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ON THE CONVERGENCE OF APPROXIMATE SOLUTIONS OF A DYNAMIC PROGRAMMING EQUATION

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The equation

$$f(p) = \sup_q F\big(p, q, f(T(p, q))\big), \quad f(\Theta) = 0, \tag{1}$$

is a generalization of the fundamental functional equation of dynamic programming [1].

In paper [2] it is shown (by the successive approximations method) that under certain quite general assumptions there exists a unique solution $\bar{f}(p)$ of equation (1).

If we will find the solution $\bar{f}(p)$ by the iterative method, we see that the ordinary iterations $\{f_n(p)\}$

$$f_{n+1}(p) = \sup_q F\big(p, q, f_n(T(p, q))\big), \quad f_0(p) = 0, \tag{2}$$

cannot be exactly computed. Hence in place of the sequence $\{f_n(p)\}$ usually another sequence $\{g_n(p)\}$ of approximate iterations is produced (for more detailed discussion of this problem, see [3] and [5]).

The purpose of this paper is to show that the conditions introduced in [2] are sufficient for the convergence to the solution $\bar{f}(p)$ of any sequence $\{g_n(p)\}$ satisfying the condition

$$\sup_{\|p\|\leqslant c} \big|\sup_q F\big(p, q, g_n(T(p, q))\big) - g_{n+1}(p)\big| \leqslant b_n(c) \tag{3}$$

with $b_n(c)$ having the property

$$\lim_{n\to\infty} b_n(c) = 0 \quad \text{and} \quad b_{n+1}(c) \leqslant b_n(c), \quad n = 0, 1, \ldots, \tag{4}$$

i.e. with $b_n(c) \searrow 0$.

We shall also prove that the following error estimation

$$\sup_{\|p\|\leqslant c} |\bar{f}(p) - g_n(p)| \leqslant u_n(c),$$

where $\{u_n(c)\}$ is an adequately defined sequence, holds true.

The sequence $\{g_n(p)\}$ will be constructed. The explicit form of $u_n(c)$'s will be given in the case where the function $\omega(u, v)$ appearing in the estimation of the modulus of increment of the function $F(p, q, x)$ is linear.

1. Assumptions, definitions and lemmas. Let us begin with the following (see [2])

ASSUMPTION H_1. 1° The function $F(p, q, x)$ is defined for $p = (p_1, \ldots, p_m)$, $p \in D \subset R^m$, $q \in S \subset R^1$, $x \in R^1$, where D is some region in R^m containing the element $\Theta = (0, \ldots, 0)$, and S is an arbitrarily fixed subset of R^1;

2° for any $(p, q, x) \in D \times S \times R^1$ we have $F(p, q, x) \in R^1$;

3° for any $q \in S$ the function $T(p, q)$ is a transformation of D into D;

4° for any $q \in S$, $F(\Theta, q, 0) = 0$ and $F(p, q, 0)$ is bounded for $q \in S$ and those $p \in D$ for which $||p|| \leqslant c_1$, where c_1 is an arbitrarily fixed positive number;

5° there exists a non-negative function $\omega(u, v)$ defined for $u, v \geqslant 0$, non-decreasing and continuous with respect to u and v, which fulfils the condition $\omega(u, 0) \equiv 0$; moreover, for any $(p, q, x_i) \in D \times S \times R^1$, $i = 1, 2$, we have the inequality

$$|F(p, q, x_1) - F(p, q, x_2)| \leqslant \omega(||p||, |x_1 - x_2|);$$

6° there exists a non-negative and non-decreasing function $a(u)$ defined and continuous for $u \geqslant 0$ and such that

$$||T(p, q)|| \leqslant a(||p||), \quad a(0) = 0,$$

for $p \in D$ and $q \in S$.

ASSUMPTION H_2. Let $\{g_n(p)\}$ be a sequence defined in D with $g_n(p) \in R^1$. The elements of the sequence $\{g_n(p)\}$ will be called the *approximate iterations* (or *solutions*) of equation (1). Assume that there exists a sequence $\{b_n(c)\}$ of non-decreasing functions $b_n(c)$ defined for $c \geqslant 0$ and satisfying condition (3) for $c \geqslant 0$, $p \in D$, $q \in S$ $(n = 0, 1, \ldots)$ and condition (4) for $c \geqslant 0$. Moreover,

1° there exist a non-negative and non-decreasing solutions $\bar{v}(c)$ and $\bar{u}(c)$ of the inequalities

$$\omega\big(c, v(a(c))\big) + b_0(c) \leqslant v(c), \quad v(0) = v(0^+) = 0,$$

$$\omega\big(c, u(a(c))\big) + \sup_{||p|| \leqslant c} \sup_{q} |F(p, q, 0)| \leqslant u(c),$$

$$u(0) = u(0^+) = 0, \ p \in D, \ q \in S,$$

defined for $c \geqslant 0$, where $\omega(u, v)$ is the function from assumption H_1,

2° in the class of functions satisfying the condition

$$0 \leqslant u(c) \leqslant \max\big(\bar{u}(c), \bar{v}(c)\big)$$

the function $u(c) \equiv 0$ is the only solution of the equation

$$u(c) = \omega\big(c, u(a(c))\big).$$

Let us define a sequence $\{u_n(c)\}$ by the relations

$$\begin{cases} u_0(c) = \bar{v}(c), \quad c \geqslant 0, \\ u_{n+1}(c) = \omega\big(c, u_n(a(c))\big) + b_n(c) \end{cases} \tag{5}$$

for $c \geqslant 0$ and $n = 0, 1, \ldots$

We have then

LEMMA 1. *If* 5° *and* 6° *of assumption* H_1, *and assumption* H_2 *are fulfilled, then*

$$0 \leqslant u_{n+1}(c) \leqslant u_n(c) \leqslant \bar{v}(c), \quad c \geqslant 0, \quad n = 0, 1, \ldots,$$

$$\lim_{n\to\infty} u_n(c) = 0 \quad \textit{for } c \geqslant 0,$$

and the convergence is uniform in each bounded set.

Proof. We see that

$$0 \leqslant u_1(c) = \omega\big(c, u_0(a(c))\big) + b_0(c)$$
$$= \omega\big(c, \bar{v}(a(c))\big) + b_0(c) \leqslant \bar{v}(c) = u_0(c), \quad c \geqslant 0.$$

Further, if we suppose that $u_n(c) \leqslant u_{n-1}(c)$, $c \geqslant 0$, then

$$0 \leqslant u_{n+1}(c) = \omega\big(c, u_n(a(c))\big) + b_n(c)$$
$$\leqslant \omega\big(c, u_{n-1}(a(c))\big) + b_{n-1}(c) = u_n(c) \leqslant \bar{v}(c), \quad c \geqslant 0.$$

Now the first part of the lemma follows by induction.

Since the sequence $\{u_n(c)\}$ is non-increasing and bounded from below, it is convergent to a certain function $\tilde{u}(c)$. According to the continuity property of the function $\omega(u, v)$, the function $\tilde{u}(c)$ satisfies the equation

$$u(c) = \omega\big(c, u(a(c))\big).$$

Now from assumption H_2 we have $\tilde{u}(c) \equiv 0$.

The uniform convergence of the sequence $\{u_n(c)\}$ follows from the monotonity of that sequence and of all functions $u_n(c)$.

LEMMA 2. *If* 5° *and* 6° *of assumption* H_1, *and assumption* H_2 *are fulfilled, and* $\{w_n(c)\}$ *is an arbitrary sequence satisfying the conditions*

$$w_0(c) \leqslant \bar{v}(c), \quad c \geqslant 0,$$

$$w_{n+1}(c) \leqslant \omega\big(c, w_n(a(c))\big) + b_n(c), \quad c \geqslant 0, \ n = 0, 1, \ldots,$$

then

$$0 \leqslant w_n(c) \leqslant u_n(c), \quad c \geqslant 0, \ n = 0, 1, \ldots,$$

$$\lim_{n\to\infty} w_n(c) = 0, \quad c \geqslant 0.$$

Proof. By the assumption we have $0 \leqslant w_0(c) \leqslant \bar{v}(c) = u_0(c)$, $c \geqslant 0$. Further, if we suppose that $w_n(c) \leqslant u_n(c)$, $c \geqslant 0$, then

$$0 \leqslant w_{n+1}(c) \leqslant \omega\Big(c, w_n\big(a(c)\big)\Big) + b_n(c)$$
$$\leqslant \omega\Big(c, u_n\big(a(c)\big)\Big) + b_n(c) = u_{n+1}(c), \quad c \geqslant 0.$$

Since the sequence $\{u_n(c)\}$ is convergent to zero,

$$\lim_{n\to\infty} w_n(c) = 0, \quad c \geqslant 0.$$

2. Convergence of approximate iterations. Using Bellman's argument ([1], p. 118) one can obtain the following

LEMMA 3. *If*

$$s_i(p) = \sup_q F_i\Big(p, q, f_i\big(T(p, q)\big)\Big), \quad i = 1, 2,$$

then

$$|s_1(p) - s_2(p)| \leqslant \sup_q \Big|F_1\Big(p, q, f_1\big(T(p, q)\big)\Big) - F_2\Big(p, q, f_2\big(T(p, q)\big)\Big)\Big|.$$

Let us recall the existence theorem from [2]:

THEOREM 1. *If assumptions* H_1 *and* H_2 *are satisfied, then there exists a solution* $\bar{f}(p)$ *of equation* (1) *being the limit of the sequence* $\{f_n(p)\}$ *defined by* (2).

Moreover, we have the estimations

$$\sup_{\|p\|\leqslant c} |\bar{f}(p) - f_n(p)| \leqslant h_n(c), \quad p \in D,\ c \geqslant 0,\ n = 0, 1, \ldots,$$

and

$$\sup_{\|p\|\leqslant c} |\bar{f}(p)| \leqslant \bar{u}(c), \quad p \in D,\ c \geqslant 0,$$

where the sequence $\{h_n(c)\}$ *is defined by* (5) *with* $\bar{u}(c)$ *instead of* $\bar{v}(c)$ *and* $b_n(c) \equiv 0$ *for* $c \geqslant 0$, $n = 0, 1, \ldots$

Now we can formulate the theorem on the convergence of approximate iterations $\{g_n(p)\}$, mentioned in assumption H_2, to the solution $\bar{f}(p)$ of equation (1).

THEOREM 2. *If assumptions* H_1 *and* H_2 *are fulfilled and*

$$\sup_{\|p\|\leqslant c} |\bar{f}(p) - g_0(p)| \leqslant \bar{v}(c), \quad c \geqslant 0,\ p \in D, \tag{6}$$

then the sequence $\{g_n(p)\}$ *is convergent to* $\bar{f}(p)$, *and the estimation*

$$\sup_{\|p\|\leqslant c} |\bar{f}(p) - g_n(p)| \leqslant u_n(c), \quad c \geqslant 0,\ p \in D,\ n = 0, 1, \ldots, \tag{7}$$

holds true.

Proof. We shall prove that the sequence $\{g_n(p)\}$ fulfils condition (7). Let us put

$$w_n(c) = \sup_{\|p\|\leqslant c} |\bar{f}(p) - g_n(p)|, \quad p \in D,\ c \geqslant 0,\ n = 0, 1, \ldots$$

Evidently,

$$w_0(c) = \sup_{\|p\|\leqslant c}|\bar{f}(p)-g_0(p)| \leqslant \bar{v}(c) = u_0(c)$$

for $p \in D$ and $c \geqslant 0$.

Further, for $p \in D$ and $c \geqslant 0$ we have

$$\begin{aligned}
&|\bar{f}(p)-g_{n+1}(p)|\\
&\qquad\leqslant \left|\sup_q F\big(p, q, \bar{f}(T(p, q))\big)-\sup_q F\big(p, q, g_n(T(p, q))\big)\right|\\
&\qquad\quad+\left|\sup_q F\big(p, q, g_n(T(p, q))\big)-g_{n+1}(p)\right|+\\
&\qquad\leqslant \sup_q \omega\Big(\|p\|, \left|\bar{f}(T(p, q))-g_n(T(p, q))\right|\Big)+b_n(c)\\
&\qquad\leqslant \omega\Big(c, w_n\big(a(c)\big)\Big)+b_n(c),
\end{aligned}$$

whence

$$w_{n+1}(c) \leqslant \omega\Big(c, w_n\big(a(c)\big)\Big)+b_n(c).$$

Now inequality (7) is implied by Lemma 2. Passing in (7) to the limit with $n \to \infty$ we finally have

$$\lim_{n\to\infty} g_n(p) = \bar{f}(p), \qquad p \in D.$$

Thus the proof of Theorem 2 is completed.

Remark 1. Condition (6) of Theorem 2 is fulfilled if

$$\bar{u}(c)+\sup_{\|p\|\leqslant c} |g_0(p)| \leqslant \bar{v}(c), \qquad p \in D,\ c \geqslant 0.$$

Indeed, for $p \in D$ and $c \geqslant 0$ we have

$$\begin{aligned}
\sup_{\|p\|\leqslant c} |\bar{f}(p)-g_0(p)| &\leqslant \sup_{\|p\|\leqslant c} |\bar{f}(p)|+\sup_{\|p\|\leqslant c}|g_0(p)|\\
&\leqslant \bar{u}(c)+\sup_{\|p\|\leqslant c}|g_0(p)|.
\end{aligned}$$

Remark 2. If we define the function $\bar{v}(c)$ as a solution of the inequality

$$\omega\Big(c, u\big(a(c)\big)\Big)+\max\big[b_0(c),\ \bar{u}(c)+\sup_{\|p\|\leqslant c}|g_0(p)|\big] \leqslant u(c),$$

where $p \in D$ and $c \geqslant 0$, then condition (6) of Theorem 2 is fulfilled.

Remark 3. If $g_0(p) = g_1(p) = 0$ for $p \in D$ and

$$b_0(c) = \sup_{\|p\|\leqslant c} \sup_q |F(p, q, 0)|, \qquad p \in D,\ q \in S,\ c \geqslant 0,$$

then
$$\bar{u}(c) = \bar{v}(c), \quad c \geqslant 0,$$
and condition (6) of Theorem 2 is satisfied.

Indeed, we have
$$\sup_{\|p\|\leqslant c} |\bar{f}(p)-g_0(p)| = \sup_{\|p\|\leqslant c} |\bar{f}(p)| \leqslant \bar{u}(c) = \bar{v}(c)$$
for $p \in D$ and $c \geqslant 0$.

3. Construction of approximate iterations. Let us note first that if $g_n(p) = f_n(p)$, $p \in D$ and $n = 0, 1, \ldots$, i.e., if $g_n(p)$ are exact iterations defined by formula (2), then Theorem 2 gives the estimations mentioned in Theorem 1. Indeed, $b_n(c) = 0$, $c \geqslant 0$ and $n = 0, 1, \ldots$

For $\bar{v}(c)$ we take $\bar{u}(c)$; it is evident that condition (6) of Theorem 2 is satisfied. Thus the assertion of Theorem 2 holds true and $u_n(c) = h_n(c)$, $c \geqslant 0$ and $n = 0, 1, \ldots$, in view of the definition of the sequence $\{h_n(c)\}$.

If we take into consideration the rounded errors, then we observe that we cannot appoint exactly the sequence $\{f_n(p)\}$ and, consequently, the estimation given by Theorem 1 is useless.

The sequence of the approximate iterations $\{g_n(p)\}$ may be defined by

$$(8) \qquad g_{n+1}(p) = \sup_q F_n\big(p, q, g_n(T(p, q))\big), \quad p \in D, q \in S, n = 0, 1, \ldots,$$

where $\{F_n\}$ is a sequence convergent to the function F and $g_0(p)$ is an arbitrarily fixed function defined for $p \in D$.

Now we get

THEOREM 3. *If assumption* H_1, 1° *and* 2° *of assumption* H_2, *and condition* (4) *are all satisfied, and if, moreover,*

1° $F_n(p, q, x)$, *where* $F_n(p, q, x) \in R^1$ *and* $n = 0, 1, \ldots$, *are functions defined on the product* $D \times S \times R^1$ *and satisfying assumption* H_1;

2° *the sequence* $\{g_n(p)\}$ *is defined by relation* (8);

3° $\sup_{\|p\|\leqslant c} |\bar{f}(p)-g_0(p)| \leqslant \bar{v}(c)$, $p \in D$, $c \geqslant 0$;

4° $\sup_{\|p\|\leqslant c} |\sup_q F_n(p, q, x)-\sup_q F(p, q, x)| \leqslant b_n(c)$, $n = 0, 1, \ldots$, *for* $p \in D$, $q \in S$, *and* $x \in R^1$ *such that*
$$\sup_{\|p\|\leqslant c} |x-g_0(p)| \leqslant \bar{v}(c);$$
then
$$\lim_{n\to\infty} g_n(p) = \bar{f}(p), \quad p \in D,$$
and estimation (7) *holds true.*

Proof. Let us put
$$m_n(c) = \sup_{\|p\|\leqslant c} |\bar{f}(p)-g_n(p)|, \quad p \in D, c \geqslant 0, n = 0, 1, \ldots$$

Then for $p \in D$, $q \in S$, $c \geqslant 0$, $n = 0, 1, \ldots$, we have

$$|\bar{f}(p) - g_{n+1}(p)|$$
$$\leqslant \left|\sup_q F\big(p, q, \bar{f}(T(p, q))\big) - \sup_q F_n\big(p, q, \bar{f}(T(p, q))\big)\right| +$$
$$+ \left|\sup_q F_n\big(p, q, \bar{f}(T(p, q))\big) - \sup_q F_n\big(p, q, g_n(T(p, q))\big)\right|$$
$$\leqslant \sup_q \omega\big(\|p\|, |\bar{f}(T(p, q)) - g_n(T(p, q))|\big) + b_n(c)$$
$$\leqslant \omega\big(c, m_n(a(c))\big) + b_n(c).$$

Hence

$$m_{n+1}(c) \leqslant \omega\big(c, m_n(a(c))\big) + b_n(c), \qquad c \geqslant 0, \ n = 0, 1, \ldots$$

From assumption 3° we get $m_0(c) \leqslant \bar{v}(c)$. Using Lemma 2 we infer that $m_n(c) \leqslant u_n(c)$ for $c \geqslant 0$, $n = 0, 1, \ldots$, and, consequently, $\lim_{n\to\infty} m_n(c) = 0$, $c \geqslant 0$. These relations show that $\lim_{n\to\infty} g_n(p) = \bar{f}(p)$, $p \in D$, and that estimation (7) is true. Thus Theorem 3 is proved.

Remark 4. The proof of Theorem 3 remains true if assumption 4° of this theorem is replaced by the relation

$$\sup_{\|p\| \leqslant c} |\sup_q F_n(p, q, \bar{f}) - \sup_q F(p, q, \bar{f})| \leqslant b_n(c)$$

for $p \in D$, $q \in S$, $n = 0, 1, \ldots$

4. Convergence of solutions of approximate equations for equation (1). Before formulating the theorem on the convergence of a sequence of solutions of approximate equations for equation (1) we need two lemmas (see [4]).

LEMMA 4. *If 5° and 6° of assumption* H_1 *and 1° of assumption* H_2 *are satisfied, then the equation*

$$u(c) = \omega\big(c, u(a(c))\big) + b_0(c), \qquad c \geqslant 0,$$

has the solution $t(c) = \overline{m}\big(\bar{v}(c), b_0(c)\big) \leqslant \bar{v}(c)$, $c \geqslant 0$, *which has the following properties*:

1° $\overline{m}\big(\bar{v}(c), b_0(c)\big) = \lim_{n\to\infty} v_n\big(\bar{v}(c), b_0(c)\big)$, *where* $v_0\big(\bar{v}(c), b_0(c)\big) = \bar{v}(c)$,

$$v_{n+1}\big(\bar{v}(c), b_0(c)\big) = b_0(c) + \omega\Big(c, v_n\big(\bar{v}(c), b_0(c)\big)\Big), \qquad c \geqslant 0, \ n = 0, 1, \ldots,$$

2° *if* $s(c) \leqslant \bar{v}(c)$ *and* $s(c) \leqslant b_0(c) + \omega\big(c, s(a(c))\big)$, $c \geqslant 0$, *then*

$$s(c) \leqslant \overline{m}\big(\bar{v}(c), b_0(c)\big), \qquad c \geqslant 0.$$

LEMMA 5. *If* 5° *and* 6° *of assumption* H_1, 1° *and* 2° *of assumption* H_2 *and condition* (4) *are all satisfied, then the equation*

$$u(c) = \omega\big(c, u(a(c))\big) + b_n(c), \quad c \geqslant 0,\ n = 0, 1, \ldots,$$

has a solution $u^*(c) = \overline{m}\big(\bar{v}(c), b_n(c)\big) \leqslant \bar{v}(c)$ *such that*

$$\overline{m}\big(\bar{v}(c), b_{n+1}(c)\big) \leqslant \overline{m}\big(\bar{v}(c), b_n(c)\big), \quad c \geqslant 0,\ n = 0, 1, \ldots$$

Moreover,

$$\overline{m}\big(\bar{v}(c), b_n(c)\big) \searrow 0, \quad c \geqslant 0.$$

Now we can formulate the following

THEOREM 4. *If assumption* H_1, 1° *and* 2° *of assumption* H_2, *and condition* (4) *are satisfied, and if, moreover,*

1° $F_n(p, q, x)$, *where* $F_n(p, q, x) \in R^1$ *and* $n = 0, 1, \ldots$, *are functions defined on the product* $D \times S \times R^1$ *and satisfying assumption* H_1;

2° $\{\tilde{g}_n(p)\}$ *is the sequence of functions with the values belonging to* R^1 *and satisfying the condition*

$$\tilde{g}_n(p) = \sup_q F_n\big(p, q, \tilde{g}_n(T(p, q))\big), \quad p \in D,\ q \in S,\ n = 0, 1, \ldots;$$

3° $\sup\limits_{\|p\| \leqslant c} |\bar{f}(p) - \tilde{g}_n(p)| \leqslant \bar{v}(c)$, $p \in D$, $c \geqslant 0$, $n = 0, 1, \ldots$;

4° $\sup\limits_{\|p\| \leqslant c} |\sup\limits_q F_n(p, q, x) - \sup\limits_q F(p, q, x)| \leqslant b_n(c)$, $n = 0, 1, \ldots$,

for $p \in D$, $q \in S$, *and* $x \in R^1$ *such that*

$$\sup_{\|p\| \leqslant c} |x - \tilde{g}_n(p)| \leqslant \bar{v}(c);$$

then

$$\lim_{n \to \infty} \tilde{g}_n(p) = \bar{f}(p), \quad p \in D,$$

and

$$\sup_{\|p\| \leqslant c} |\tilde{g}_n(p) - \bar{f}(p)| \leqslant \overline{m}\big(\bar{v}(c), b_n(c)\big)$$

for $p \in D$, $c \geqslant 0$, $n = 0, 1, \ldots$, *where* $\overline{m}(a, b)$ *is defined in Lemma* 4.

Proof. Put

$$m_n(c) = \sup_{\|p\| \leqslant c} |\tilde{g}_n(p) - \bar{f}(p)|, \quad p \in D,\ c \geqslant 0,\ n = 0, 1, \ldots;$$

thus for $p \in D$, $q \in S$, $c \geqslant 0$, $n = 0, 1, \ldots$, we have

$$\begin{aligned} |\tilde{g}_n(p) - \bar{f}(p)| &\leqslant \Big|\sup_q F_n\big(p, q, \tilde{g}_n(T(p, q))\big) - \sup_q F\big(p, q, \tilde{g}_n(T(p, q))\big)\Big| + \\ &\quad + \Big|\sup_q F\big(p, q, \tilde{g}_n(T(p, q))\big) - \sup_q F\big(p, q, \bar{f}(T(p, q))\big)\Big| \\ &\leqslant \omega\big(c, m_n(a(c))\big) + b_n(c). \end{aligned}$$

Hence we get

$$m_n(c) \leqslant \omega\big(c, m_n(a(c))\big) + b_n(c), \quad c \geqslant 0,\ n = 0, 1, \ldots$$

From assumption 3° and from the last inequality of Lemma 4 we have

$$m_n(c) \leqslant \overline{m}\big(\bar{v}(c), b_n(c)\big), \quad c \geqslant 0,\ n = 0, 1, \ldots,$$

and so the first part of the theorem is proved.

Since $b_n(c) \searrow 0$, we have in view of Lemma 5, $\overline{m}\big(\bar{v}(c), b_n(c)\big) \searrow 0$, $c \geqslant 0$, and this shows that

$$\lim_{n\to\infty} \tilde{g}_n(p) = \bar{f}(p), \quad p \in D.$$

5. Case of the function $\omega(u, v)$ linear in v. Now we shall discuss the question of convergence of the sequence $\{g_n(p)\}$ in a special case, where the function $\omega(u, v)$ is linear with respect to v, i.e., fulfilling the condition

$$\omega(u, v) = k(u)v, \quad k(u) \geqslant 0.$$

Let us define the sequences $\{a_n(c)\}$ and $\{k_n(c)\}$ by the following relations:

$$a_0(c) = c, \quad a_{n+1}(c) = a\big(a_n(c)\big), \quad c \geqslant 0,\ n = 0, 1, \ldots,$$

$$k_0(c) = 1, \quad k_n(c) = \prod_{i=0}^{n-1} k\big(a_i(c)\big), \quad c \geqslant 0,\ n = 1, 2, \ldots$$

LEMMA 6. *The condition*

(*) $$\sum_{n=0}^{\infty} k_n(c) b_0\big(a_n(c)\big) < \infty$$

is necessary and sufficient for the equation

(9) $$u(c) = k(c) u\big(a(c)\big) + b_0(c)$$

to have a non-negative solution $\bar{v}(c)$.

If condition (*) *is fulfilled, then*

(10) $$\bar{v}(c) = \sum_{n=0}^{\infty} k_n(c) b_0\big(a_n(c)\big),$$

and the solution $\bar{v}(c)$ satisfies the relation

(11) $$\lim_{n\to\infty} k_n(c)\bar{v}\big(a_n(c)\big) = 0.$$

There is no other solution of equation (9) *in the class of functions* $0 \leqslant u(c) \leqslant \bar{v}(c)$.

Proof. In paper [2] it is shown that condition (*) is necessary and sufficient for the existence of the solution of equation (9). It is obvious that $\bar{v}(c)$ defined by (10) is the solution of equation (9).

Now we shall prove condition (11). It is easy to see [2] that the function $\bar{v}(c)$ fulfills the condition

$$\bar{v}(c) = \sum_{n=0}^{l} k_n(c) b_0(a_n(c)) + k_{l+1}(c)\bar{v}(a_{l+1}(c)), \quad l = 0, 1, \ldots \tag{12}$$

If $\bar{v}(c)$ has the form (10), than by $l \to \infty$ we get

$$\lim_{l\to\infty} k_l(c)\bar{v}(a_l(c)) = 0.$$

It remains to show that no other function $\bar{\bar{v}}(c)$, $0 \leqslant \bar{\bar{v}}(c) \leqslant \bar{v}(c)$, is a solution of equation (9). Assume a contrario that some $\bar{\bar{v}}(c)$ satisfying the last inequality and distinct from $\bar{v}(c)$ is also a solution of (9). Since (12) is fulfilled for the solution $\bar{\bar{v}}(c)$, we get

$$0 \leqslant \bar{v}(c) - \bar{\bar{v}}(c) \leqslant k_{l+1}(c)\{\bar{v}(a_{l+1}(c)) - \bar{\bar{v}}(a_{l+1}(c))\}$$
$$\leqslant k_{l+1}(c)\bar{v}(a_{l+1}(c)).$$

Since $\bar{v}(c)$ is a solution of equation (9), condition (11) is fulfilled, and therefore we get $\bar{\bar{v}}(c) \equiv \bar{v}(c)$. The received contradiction proves the uniqueness of the solution $\bar{v}(c)$ of equation (9) in the class of functions $0 \leqslant u(c) \leqslant \bar{v}(c)$.

LEMMA 7 (see [2]). *If*

$$\sum_{n=0}^{\infty} k_n(c) b_0(a_n(c)) < \infty$$

and if $\bar{v}(c)$ has the form (10), *then in the class of functions $0 \leqslant u(c) \leqslant \bar{v}(c)$ the function $\tilde{u}(c) \equiv 0$ is the only solution of the inequality*

$$u(c) \leqslant k(c) u(a(c)). \tag{13}$$

Now we can formulate

THEOREM 5. *If assumption* H_1 *is fulfilled and if*

1° $\omega(u, v) = k(u)v$, $k(u) \geqslant 0$;

2° *the sequences* $\{g_n(p)\}$ *and* $\{b_n(c)\}$ *are defined in assumption* H_2;

3° $\bar{v}(c) = \sum_{n=0}^{\infty} k_n(c) b_0(a_n(c)) < \infty$;

4° $\bar{u}(c) = \sum_{n=0}^{\infty} k_n(c) z(a_n(c)) < \infty$, *where*

$$z(c) = \sup_{\|p\| \leqslant c} \sup_{q} |F(p, q, 0)|$$

for $p \in D$, $q \in S$, $c \geqslant 0$;

5° $\sup_{\|p\| \leqslant c} |\bar{f}(p) - g_0(p)| \leqslant \bar{v}(c)$, $p \in D$, $c \geqslant 0$;

then

$$\lim_{n\to\infty} g_n(p) = \bar{f}(p), \qquad p \in D.$$

Moreover,

$$\sup_{\|p\| \leqslant c} |\bar{f}(p) - g_n(p)| \leqslant u_n(c), \qquad p \in D,\ c \geqslant 0,\ n = 0, 1, \ldots,$$

where

$$\begin{cases} u_0(c) = \bar{v}(c), \\ u_n(c) = \sum_{i=n}^{\infty} k_i(c) b_0\big(a_i(c)\big) + \sum_{j=1}^{n} k_{n-j}(c) b_{j-1}\big(a_{n-j}(c)\big) \end{cases} \tag{14}$$

for $c \geqslant 0$, $n = 1, 2, \ldots$

Proof. From Lemmas 6 and 7 and from assumptions 1°-5° it follows that the assumptions of Theorem 2 are fulfilled, thus Theorem 5 is a consequence of Theorem 2.

It remains to prove that under the conditions of Theorem 5 the $u_n(c)$ takes the form (14).

Indeed, now we have

$$\begin{aligned} u_1(c) &= k(c) u_0\big(a(c)\big) + b_0(c) \\ &= k(c) \sum_{i=0}^{\infty} k_i\big(a(c)\big) b_0\Big(a_i\big(a(c)\big)\Big) + b_0(c) \\ &= \sum_{i=1}^{\infty} k_i(c) b_0\big(a_i(c)\big) + b_0(c). \end{aligned}$$

This means that relation (14) is valid for $n = 1$.

Let us suppose that formula (14) is true for $n \geqslant 1$. By the definitions of $u_{n+1}(c)$, $k_n(c)$, and $a_n(c)$ we get

$$\begin{aligned} u_{n+1}(c) &= k(c) u_n\big(a(c)\big) + b_n(c) = k(c) \sum_{i=n}^{\infty} k_i\big(a(c)\big) b_0\Big(a_i\big(a(c)\big)\Big) + \\ &\quad + k(c) \sum_{j=1}^{n} k_{n-j}\big(a(c)\big) b_{j-1}\Big(a_{n-j}\big(a(c)\big)\Big) + b_n(c) \\ &= \sum_{i=n+1}^{\infty} k_i(c) b_0\big(a_i(c)\big) + \sum_{j=1}^{n} k_{n+1-j}(c) b_{j-1}\big(a_{n+1-j}(c)\big) + b_n(c) \\ &= \sum_{i=n+1}^{\infty} k_i(c) b_0\big(a_i(c)\big) + \sum_{j=1}^{n+1} k_{n+1-j}(c) b_{j-1}\big(a_{n+1-j}(c)\big). \end{aligned}$$

Hence, by induction, we get (14) for $n = 0, 1, 2 \ldots$

Example. Let us take in Theorem 6: $b_0(c) = dc$, $k(c) = k$, $a(c) = ac$, $k \geqslant 0$, $a \geqslant 0$, $0 \leqslant ak < 1$, and $z(c) = ec$, $g_0(p) = 0$, $0 \leqslant e \leqslant d$, $c \geqslant 0$.

We see that all assumptions of this theorem are satisfied. We have

$$a_n(c) = a^n c, \quad k_n(c) = k^n,$$

$$\bar{v}(c) = \frac{cd}{1-ak}, \quad \sup_{\|p\| \leqslant c} |\bar{f}(p)| \leqslant \frac{cd}{1-ak}, \quad p \in D.$$

Let us suppose that the sequence $\{b_n(c)\}$ fulfills the condition

$$b_n(c) \leqslant \frac{b_0(c)}{10^{\alpha n}},$$

where α is a positive integer, $ka \cdot 10^{\alpha} > 1$.

Under these assumptions we get

$$u_n(c) \leqslant \sum_{i=n}^{\infty} k^i da^i c + \sum_{j=1}^{n} k^{n-j} \frac{da^{n-j}c}{10^{\alpha(j-1)}}$$

$$\leqslant dc(ka)^n \left[\frac{1}{1-ka} + 10^{\alpha} \frac{1}{10^{\alpha} ka - 1}\right],$$

and, consequently,

$$\sup_{\|p\| \leqslant c} |\bar{f}(p) - g_n(p)| \leqslant dc(ka)^n \left[\frac{1}{1-ka} + \frac{10^{\alpha}}{10^{\alpha} ka - 1}\right]$$

for $p \in D$, $n = 1, 2, \ldots$

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