),$$

$Y(1) = 0$, which has a formal solution $Y = C\exp(\delta(x))$ (which is impossible to interpret as a distribution over any of the well-known test spaces $(D, E, Z, \ldots)$).

In what follows $\delta(x)$ will denote any pre-distribution over $K(\Omega)$ in the Dirac delta class.

4.5. LEMMA. *$x^\alpha\delta(x) \approx \emptyset$ iff $\alpha > 0$, and $\alpha \not\approx 0$.*

Proof. Choose any function $\psi(x)$ having only near standard values in some open standard neighbourhood of zero. Then $\int_y^z x^\alpha \delta(x)\varphi(x) \approx 0$ for any $y, z \in \mu(0)$. But if $\alpha \leqslant 0$, then choose $\varphi(x) \equiv 1$ and check that $\int_y^\varepsilon x^0 \delta(x)\varphi(x)\,dx = \int_y^z \delta(x)\varphi(x)\,dx \approx 1$, for a suitable choice of $y, z \in \mu(0)$. Since $\int_y^z |x^\alpha \delta(x)\varphi(x)|\,dx \gtrapprox \int_y^z |\delta(x)\varphi(x)|\,dx$ if $\alpha \lesssim 0$, the proof is complete.

(Note. The symbol $\lesseqgtr$ denotes: either $<$ or $\simeq$.)

THEOREM 4.5. *A pre-distribution $f(x)$ having $m > 0$ essential oscillations at zero is a derivative of order m of a pre-distribution having the following form in the monad of zero $F(x) = \varphi(x) + C\delta(x)$, where $\varphi(x)$ is an absolutely continuous near-standard function in some standard neighbourhood of zero, C is a near standard number, and $\delta(x)$ is an element of the Dirac-delta distribution.*

Proof. We make the following observation. Let $x_1, x_2, \ldots, x_m$ denote the location of zeros of $f(x)$ in the monad of zero. Then there exist points x_0, $x_{m+1} \in \mu(0)$ such that

$$\int_{x_0}^{x_1} f(x)\,dx \not\approx 0, \qquad \int_{x_1}^{x_2} f(x)\,dx \not\approx 0, \qquad \ldots, \qquad \int_{x_m}^{x_{m+1}} f(x)\,dx \not\approx 0,$$

but

$$\int_\xi^{x_0} f(x)\,dx \approx 0, \qquad \int_{x_{m+1}}^{\eta} f(x)\,dx \approx 0 \qquad \text{for any } \xi < x_0,\ \eta > x_{m+1},\ \xi, \eta \in \mu(0).$$

In fact if $m > 0$, then $\int_{x_i}^{x_{i+1}} f(x)\,dx \in {}^*\boldsymbol{R}_\infty$, $i = 0, 1, \ldots, m$, but we we shall not make use of this property at this time.

This observation implies that $F(x) = \int_y^x f(\xi)\,d\xi$ $(y < x_0)$ has at most $m-1$ essential zeros in $\mu(0)$.

Next we observe that the only distributions which have zero as an essential point of support and have zero oscillation number at zero are of the form $C\delta(x)$, where C is a near-standard number and $\delta(x)$ is Dirac-delta equivalence class. Hence

$$\int_y^x \int_y^{\bar{x}_{m-1}} \cdots \int_y^{\bar{x}_1} f(\xi)\,d\xi = \frac{1}{(m-1)!}\int_y^x (x-\xi)^{m-1} f(\xi)\,d\xi$$

is of the form $C\delta(x)+\varphi(x)$, where $\varphi(x)$ is a function which is regular at zero, and C is a near-standard number.

Remark. It follows easily that any pre-distribution $f(x)$ having oscillation number $m>0$ at zero is of the form: $f(x) = \sum_{k=0}^{m} C_k\delta^{(k)}(x)+\psi(x)$, where ψ is regular at zero, and $C_m \not\approx 0$.

We shall introduce the following notation.

Let $f(x)$ be a pre-distribution having its essential support in the monad of zero contained in an interval $[-a, +a]$. Suppose $x_1, x_2, \ldots, x_m$ are the zeros of $f(x)$ which are essential zeros in $\mu(0)$. We denote

$$F_0 = \int_{-a}^{x_1} f(\xi)\,d\xi, \quad F_1 = \int_{x_1}^{x_2} f(\xi)\,d\xi, \ldots, \quad F_m = \int_{x_m}^{+a} f(\xi)\,d\xi;$$

$$\bar{x}_0 = \frac{\int_{-a}^{x_1} xf(x)\,dx}{F_0}, \quad \bar{x}_1 = \frac{\int_{x_1}^{x_2} xf(x)\,dx}{F_1}, \ldots, \quad \bar{x}_m = \frac{\int_{x_m}^{a} xf(x)\,dx}{F_m};$$

$$\bar{x}_0^{[2]} = \frac{\int_{-a}^{x_1} x^2 f(x)\,dx}{F_0}, \ldots, \quad \bar{x}_m^{[2]} = \frac{\int_{x_m}^{+a} x^2 f(x)\,dx}{F_m},$$

$$x_0^{[m]} = \frac{\int_{-a}^{x_1} x^m f(x)\,dx}{F_0}, \ldots, \quad x_m^{[m]} = \frac{\int_{x_m}^{a} x^m f(x)\,dx}{F_m},$$

and f_+ denotes the function $f_+(x) = f(x)$ if $f(x)>0$, $f_+(x) = 0$ if $f(x) \leqslant 0$, f_- is defined identically.

4.6. THEOREM. *A distribution $f(x)$ having m zeros and areas $F_0, F_1, \ldots, F_m$ over the essential part of the monad of zero is equivalent to any distribution $g(x)$ having m zeros and areas*

(i) $F'_0 \approx F_0$, $F'_1 \approx F_1, \ldots, F'_m \approx F_m$ *with the moments relationships,*

(ii) $F_0\bar{x}_0 \approx F'_0\bar{x}'_0$, $F_1\bar{x}_1 \approx F'_1\bar{x}'_1, \ldots, F_m\bar{x}_m \approx F'_m\bar{x}'_m$, *and*

(iii) $F_0\bar{x}_0^{[2]} \approx F'_0\bar{x}_0'^{[2]}, \ldots, (m+1)\ F_0\bar{x}_0^{[m]} \approx F'_0 x_0'^{[m]}, \ldots, F_m\bar{x}_m^{[m]} \approx F'_m x_m'^{[m]}$.

Proof. Choose any $\varphi \in K(r)$ having at least $m+1$ continuous derivatives at zero. Let $[-a, +a]$ denote the interval of essential support in the monad of zero for both $f(x)$ and $g(x)$.

Compute $\int_{-a}^{+a} f(x)\varphi(x)\,dx$ and $\int_{-a}^{+a} g(x)\varphi(x)\,dx$. Let us represent $\varphi(x)$ by Taylor series expansion with the remainder term:

$$\varphi(x) = \varphi(0) + \frac{\varphi'(0)\,x}{1} + \ldots + \frac{\varphi^{(m)}(0)\,x^m}{m!} + \frac{\varphi^{m+1}(\xi)\,x^{m+1}}{(m+1)!}, \qquad \xi \in \mu(0),$$

$$\int_{-a}^{+a} f(x)\varphi(x)\,dx = \int_{-a}^{+a} f(x)\varphi(0)\,dx + \int_{-a}^{+a} (xf(x))_+\,\varphi'(0)\,dx +$$

$$+ \int_{-a}^{+a} (xf(x))_-\,\varphi'(0)\,dx + \ldots + \frac{1}{m!}\Big\{\int_{-a}^{+a} (x^m f(x))_+\,\varphi^m(0)\,dx + \int_{-a}^{+a} (x^m f(x))_-\,\varphi^m(0)\,dx\Big\} +$$

$$+ \frac{1}{(m+1)!}\Big\{\int_{-a}^{+a} x^{m+1} f(x)\,\varphi^{m+1}(0)\,dx\Big\}.$$

The last term is infinitesimal and can be ignored.

The remainder of the proof is routine. Relation (ii) implies

$$\int_{-a}^{+a} (xg(x)\varphi'(0))_+\,dx = \int_{-a}^{+a} (xf(x))_+\,\varphi'(0)\,dx,$$

and

$$\int_{-a}^{+a} (xg(x)\varphi'(0))_-\,dx = \int_{-a}^{+a} (xf(x))_-\,\varphi'(0)\,dx, \qquad \text{etc.}$$

4.7. A recurrence relation. In distribution theory the following occurrence relation is well known (see for example Gelfand and Shilov [5], vol. 1),

$$x\delta^{(k)}(x) + k\delta^{(k-1)}(x) = 0.$$

This relation needs to be restated for pre-distributions. Clearly $x\delta(x) \approx \emptyset$ implies $\int_S x\delta(x)\varphi(x)\,dx \approx 0$ for any $\varphi \in K(\Omega)$, and any set $S \subset {}^*\boldsymbol{R}_{\mathrm{bd}}$. Of course, the usual distributional argument, namely $x\delta(x) \equiv 0 \Rightarrow \delta(x) + x\delta'(x) \equiv 0$ does not apply, since $x\delta(x) \approx 0$ (essentially) is the best available statement. And just because $x\delta(x)$ is essentially infinitesimal, it does not follow that $\frac{d}{dx}(x\delta(x))$ is also essentially infinitesimal. However, if we compute $\int_S (x\delta(x))'\varphi(x)\,dx$ (where by results of Section 3 it suffices to consider only some interval $S = [-A, +A]$ inside the monad of zero, A being a sufficiently large infinitesimal), a similar recurrence relation emerges. We have in fact

$$\int_{-A}^{+A} (x\delta(x))' \varphi(x)\, dx \approx - \int_{-A}^{+A} \delta'(x)\big(x\varphi(x)\big) \approx 0.$$

Hence

$$\int_{-A}^{+A} \big(x\delta'(x)\varphi(x)\big)dx = \int_{-A}^{+A} \delta'(x)\big(x\varphi(x)\big)dx$$

$$= \delta(x)\cdot x\cdot \varphi(x)|_{-A}^{+A} - \int_{-A}^{+A} \delta(x)\cdot[x\varphi'(x)+\varphi(x)]\,dx.$$

Our choice of $-A$, $+A$ can be made such that $\delta(x)\cdot x\cdot\varphi(x)|_{-A}^{+A} \approx 0$, and

$$\int_{-A}^{+A} \big(x\delta'(x)\varphi(x)\big)dx \approx - \int_{-A}^{+A} \delta(x)\varphi(x)\,dx.$$

Hence $x\delta'(x) \overset{\alpha}{\approx} -\delta(x) \in K'(\Omega)$. The relation $x\delta^{(k)}(x)+k\delta^{(k-1)}(x) \approx 0$ results from an identical argument following integration by parts:

$$\int_{-A}^{+A} \big(x\delta^{(k)}(x)\varphi(x)\big)dx \approx -k \int_{-A}^{+A} \delta^{(k-1)}(x)\varphi(x)\,dx.$$

The following lemma follows easily from this recurrence relation.

4.8. LEMMA. *$\delta^{(k)}(x)$ is of order k at zero.*

4.9. THEOREM. *If a pre-distribution $f(x)$ has an order $m \geqslant 0$ at zero, then $f(x)$ has oscillation number m at zero.*

Proof. Suppose that the conclusion is false, and that $f(x)$ has n essential zeros in the monad of zero, $n \neq m$. By remark A, $f(x) = \psi(x) + \sum_{k=0}^{n} C_k\delta^{(k)}(x)$. If $n < m$, it follows that $f(x)$ is at most of order n at zero which is a contradiction. If $n > m$, and $C_n \not\approx 0$, then $f(x)$ is of order greater than m, again contradicting the hypothesis. Hence $n = m$, as required. Combining theorems (4.6) and (4.7) we have this result:

4.10. THEOREM. *At any point of its essential support the oscillation number and the order of a distribution are equal.*

COROLLARY. *Every distribution $f(x)$ over $K(\Omega)$ is equivalent in the monad of zero to the distribution $\sum_{k=0}^{N} C_k\delta^{(k)}(x)+\psi(x)$, where N is a standard integer. C_k, $k = 1, 2, \ldots, N$, are standard real numbers, and $\psi(x)$ is regular at zero.*

Proof. This follows from Theorem (4.6) and (4.7), (4.10) and the fact that every distribution is of (standard) finite order at zero.

Remark. We could define the order of a distribution α by stating that a (k, Ω) distribution α is of order zero at zero if and only if $f \in \alpha$ implies that $\int_{\varepsilon_1}^{\varepsilon_2} f(x)\,dx \approx 0$ for any $\varepsilon_1 \approx \varepsilon_2 \approx 0$, and is of order one if it is not of order zero, and if for any $\varepsilon_1 \approx \varepsilon_2 \approx 0$ $\int_{\varepsilon_1}^{\varepsilon_2} xf(x)\,dx \approx 0$. $f \in \alpha$ is called *of order zero* if α is order zero. We introduce an inductive definition of the order of a pre-distribution. A pre-distribution $f \in \alpha$ is said to be *of order* $n > 1$ at zero if α is not of order $(n-1)$ and for every pre-distribution $g_\alpha \in \varepsilon$ and every $\varepsilon_1 \approx \varepsilon_2 \approx 0$ it is true that $\int_{\varepsilon_1}^{\varepsilon_2} x^n g_\alpha(x)\,dx \approx 0$.

Note. Since α is not of order $n-1$ at zero there exists some $f \in \alpha$, some $\varepsilon_1, \varepsilon_2 \approx 0$ such that $\int_{\varepsilon_1}^{\varepsilon_2} x^{n-1} f(x)\,dx \not\approx 0$. It only involves an elementary argument to show that the two definitions of order are equivalent.

5. Some non-counter-examples

(a) The following "distribution" appears to be a counterexample to Lemma 1.2, and to the corollary to Theorem 4.10, $f(x) = \sum_{n=1}^{\infty} (\delta(x-1/n)/n)$, where as usual δ stands for any pre-distribution belonging to the Dirac-delta distribution. The trouble with this "counterexample" is that $f(x)$ is not a distribution over $\mathscr{D}$, since it fails to be a bounded linear functional over $\mathscr{D}$.

(b) The following pre-distribution $f(x) = \varphi(x)\sin(1/x)$, with $\varphi(x) > 0$ in some neighbourhood of zero seems to contradict the finite oscillation theorem. Of course, it does not contradict it, since it is indistinguishable from any other bounded function inside the monad of zero, and in any closed interval $[-\varepsilon, +\varepsilon] \subset \mu(0)$, $f(x)$ is indistinguishable from $\tilde{f}(x) \equiv 1$. Hence the oscillation number of $f(x)$ at zero is zero.

(c) Choose any representative of $\delta(x-p)$, where p is a standard point and define $f(x) = \exp(\delta(x-p))$ on a standard interval $p-a \leqslant x \leqslant p+ +a$, $a \neq 0$. Then $f'(x)$ has oscillation number equal to one, but it is not of order one. The answer is that neither $f(x)$ nor $f'(x)$ is a pre-distribution over any test space except the trivial one consisting of the zero function only. Hence Theorem 4.10 is not contradicted by this example. The last "non-counter-example" suggests the following imprecisely stated problem. Can some estimates on the order of the galaxy of $\int_{-\varepsilon}^{+\varepsilon} f(x)\,dx$ be made, which imply that $f(x)$ cannot be a pre-distribution over any test space?

6. Concluding remarks

In this paper we have set out to achieve two objectives. Characterize $*C^{\alpha}$ functions which are candidates for pre-distributions, and relate their local behavior to the corresponding global behavior of the distributions. Having answered some questions we have created in the process some more question which need answers. We have deliberately avoided related algebraic questions, topological problems, or the deeper aspects of approximation theory. The basic setting was $L_1(\Omega)$, which was "lifted" to $*L_1(*\Omega)$. We have deliberately avoided the approach suggested by Robinson [12] of extending the $L_2(\Omega)$ theory to the non-standard setting. The equivalence classes we have introduced here do not survive the operation of pointwise multiplication preceeding the integration and a much more detailed analysis is required to even define the meaning of the commonly used "standard" procedures.

Appendix I. Universe $\hat{R}$ and formal language $\mathscr{L}$

The setting for all *standard* definitions and results in the paper is the collection $\hat{R}$ consisting of the real numbers $\boldsymbol{R}$ together with all the sets of finite rank based on $\boldsymbol{R}$ as the domain of individuals. Precisely, $\hat{R}$ is defined thus:

$\hat{R}(0) = \boldsymbol{R}$;

$\hat{R}(n+1) = P(\bigcup_{k \leq n} \hat{R}(k))$, where $P(A)$, for any set A, denotes the powerset of A;

$\hat{\boldsymbol{R}} = \bigcup_{n} \hat{R}(n)$.

In Appendix II, we shall "axiomatically" describe the non-standard extension $\tilde{\boldsymbol{R}}$ of $\hat{R}$ which is the setting for the arguments in the paper; first, however, we must indicate the language $\mathscr{L}$ relative to which $\tilde{\boldsymbol{R}}$ will be an "elementary extension" of $\hat{R}$.

$\mathscr{L}$ is to be a first-order language with identity, of the usual kind (for the basic syntax and semantics of such languages, we refer the reader to [2] or to any other of the many available sources on non-infinitary first-order logic and its model theory). The binary relation symbol $\in$ and the ternary relation symbol Pr are to be present in $\mathscr{L}$ as its only "non-logical" relation constants. For each x in $\hat{R}$ there shall be in $\mathscr{L}$ corresponding constant symbol $\boldsymbol{x}$ which "names" x. In $\hat{R}$, the sentences "$\boldsymbol{x} \in \boldsymbol{y}$" and "$\mathrm{Pr}(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{y})$" are to be interpreted, respectively, as asserting that x is a member of the set y and that $z = \langle x, y \rangle = \{\{x\}, \{x, y\}\}$. (Inclusion of the symbol Pr, in $\mathscr{L}$, is a convenience item.) "$\boldsymbol{x} = \boldsymbol{y}$" is, of course, interpeted in $\hat{R}$ as the assertion that $x = y$. The only other symbols for $\mathscr{L}$ are: *variables* $v_0, v_1, \ldots$; propositional connectives &, $\wedge$, $\rightarrow$, and ¬ (for *and or*, *implies*, and *not*, respectively); quantifiers ∃ and ∀ (for *exists* and *for all*, respectively); and, finally, "punctuation" symbols [,], and comma. (The reader will note that the *formal* symbols ∈ and = of $\mathscr{L}$ are here written in the same way exactly as their respective $\hat{R}$-*interpretations* of membership equality; this is a matter of convenience which can hardly cause any real confusion.)

Appendix II. $\hat{\boldsymbol{R}}$

The *non-standard* context of our work is a non-Archimedean ordered fieldextension $\hat{\boldsymbol{R}}$, of $\hat{R}$, satisfying the following "axioms":

1. $\hat{\boldsymbol{R}}$ is an $\mathscr{L}$-structure (as defined, say, in [2]).

2. $\hat{\boldsymbol{R}}$ is an *elementary extension* of $\hat{R}$ with respect to $\mathscr{L}$; that is, for each sentence Φ of the language $\mathscr{L}$ we have that Φ holds in $\hat{R}$ if and only if Φ holds in $\hat{\boldsymbol{R}}$ (note that *each* element of $\hat{R}$ has a name in $\mathscr{L}$).

3. $\hat{\boldsymbol{R}}$ is an enlargement of $\hat{R}$; i.e., if b is any binary relation such that (i) $b \in \hat{R}$ and (ii) whenever F is a finite set of elements of the domain of b we have $b(x, y)$ for some fixed y and all $x \in F$, then there is a fixed element y' of $\hat{\boldsymbol{R}}$ such that y' satisfies the formula $b(x, v_0)$ in $\hat{\boldsymbol{R}}$ for all $x \in$ domain (b). (Note that, on account of 2, y' cannot in general be the $\hat{\boldsymbol{R}}$-interpretation of a constant z of $\mathscr{L}$.)

It is proven in (for example) [10] that $\hat{\boldsymbol{R}}$, satisfying conditions 1, 2, and 3, can be obtained as an *ultrapower of* $\hat{R}$; for a general discussion of ultrapowers and of "Łoś' Theorem" (which provides condition 2 for $\hat{\boldsymbol{R}}$), see [2]. An important special class of objects in $\hat{\boldsymbol{R}}$ are the *strongly internal* objects; they are just those elements x of $\hat{\boldsymbol{R}}$ such that x satisfies (in $\hat{\boldsymbol{R}}$) the $\mathscr{L}$-formula $v_0 \in y$ for some $y \in \hat{R}$. The most fundamental examples of strongly internal objects which are not, themselves, $\hat{\boldsymbol{R}}$-interpretations of constants of $\mathscr{L}$ are the so-called *infinitesimal* $\hat{\boldsymbol{R}}$*-reals* (see Section 1). A very useful property possessed by our ultrapower $\hat{\boldsymbol{R}}$, in addition to 2 and 3, is the property of *countable saturation* (for a general discussion of saturation, and for material from which the countable saturation of $\hat{\boldsymbol{R}}$ in the sense to be defined is derivable, see [2] and [10]): this means that if $\mathscr{L}^D$ is obtained from $\mathscr{L}$ by adjoining new constants denoting the elements of D, where D is an arbitrary *countable* subset of $\boldsymbol{R}$, and if C is a countable set of formulas of $\mathscr{L}^D$ such that each $\zeta \in C$ has v_0 as its unique free variable, and if each *finite* subset C_F of C is simultaneously satisfiable in $\boldsymbol{R}$, then C is simultaneously satisfied by some $z \in \hat{\boldsymbol{R}}$. (Thus, countable saturation is an "internal compactness" phenomenon.) As an example of countable saturation, consider the following:

PROPOSITION. *Let* $\{r_i | \ i \in N\}$ *be any non-empty countable set of positive elements of* $^*\boldsymbol{R}$. *Then there exists* $\beta \in {}^*\boldsymbol{R}$ *such that* $\beta > 0$ *and* β / r_i *is infinitesimal for all* i.

Proof. Let $T = \{r_i | \ i \in N\}$; and let D = a set of new constants d_i, one for each $r_i \in T$. Clearly, each finite subset of the following set of $\mathscr{L}^D$-formulas F_n is simultaneously satisfiable in $\boldsymbol{R}$:

F_n is the formula $v_0 \in \boldsymbol{R}$ and $0 < v_0$ and $v_0/d_i < 1/n$; here $n \in N^+$ = the set of all (standard) positive integers. By countable saturation, all F_n are satisfiable by a fixed $\beta \in {}^*\boldsymbol{R}$; clearly such a β is as required by the proposition.

Note that the above proposition both establishes the mere *existence* of infinitesimals in ${}^*\boldsymbol{R}$ and, at the same time, shows that if C is any (standardly) countable collection of infinitesimals then there is an infinitesimal which is simultaneously "of higher order" than all elements of C. (Other examples of the utility of countable saturation abound; for instance, consider several of the proofs given in Sections 3–5 of the present paper.)

Finally, we observe that higher levels of saturation can be built into enlargements of $\hat{\boldsymbol{R}}$ by intertwining the enlargement-yielding construction with a saturation-producing construction due to Keisler, and appealing to the well-known Tarski–Vaught theorem on unions of chains of elementary extensions. Also, by merely iterating the enlargement-yielding construction *once* (replacing $\hat{R}$ by $\hat{\boldsymbol{R}}$ at the beginning of the iteration step), we obtain an elementary extension of $\hat{\boldsymbol{R}}$ (with respect to a language containing names for all elements of $\hat{\boldsymbol{R}}$, and taking into account all memberships and pairings that occur in $\hat{\boldsymbol{R}}$) in which a non-standard copy of $\mu(p)$ occurs as an object, for each $p \in \boldsymbol{R}$. No particular uses for such "second-level extensions", however, are apparent in connection with the investigations of the present paper.

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