ON THE REDUCIBILITY OF CONNECTIONS
ON THE PROLONGATIONS OF VECTOR BUNDLES

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It is well known that a connection on a vector bundle \((E, p, B)\) can be defined as a splitting of the exact sequence

\[
0 \to E \otimes T^*(B) \to J^1 E \to E \to 0.
\]

Since the \(r\)-th non-holonomic prolongation \(\tilde{J}^r E\) of \(E\) is also a vector bundle, it is natural to introduce a connection on \(\tilde{J}^r E\) as a splitting of the exact sequence

\[
0 \to \tilde{J}^r E \otimes T^*(B) \to \tilde{J}^{r+1} E \to \tilde{J}^r E \to 0.
\]

However, accepting this point of view, we neglect entirely the fact that \(\tilde{J}^r E\) is a special vector bundle constructed by means of the successive jet prolongations of \(E\). Let \(\Phi(E)\) be the groupoid of all linear isomorphisms between the fibres of \(E\). Then, by the general theory of prolongations of fibre bundles, \([1]\), the \(r\)-th non-holonomic prolongation \(\tilde{\Phi}^r(E)\) of \(\Phi(E)\) is a groupoid of operators on \(\tilde{J}^r E\). Since \(\tilde{\Phi}^r(E)\) is evidently a proper subgroupoid of the groupoid \(\Phi(\tilde{J}^r E)\) of all linear isomorphisms between the fibres of \(\tilde{J}^r E\), there appears a natural question: under what conditions a connection on \(\Phi(\tilde{J}^r E)\) given by an arbitrary splitting of (2) can be reduced to \(\tilde{\Phi}^r(E)\)? In the present paper, we solve this problem for \(r = 1\) and \(r = 2\), but we hope that one meets all essential features of the general problem already in the case \(r = 2\) and we feel inconvenient to treat directly the general case because of a great number of different conditions which should appear there. After that, we investigate the reducibility of a connection on \(\tilde{\Phi}^2(E)\) to some natural subgroupoids of \(\tilde{\Phi}^2(E)\). We should like to underline the remarkable role played in our considerations by the so-called lateral projections of non-holonomic jets introduced recently in \([6]\). The standard terminology and notations of the theory of jets are used throughout the paper. In addition, \(j^s_r\) means the usual projection of \(r\)-jets into \(s\)-jets, \(s < r\). Our considerations are in the category \(C^\infty\).
1. Let $(E, p, B)$ be a vector bundle over $B$ of fibre dimension $m$ and let $n = \dim B$. Further, let $\Phi(E)$ be the groupoid of all linear isomorphisms between the fibres of $E$ and let $a: \Phi(E) \to B$ or $b: \Phi(E) \to B$ be its source or target projection, respectively (i.e., if $\theta \in \Phi$ is a linear isomorphism of $E_x$ into $E_t$, then $a(\theta) = x$, $b(\theta) = t$). Obviously, $\Phi(E)$ is a Lie groupoid over $B$ [8]. A connection on $E$ can be equivalently defined either as a cross-section $C: B \to Q^1(\Phi(E))$, where $Q^1(\Phi(E))$ means the fibre bundle of all elements of connection of the first order on $\Phi(E)$, [2], or as a splitting $\gamma: E \to J^1E$ of the exact sequence (1), see [8]. If a connection $C$ is given, then the value of the corresponding splitting $\gamma$ on a vector $v \in E_x$ is determined as follows. If $C(x) = j^1_x \circ t(t)$, where $t$ is a mapping of a neighbourhood $U$ of $x \in B$ into $\Phi(E)$ such that $a(t) = x$, $b(t) = t$, $t(x) = \text{id}_{E_x}$, $t \in U$, then $t \mapsto \circ t(t)(v)$ is a local cross-section of $E$ over $U$. By [8], $\gamma(v)$ is the 1-jet of $\circ t(t)(v)$ at $x$, i.e.

$$\gamma(v) = j^1_x[\circ t(t)(v)].$$

In what follows we systematically denote a cross-section of $Q^1(\Phi(E))$ by a capital Roman letter and we shall use the corresponding lower-case Greek letter for the associated splitting $E \to J^1E$.

We use frequently the evaluations in some local coordinates. For the sake of simplicity, we assume in such a situation that $E$ is a trivial vector bundle $\mathbb{R}^n \times \mathbb{R}^m$ over $\mathbb{R}^n = B$ and that

$$x^i, y^a, \quad \begin{align*} i, j, \ldots &= 1, \ldots, n, \\ a, \beta, \ldots &= 1, \ldots, m, \end{align*}$$

are the natural coordinates on $\mathbb{R}^n \times \mathbb{R}^m$. Then $\Phi(\mathbb{R}^n \times \mathbb{R}^m) = \mathbb{R}^n \times L_m \times \mathbb{R}^n$. This induces the coordinates

$$\bar{x}^i, \bar{b}^a_\beta, x^i,$$

det $|\bar{b}^a_\beta| \neq 0$, on $\Phi(\mathbb{R}^n \times \mathbb{R}^m)$. We have $a(\bar{x}^i, \bar{b}^a_\beta, x^i) = (x^i), b(\bar{x}^i, \bar{b}^a_\beta, x^i) = (\bar{x}^i)$ and the action of $\Phi(\mathbb{R}^n \times \mathbb{R}^m)$ on $\mathbb{R}^n \times \mathbb{R}^m$ is given by

$$a^i_j(x^i, y^a, \bar{b}^a_\beta) = (x^i, y^a, \bar{b}^a_\beta).$$

On $J^1(\mathbb{R}^n \times \mathbb{R}^m)$ there are further coordinates $y^a_i$ determined by

$$y^a_i(j^1_x \sigma) = \partial_i y^a(\sigma(x)),$$

provided $\partial_i$ denotes the partial differentiation with respect to $x^i$. Thus, if $C(x) = j^1_x(\sigma^i, \bar{b}^a_\beta(t), \bar{x}^i)$, $\bar{b}^a_\beta(x) = \delta^a_\beta$, then

$$y^a_i(x^i, y^a, \bar{b}^a_\beta) = (x^i, y^a, \Gamma^a_{\beta i}(x) y^\beta),$$

where $\Gamma^a_{\beta i}(x) = \partial_i b^a_\beta(x)$. The functions $\Gamma^a_{\beta i}$ are called the Christoffel's symbols of $C$. In short, we say that the splitting (5) is given by

$$y^a_i = \Gamma^a_{\beta i} y^\beta.$$
2. Consider the tensor product $E_1 \otimes E_2$ of two vector bundles over $B$. Let $C_a$ be a connection on $E_a$, $a = 1, 2$. Then $C_1$ and $C_2$ determine a connection $C_1 \otimes C_2$ on $E_1 \otimes E_2$ in the following well-known manner. If $C_a(x) = j^a_x \mathcal{E}_a(t)$, then $\mathcal{E}_1(t) \otimes \mathcal{E}_2(t) \in \Phi(E_1 \otimes E_2)$ and we write

\[(C_1 \otimes C_2)(x) = j^1_x(\mathcal{E}_1(t) \otimes \mathcal{E}_2(t)).\]

In coordinates, if

\[
\begin{align*}
    y_i^a &= \Gamma^a_{\beta i} y^\beta, & m - \text{the fibre dimension of } E_1, \\
    y_i^\lambda &= \Gamma^\lambda_{\mu i} y^\mu, & \lambda, \mu, \ldots = m + 1, \ldots, m + \text{the fibre dimension of } E_2,
\end{align*}
\]

are the corresponding splittings, $\Gamma^a_{\beta i}(x) = \partial_i b^a_\beta(x)$, $\Gamma^\lambda_{\mu i}(x) = \partial_i b^\lambda_\mu(x)$, then the Christoffel's symbols of $C_1 \otimes C_2$ satisfy $\Gamma^{a\lambda}_{\beta\mu i}(x) = \partial_i (b^a_\beta b^\lambda_\mu)(x)$. This implies

\[
\Gamma^{a\lambda}_{\beta\mu i} = \Gamma^a_{\beta i} \delta^\lambda_\mu + \delta^a_\beta \Gamma^\lambda_{\mu i}.
\]

A connection $C$ on $E_1 \otimes E_2$ will be said to be **decomposable** if there exist connections $C_1$ and $C_2$ such that $C = C_1 \otimes C_2$. The decomposability of $C$ is characterized by linear equations (8) with unknown $\Gamma^a_{\beta i}, \Gamma^\lambda_{\mu i}$. Since the general solution of the homogeneous system depends on one parameter, the factors $C_1$ and $C_2$ of a decomposable connection are not uniquely determined. We shall further say that a connection $C$ on $E_1 \otimes E_2$ is decomposable with respect to a connection $C_1$ on $E_1$, if there exists a connection $C_2$ on $E_2$ such that $C = C_1 \otimes C_2$. By (8), we obtain

**Lemma 1.** If a connection $C$ on $E_1 \otimes E_2$ is decomposable with respect to a connection $C_1$ on $E_1$, then the second factor $C_2$ on $E_2$ is uniquely determined.

3. We shall state an algebraic lemma generalizing Lemma 2 of [7]. Consider a commutative diagram with exact rows and columns of vector bundles over the same base and of their base-preserving homomorphisms:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & A_1 & B_1 & C_1 \\
\varphi_1 & \downarrow i_1 & \downarrow p_1 & \zeta_1 \\
\downarrow & \downarrow & \downarrow & \\
0 & A_2 & B_2 & C_2 \\
\varphi_2 & \downarrow i_2 & \downarrow p_2 & \zeta_2 \\
\downarrow & \downarrow & \downarrow & \\
0 & A_3 & B_3 & C_3 \\
\varphi_3 & \downarrow i_3 & \downarrow p_3 & \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

Let $\gamma: C_2 \rightarrow B_2$ be a splitting in the middle row. By exactness, the values of the composition $\omega(\gamma) = \varphi_2 \gamma \varphi_1$ lie in $A_3 \subset B_3$. Hence $\omega(\gamma)$ is an element of Hom$(C_1, A_3)$, which will be said to be the **obstruction** associated with $\gamma$. 
Lemma 2. A necessary and sufficient condition for the existence of a splitting \( \gamma_0 \colon C_3 \to B_3 \) or \( \gamma^0 \colon C_1 \to B_1 \) compatible with diagram (9) is \( \omega(\gamma) = 0 \in \text{Hom}(C_1, A_3). \)

Proof. Necessity is obvious. Let \( \omega(\gamma) = 0. \) If \( x + \zeta_1(C_1) \in C_3, \) then we define \( \gamma_0(x + \zeta_1(C_1)) = \psi_2 \gamma(x). \) Since \( \psi_2 \gamma \zeta_1(C_1) = 0, \) this definition is correct. Further, we have

\[
p_3 \gamma_0(x + \zeta_1(C_1)) = p_3 \psi_2 \gamma(x) = \zeta_2 p_2 \gamma(x) = x + \zeta_1(C_1),
\]

so that \( \gamma_0 \) is a splitting. Moreover, if \( y \in C_1, \) then \( \omega(y) = 0 \) implies \( \gamma \zeta_1(y) \in \text{Im}(\psi_1). \) Hence there is a unique \( \gamma^0(y) \in B_1 \) with \( \psi_1 \gamma^0(y) = \gamma \zeta_1(y). \) Then

\[
\zeta_1 p_1 \gamma^0(y) = p_2 \psi_1 \gamma^0(y) = p_2 \gamma \zeta_1(y) = \zeta_1(y).
\]

4. The first prolongation \( J^1 E \) of \( E \) is a vector bundle of fibre dimension \( N = m(n + 1). \) Let \( \Phi(J^1 E) \) be the groupoid of all linear isomorphisms between the fibres of \( J^1 E. \) In the trivial case,

\[
\Phi(J^1(R^n \times R^m)) = R^n \times L^1_N \times R^n.
\]

Taking into account that \( J^1 E \) is the prolongation of \( E, \) we shall say that an element \( \theta \in \Phi(J^1 E) \) with source \( x \) and target \( t \) is projectable, if there exists an element \( \varphi \in \Phi(E) \) such that the diagram

\[
\begin{array}{ccc}
J^1 E & \xrightarrow{\theta} & J^1 E \\
\downarrow j^0 \downarrow & & \downarrow j^0 \\
E_x & \xrightarrow{\varphi} & E_t
\end{array}
\]

(10)

commutes.

Denote by \( \Phi_p(J^1 E) \) the subgroupoid of all projectable elements of \( \Phi(J^1 E). \) If the coordinate expression of \( \theta \) is

\[
(x^i, y^a, y^0_i) \mapsto (t^i, c_a^i y^a + c_{\bar{a}}^i y^0_{\bar{a}}, c_{\bar{a}}^i y^0 + e_{\bar{a}}^i y^0),
\]

then

(11)

\[
\theta \in \Phi_p(J^1 E) \text{ if and only if } e_{\bar{a}}^i = 0.
\]

In general, if a Lie groupoid \( \Psi \) is a subgroupoid of a Lie groupoid \( \Phi \) over the same base \( B, \) then a connection \( \gamma : B \to Q^1(\Phi) \) is said to be reducible to \( \Psi \) if \( C(B) \subset Q^1(\Psi). \) We shall first investigate the reducibility of a connection \( \gamma : J^1 E \to J^2 E \) to \( \Phi_p(J^1 E). \)

We recall that beside the usual projection \( j^1_2 : J^2 E \to J^1 E \) there is another canonical projection \( l^1_2 : J^2 E \to J^1 E \) defined by \( l^1_2(j^1_2 \sigma) = j^1_2(j^0_2 \sigma); \) this projection is said to be lateral, [6]. On \( J^2(R^n \times R^m) \) there are further coordinates \( y^a_{\bar{a}i}, y_{\bar{a}ij} \) given by

\[
y^a_{\bar{a}i}(j^1_2 \sigma) = \partial_i y^a(\sigma(x)), \quad y_{\bar{a}ij}(j^1_2 \sigma) = \partial_j y^a(\sigma(x)).
\]
According to [6], if \( x^i, y^a, \dot{y}^a, y^a_{0i}, y^a_{ij} \) are the coordinates of an element \( Y \in \tilde{J}^2(R^n \times R^m) \), then

\[
\begin{align*}
\tilde{f}_2^1(Y) &= (x^i, y^a, \dot{y}^a), \\
\tilde{l}_2^1(Y) &= (x^i, y^a, y^a_{0i}).
\end{align*}
\]

**Proposition 1.** A connection \( \gamma : J^1E \to \tilde{J}^2E \) is reducible to \( \Phi_\rho(J^1E) \) if and only if there exists a connection \( \gamma_0 : E \to J^1E \) compatible with the diagram

\[
\begin{array}{cccccccccc}
0 & \quad & 0 & \quad & 0 \\
\downarrow & \quad & \downarrow & \quad & \downarrow \\
0 & \quad & \rightarrow & E \otimes (\otimes T^*(B)) & \rightarrow J^1(E \otimes T^*(B)) & \rightarrow E \otimes T^*(B) & \rightarrow 0 \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
0 & \quad & \rightarrow & J^1E \otimes T^*(B) & \rightarrow \tilde{J}^2E & \rightarrow J^1E & \rightarrow 0 \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
0 & \quad & \rightarrow & E \otimes T^*(B) & \rightarrow J^1E & \rightarrow E & \rightarrow 0 \\
\downarrow & \quad & \downarrow & \quad & \downarrow \\
0 & \quad & \rightarrow & 0 & \rightarrow 0 & \rightarrow 0 & \rightarrow 0
\end{array}
\]

**Proof.** In general, the coordinate form of \( \gamma \) is

\[
\begin{align*}
y^a_{0i} &= \Gamma^a_{\beta0i} y^\beta + \Gamma^a_{\beta0j} y^\beta j^1_i, \\
y^a_{ij} &= \Gamma^a_{\betaij} y^\beta + \Gamma^a_{\betaik} y^\beta k^1.
\end{align*}
\]

The obstruction \( \omega(\gamma) \in \text{Hom}(E \otimes T^*(B), E \otimes T^*(B)) \) vanishes if and only if \( \Gamma^a_{\betaij} = 0 \). Comparing with (11), we obtain our assertion.

The connection \( C_0 \) can be described geometrically in the following simple way. The mapping \( \pi : \Phi_\rho(J^1E) \to \Phi(E) \), \( \theta \to \varphi \) is a functor. In general, if \( \Phi \) and \( \Psi \) are two Lie groupoids over \( B \) and if \( \lambda : \Phi \to \Psi \) is a base-preserving functor, then every connection \( C : B \to Q^1(\Phi) \) is transformed into a connection \( \lambda_*(C) : B \to Q^1(\Psi) \). In particular, in the situation of Proposition 1, we have \( C_0 = \pi_*(C) \).

5. By the general theory of prolongations of fibre bundles, if \( \Phi \) is a groupoid of operators on a fibred manifold \((S, q, B)\), then the first prolongation \( \Phi^1 \) of \( \Phi \) is a groupoid of operators on \( J^1S, [1] \). In particular, the first prolongation \( \Phi^1(E) \) of \( \Phi(E) \) is a groupoid of operators on \( J^1E \). Obviously, \( \Phi^1(E) \subset \Phi_\rho(J^1E) \). In the trivial case,

\[
\Phi^1(R^n \times R^m) = R^n \times T^1_n(L^1_m) \times L^1_n \times R^n,
\]
where \( T^1_n(L^1_m) \times L^1_n \) means the semi-direct product of \( T^1_n(L^1_m) \) and \( L^1_n \) with respect to the action \((S, A) \mapsto SA \) of \( L^1_n \) on \( T^1_n(L^1_m) \), \( S \in T^1_n(L^1_m) \), \( A \in L^1_n \). (In the other words, the multiplication in \( T^1_n(L^1_m) \times L^1_n \) is given by
\((S_1, A_1) (S_2, A_2) = ((S_1 A_2) \cdot S_2, A_1 A_2)\), where the dot denotes the multiplication in \(T^1_n (L^1_m)\). We shall study the reducibility of connection (14) to \(\Phi^1(E)\). On \(\Phi^1(R^n \times R^m)\) there are coordinates
\[
\tilde{x}^i, b^a_{\tilde{t}}, b^a_{\tilde{t}}, a^i_{\tilde{t}}, x^i,
\]
det \(a^i_{\tilde{t}} \neq 0\), and the action of \(\Phi^1(R^n \times R^m)\) on \(J^1(R^n \times R^m)\) is given by
\[
(\tilde{x}^i, b^a_{\tilde{t}}, b^a_{\tilde{t}}, a^i_{\tilde{t}}, x^i, \xi^a, \nu^a) = (\tilde{x}^i, b^a_{\tilde{t}} y^a, (b^a_{\tilde{t}} y^a + b^a_{\tilde{t}} y^a A^i_{\tilde{t}}))
\]
provided \(a^i_{\tilde{t}} A^i_{\tilde{t}} = \delta^i_k\) (cf. [4]).

Let C be a connection on \(\Phi^1(E)\),
\[
C(x) = j^l_i(t^r, b^a_{\tilde{t}}(t), b^a_{\tilde{t}}(t), a^i_{\tilde{t}}(t), x^i).
\]

Using (15) and \(b^a_{\tilde{t}}(x) = \delta^a_{\tilde{t}}, b^a_{\tilde{t}}(x) = 0, a^i_{\tilde{t}}(x) = a^i_{\tilde{t}}\), one finds easily that the corresponding splitting \(J^1E \to \tilde{J}^2E\) is
\[
y^a_{\tilde{t}i} = \Gamma^a_{\tilde{t}ij} y^a_j + \Gamma^a_{\tilde{t}ij} y^a_j - \Gamma^a_{\tilde{t}ij} y^a_k,
\]
where
\[
(17) \quad \Gamma^a_{\tilde{t}ij}(x) = \partial_j b^a_{\tilde{t}}(x), \quad \Gamma^a_{\tilde{t}ij}(x) = \partial_j b^a_{\tilde{t}}(x), \quad \Gamma^a_{\tilde{t}ij}(x) = \partial_k a^i_{\tilde{t}}(x).
\]

Consider now an arbitrary connection (14) on \(J^1E\). If (14) is reducible to \(\Phi^1(J^1E)\), then there is a splitting \(\gamma^0: E \otimes T^* (E) \to J^1(E \otimes T^* (E))\) in the top row of (13) compatible with the diagram. By (16), if the connection (14) is further reducible to \(\Phi^1(E)\), then \(C^0\) is decomposable with respect to \(C_0\), i.e., there is a linear connection \(L\) on \(B\) such that \(C^0 = C_0 \otimes L\). One sees easily that this condition is also sufficient. Hence we obtain

**Proposition 2.** A connection \(\gamma: J^1E \to \tilde{J}^2E\) is reducible to \(\Phi^1(E)\) if and only if there exists a connection \(\gamma_0: E \to J^1E\) compatible with diagram (13) and the induced connection \(C^0\) on \(E \otimes T^* (B)\) is decomposable to \(C_0\).

**6.** We shall now discuss the similar problem for \(r = 2\). Since the lateral projections play a fundamental role even in this case, we shall first give a survey of some of their properties. According to [6] there are three canonical projections \(j^2_3, l^2_3, \bar{l}^2_3\) of \(\tilde{J}^2E\) into \(\tilde{J}^2E\). We shall state the coordinate form of these projections. On \(\tilde{J}^2(R^n \times R^m)\) there are further coordinates \(y^a_{\tilde{t}ii}, y^a_{\tilde{t}oi}, y^a_{\tilde{t}ij}, y^a_{\tilde{t}ik}\) given by
\[
(18) \quad y^a_{\tilde{t}ii}(j^2_3 \sigma) = \partial_i y^a(\sigma(x)), \quad y^a_{\tilde{t}oi}(j^2_3 \sigma) = \partial_i y^a_i(\sigma(x)),
\]
\[
y^a_{\tilde{t}ij}(j^2_3 \sigma) = \partial_j y^a_i(\sigma(x)), \quad y^a_{\tilde{t}ik}(j^2_3 \sigma) = \partial_k y^a_i(\sigma(x)).
\]

If \(x^i, y^a, y^a_{\tilde{t}ii}, y^a_{\tilde{t}oi}, y^a_{\tilde{t}ij}, y^a_{\tilde{t}ik}\) are the coordinates of an element \(Y \in \tilde{J}^2(R^n \times R^m)\), then
\[
j^2_3(Y) = (x^i, y^a, y^a_{\tilde{t}ii}, y^a_{\tilde{t}oi}, y^a_{\tilde{t}ij}),
\]
\[
l^2_3(Y) = (x^i, y^a, y^a_{\tilde{t}oi}, y^a_{\tilde{t}ij}),
\]
\[
\bar{l}^2_3(Y) = (x^i, y^a, y^a_{\tilde{t}oi}, y^a_{\tilde{t}ij}).
\]
7. Considering an arbitrary splitting $\gamma: \tilde{J}^2 E \to \tilde{J}^3 E$ of the form
\begin{align}
y^a_{l_{0i}} &= \Gamma^a_{\beta\alpha l} y^\beta + \Gamma^a_{\alpha\beta l} y^\beta + \Gamma^a_{\rho_{0i} l} y^\rho + \Gamma^a_{\rho_{0i} l} y^\rho,
y^a_{l_{0j}} &= \Gamma^a_{\beta\alpha j} y^\beta + \Gamma^a_{\alpha\beta j} y^\beta + \Gamma^a_{\rho_{0j} l} y^\rho + \Gamma^a_{\rho_{0j} l} y^\rho,
y^a_{l_{0i}} &= \Gamma^a_{\beta\alpha i} y^\beta + \Gamma^a_{\alpha\beta i} y^\beta + \Gamma^a_{\rho_{0i} l} y^\rho + \Gamma^a_{\rho_{0i} l} y^\rho,
y^a_{l_{ij}} &= \Gamma^a_{\beta\alpha j} y^\beta + \Gamma^a_{\alpha\beta j} y^\beta + \Gamma^a_{\rho_{ij} l} y^\rho + \Gamma^a_{\rho_{ij} l} y^\rho + \Gamma^a_{\rho_{ij} l} y^\rho,
y^a_{l_{ik}} &= \Gamma^a_{\beta\alpha k} y^\beta + \Gamma^a_{\rho_{ijk} l} y^\rho + \Gamma^a_{\rho_{ijk} l} y^\rho + \Gamma^a_{\rho_{ijk} l} y^\rho + \Gamma^a_{\rho_{ijk} l} y^\rho,
\end{align}
we investigate a connection on $\Phi(\tilde{J}^2 E)$. By [1], the second non-holonomic prolongation $\tilde{\Phi}^2(E)$ of $\Phi(E)$ is a groupoid of operators on $\tilde{J}^2 E$ satisfying $\tilde{\Phi}^2(E) \subset \Phi(\tilde{J}^3 E)$. We study the reducibility of the connection (20) to $\tilde{\Phi}^2(E)$. (Similarly to Proposition 1, some of the following conditions can be explained even separately. Nevertheless, we shall not formulate explicitly any of the corresponding assertions.) In the trivial case we have
\[ \tilde{\Phi}^2 \left( R^n \times R^m \right) = R^n \times \tilde{T}^2_n \left( L_m^1 \right) \times \tilde{L}^2_n \times R^n, \]
where $\tilde{T}^2_n \left( L_m^1 \right) \times \tilde{L}^2_n$ means the semi-direct product with respect to the action $(S, A) \mapsto SA$ of $\tilde{L}^2_n$ on $\tilde{T}^2_n \left( L_m^1 \right)$, $S \in \tilde{T}^2_n \left( L_m^1 \right)$, $A \in \tilde{L}^2_n$. On $\tilde{\Phi}^2 \left( R^n \times R^m \right)$ there are coordinates
\[ \bar{x}^i, b^a_{\beta i}, b^a_{\beta j}, b^a_{\beta k}, a^i_{\beta}, a^i_{\gamma}, a^i_{\delta}, a^i_{\kappa}, a^i_{\lambda}, \]
det $| a^i_{\beta j} | \neq 0$, and the action of $\tilde{\Phi}^2 \left( R^n \times R^m \right)$ on $\tilde{J}^2 \left( R^n \times R^m \right)$ is given by
\begin{align}
(\bar{x}^i, b^a_{\beta i}, b^a_{\beta j}, b^a_{\beta k}, a^i_{\beta}, a^i_{\gamma}, a^i_{\delta}, a^i_{\kappa}, a^i_{\lambda}), & (x^i, y^a, y^b, y^c) \\
&= (\bar{x}^i, b^a_{\beta i} y^a, (b^a_{\beta j} y^a + b^a_{\beta k} y^a) A^i_{\beta j}, (b^a_{\beta j} y^a + b^a_{\beta k} y^a) A^i_{\beta k}, b^a_{\beta j} y^b + b^a_{\beta k} y^b + b^a_{\beta l} y^c) A^i_{\kappa l} A^i_{\alpha j} - (b^a_{\beta j} y^k + b^a_{\beta k} y^b) A^i_{\gamma k} a^i_{\gamma k} A^i_{\alpha j},
\end{align}
provided $a^i_{\beta} A^i_{\beta} = \delta^i_{\beta}$, $a^i_{\gamma} A^i_{\gamma} = \delta^i_{\gamma}$ (cf. [4]).
Let $C$ be a connection on $\tilde{\Phi}^2(E)$,
\[ C(x) = j^2 \left( t^i, b^a_{\beta i}(t), b^a_{\beta j}(t), b^a_{\beta k}(t), a^i_{\beta}(t), a^i_{\gamma}(t), a^i_{\delta}(t), a^i_{\kappa}(t), a^i_{\lambda}(t), x^i \right), \]
where
\[ b^a_{\beta i}(x) = \delta^a_{\beta}, \quad b^a_{\beta j}(x) = b^a_{\rho_{0j} l}(x) = 0, \]
\[ b^a_{\beta j}(x) = 0, \quad a^i_{\beta}(x) = a^i_{\gamma}(x) = \delta^i_{\beta}, \quad a^i_{\gamma}(x) = 0. \]
Introduce the corresponding $\Gamma$'s analogously to (17). Then (21) implies that $C$ determines a splitting $\tilde{J}^2 E \to \tilde{J}^3 E$ of the form
\begin{align}
y^a_{l_{0i}} &= \Gamma^a_{\beta l} y^\beta, 
y^a_{l_{0j}} &= \Gamma^a_{\beta j} y^\beta + \Gamma^a_{\rho_{0j} l} y^\rho - \Gamma^a_{\rho_{0j} l} y^\rho, 
y^a_{l_{ij}} &= \Gamma^a_{\beta ij} y^\beta + \Gamma^a_{\rho_{ij} l} y^\rho - \Gamma^a_{\rho_{ij} l} y^\rho, 
y^a_{l_{ik}} &= \Gamma^a_{\rho_{ijk} l} y^\beta + \Gamma^a_{\rho_{ijk} l} y^\rho - \Gamma^a_{\rho_{ijk} l} y^\rho + \Gamma^a_{\rho_{ijk} l} y^\rho - \Gamma^a_{\rho_{ijk} l} y^\rho - \Gamma^a_{\rho_{ijk} l} y^\rho, 
y^a_{l_{ij}} &= \Gamma^a_{\beta l} y^\beta + \Gamma^a_{\rho_{ij} l} y^\rho + \Gamma^a_{\rho_{ij} l} y^\rho + \Gamma^a_{\rho_{ij} l} y^\rho.
\end{align}
Our next investigation will be based on the comparison of (20) and (22). Naturally, we shall deduce all conditions in an invariant form.

8. Consider first the lateral projection \( l_2^* \) and a commutative exact diagram

\[
\begin{array}{ccccccc}
0 & \to & J^1E \otimes (\otimes T^*(B)) & \to & J^1(J^1E \otimes T^*(B)) & \xrightarrow{\gamma_0} & J^1E \otimes T^*(B) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tilde{J}^2E \otimes T^*(B) & \to & \tilde{J}^2E & \xrightarrow{\gamma} & \tilde{J}^2E & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & J^1E \otimes T^*(B) & \to & \tilde{J}^2E & \xrightarrow{\gamma_1} & J^1E & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

(23)

By the first two formulas in (22) there should be a splitting \( \gamma_1: J^1E \to \tilde{J}^2E \) compatible with the diagram. By Lemma 2, the obstruction is an element of \( \text{Hom}(J^1E \otimes T^*(B), J^1E \otimes T^*(B)) \), which will be denoted by \( \omega_1(\gamma) \). In coordinates, \( \omega_1(\gamma) = 0 \) means

\[
\begin{align*}
\Gamma_{\rho \kappa \lambda}^{\alpha \beta \gamma} &= 0, \\
\Gamma_{\rho \kappa \lambda}^{\alpha \beta \gamma} &= 0, \\
\Gamma_{\rho \kappa \lambda}^{\alpha \beta \gamma} &= 0, \\
\Gamma_{\rho \kappa \lambda}^{\alpha \beta \gamma} &= 0.
\end{align*}
\]

(24)

Taking into account the second lateral projection \( 2l_2^* \), we obtain a commutative exact diagram:

\[
\begin{array}{ccccccc}
0 & \to & J^1(E \otimes T^*(B)) \otimes T^*(B) & \to & \tilde{J}^2(E \otimes T^*(B)) & \xleftarrow{\gamma_2^0} & J^1(E \otimes T^*(B)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tilde{J}^2E \otimes T^*(B) & \to & \tilde{J}^2E & \xrightarrow{\gamma} & \tilde{J}^2E & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & J^1E \otimes T^*(B) & \to & \tilde{J}^2E & \xrightarrow{\gamma_2} & J^1E & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

(25)

By the first and third formulas in (22) there should be a splitting \( \gamma_2: J^1E \to \tilde{J}^2E \) compatible with the diagram. The obstruction \( \omega_2(\gamma) \) is an element of \( \text{Hom}(J^1(E \otimes T^*(B)), J^1E \otimes T^*(B)) \). In coordinates, \( \omega_2(\gamma) = 0 \) is equivalent to

\[
\begin{align*}
\Gamma_{\rho \kappa \lambda}^{\alpha \beta \gamma} &= 0, \\
\Gamma_{\rho \kappa \lambda}^{\alpha \beta \gamma} &= 0, \\
\Gamma_{\rho \kappa \lambda}^{\alpha \beta \gamma} &= 0, \\
\Gamma_{\rho \kappa \lambda}^{\alpha \beta \gamma} &= 0.
\end{align*}
\]

(26)
In what follows, we assume $\omega_1(\gamma) = 0$, $\omega_2(\gamma) = 0$. Then (24) and (26) imply that there is a splitting $\gamma_0$: $E \to J^1E$ compatible with the following diagram ($a = 1, 2$):

\[
\begin{array}{ccc}
0 & \to & E \otimes (\otimes T^*(B)) \to J^1(E \otimes T^*(B)) \to \gamma_0 \otimes \delta_a \to 0 \\
\downarrow & & \downarrow & & \downarrow \gamma_a \\
0 & \to & J^1E \otimes T^*(B) \to J^2E \to \gamma_a^{0} \to J^1E \to 0 \\
\downarrow & & \downarrow l_a^1 & & \downarrow j_a^1 \\
0 & \to & E \otimes T^*(B) \to J^1E \to \gamma_0 \to E \to 0 \\
\downarrow & & \downarrow j_1^0 & & \downarrow 0 \\
0 & & 0 & & 0
\end{array}
\]

By the first three formulas in (22), the induced connection on $E \otimes T^*(B)$ should be reducible to the connection $C_0$ on $E$, i.e., there should exist a linear connection $L_a$ on $B$ such that the induced connection in the top row of (27) is of the form $C_0 \otimes L_a$. Further, by the third and fourth formulas in (22), we conclude that the connection $C_i^0$ of the top row of (23) should satisfy

\[
(28) \quad C_i^0 = C_1 \otimes L_2.
\]

(In particular, the connection induced on the intersection $E \otimes (\otimes T^*(B))$ of the kernels of both projections $j_2^1$ and $l_2^1$ is $C_0 \otimes L_1 \otimes L_2$.)

It remains to deduce an analogous condition for $C_2^0$, but this requires an auxiliary consideration.

9. Let $E_1$ and $E_2$ be two vector bundles over $B$, let $C_a$ be a connection on $J^1E_a$ reducible to $\Phi^1(E_a)$, $a = 1, 2$, and let both $C_1$ and $C_2$ have a common underlying linear connection $L$. Then the splitting $\gamma_1$ or $\gamma_2$ is of the form

\[
\begin{align*}
y_{a1}^i & = \Gamma_{b_1}^i y^b, \\
y_{a2}^i & = \Gamma_{b_2}^i y^b + \Gamma_{b_2}^i y_j^b - \Gamma_{b_2}^i y_k^b
\end{align*}
\]

or

\[
\begin{align*}
y_{a1}^i & = \Gamma_{b_1}^i y^b, \\
y_{a2}^i & = \Gamma_{b_2}^i y^b + \Gamma_{b_2}^i y_j^b - \Gamma_{b_2}^i y_k^b
\end{align*}
\]

respectively. Hence we may write

\[
\begin{align*}
C_1(x) & = j_1^i(t^j, b_1^a(t), b_2^a(t), a_1^j(t), x^i) = j_2^i v_1(t), \\
C_2(x) & = j_1^i(t^j, b_1^a(t), b_2^a(t), a_1^j(t), x^i) = j_2^i v_2(t).
\end{align*}
\]

Prolonging the action of the product $\Phi(E_1) \times \Phi(E_2)$ on $E_1 \otimes E_2$, we obtain an action of the first prolongation $(\Phi(E_1) \times \Phi(E_2))^1$ on $J^1(E_1 \otimes E_2)$. Since the sections $v_1$ and $v_2$ determine a section $(v_1, v_2)$ of $(\Phi(E_1) \times \Phi(E_2))^1$,
\[ C(x) = j_x^i (\varphi_1 (t), \varphi_2 (t)) \] is an element of connection on \( (\Phi (E_1) \times \Phi (E_2))^1 \). Thus we obtain a connection on \( J^1 (E_1 \otimes E_2) \), which will be said to be the tensor product of \( C_1 \) and \( C_2 \) over \( L \) and will be denoted by

\[ C = C_1 \otimes L C_2. \]

By the definition, the coordinate form of the above-given action of \( (\Phi (E_1) \times \Phi (E_2))^1 \) on \( J^1 (E_1 \otimes E_2) \) is

\[
(x^i, b^a_{\beta}, b^a_{\mu}, b^a_{\mu}, a^i_j, x^i) (x^i, y_{\alpha i}, y_{\alpha j}) = (x^i, b^a_{\beta} b^a_{\mu} y_{\beta \mu}, (b^a_{\mu} b^a_{\mu} y_{\beta \mu} + b^a_{\mu} b^a_{\mu} y_{\beta \mu} + b^a_{\mu} b^a_{\mu} y_{\beta \mu}) A^i_j).
\]

Using (31), we deduce that \( C_1 \otimes L C_2 \) determines a splitting

\[ J^1 (E_1 \otimes E_2) \to \tilde{J}^2 (E_1 \otimes E_2) \]

of the form

\[
y_{\alpha i} = \Gamma_{\beta i}^\alpha y_{\beta i} + \Gamma_{\mu i}^\alpha y_{\mu i},
\]

\[
y_{\alpha j} = \Gamma_{\beta j}^\alpha y_{\beta j} + \Gamma_{\mu j}^\alpha y_{\mu j} + \Gamma_{\mu i}^\alpha y_{\beta i} - \Gamma_{\beta j}^\alpha y_{\mu j}.
\]

Further, let \( C \) be a connection on \( J^1 (E_1 \otimes E_2) \) and let \( C_1 \) be a connection on \( \Phi^1 (E_1) \) with an underlying linear connection \( L \). We say that \( C \) is decomposable with respect to \( C_1 \), if there exists a connection \( C_2 \) on \( \Phi^1 (E_2) \) with the same underlying linear connection \( L \) such that

\[ C = C_1 \otimes L C_2. \]

By (32), if \( C \) is decomposable with respect to \( C_1 \), then the second factor \( C_2 \) is uniquely determined.

10. By the second and fourth formulas in (22), the connection \( C_2^0 \) on \( J^1 (E \otimes T^* (B)) \) should be decomposable with respect to \( C_2 \), i.e., there should exist a reducible connection \( M \) on \( J^1 (T^* (B)) \) with the underlying linear connection \( L_2 \) such that

\[
C_2^0 = C_2 \otimes L_2 M.
\]

Comparing now (20) and (22), we see that we have found all relations characterizing (22) with respect to (20). Hence we have proved

**Proposition 3.** A connection \( \gamma : \tilde{J}^2 E \to \tilde{J}^3 E \) is reducible to \( \tilde{J}^2 (E) \) if and only if all the following conditions are satisfied:

(a) \( \omega_1 (\gamma) = 0, \omega_2 (\gamma) = 0; \)

(b) the connection of the top row of (27) is of the form \( C_0 \otimes L_a \), where \( C_0 \) is the connection of the bottom row of (27) and \( L_a \) is a convenient linear connection on \( B, a = 1, 2; \)
(c) the connection $C_1^q$ of the top row of (23) is of the form $C_1 \otimes L_2$, where $C_1$ is the connection of the bottom row of (23);

(d) the connection $C_2^q$ of the top row of (25) is of the form $C_2 \otimes M$, where $L_2$ is the connection of the bottom row of (25) and $M$ is a convenient connection on $J^1(T^*(B))$.

11. Assume in the sequel that $C$ is reducible to $\tilde{\Phi}^2(E)$. Beside $\tilde{\Phi}^2(E)$ and $\Phi^2(E)$, [1], one can naturally introduce also the following subgroupoids of $\tilde{\Phi}^2(E)$:

$$\tilde{\Phi}_s^2(E) = \{ \theta \in \tilde{\Phi}^2(E); \ b \theta \in \tilde{\Pi}^2(B) \},$$

$$\tilde{\Phi}_h^2(E) = \{ \theta \in \tilde{\Phi}^2(E); \ b \theta \in \Pi^2(B) \},$$

$$\Phi^2_h(E) = \{ \theta \in \Phi^2(E); \ b \theta \in \Pi^2(B) \};$$

$\tilde{\Pi}^2(B)$ or $\Pi^2(B)$ means the groupoid of all invertible semi-holonomic or holonomic 2-jets of $B$ into $B$, respectively.

In the trivial case we have

$$\tilde{\Phi}_s^2(R^n \times R^m) = R^n \times \tilde{T}^2_n(L_m) \otimes \tilde{L}_n^2 \times R^n,$$

$$\tilde{\Phi}_h^2(R^n \times R^m) = R^n \times \tilde{T}^2_n(L_m) \otimes \tilde{L}_n^2 \times R^n,$$

$$\Phi^2(R^n \times R^m) = R^n \times \tilde{T}^2_n(L_m) \otimes \tilde{L}_n^2 \times R^n,$$

$$\Phi^2_h(R^n \times R^m) = R^n \times T^2_n(L_m) \otimes \tilde{L}_n^2 \times R^n.$$

(34)

By (22), we obtain immediately

PROPOSITION 4. A connection $C$ on $\tilde{\Phi}^2(E)$ is reducible to

(a) $\tilde{\Phi}_s^2(E)$ if and only if the connections $L_1$ and $L_2$ coincide,

(b) $\tilde{\Phi}_h^2(E)$ if and only if the connections $C_1$ and $C_2$ coincide.

12. Assume $L_1 = L_2$. Then (22) implies that $C$ is reducible to $\Phi^2_h(E)$ if and only if $L_{[jk]}^i = 0$, provided the square brackets denote antisymmetrization. We shall give an invariant explanation of this condition.

Libermann ([5], p. 159) has established an identification

$$J^1T^*(B) \approx \tilde{T}^2*(B),$$

(35)

There $\tilde{T}^2*(B)$ is the vector bundle of all semi-holonomic 2-jets of $B$ into $R$ with target 0. Consider the injection $i: T^2*(B) \to \tilde{T}^2*(B)$ of the subspace $w^2*(B) \subset \tilde{T}^2*(B)$ of all holonomic jets. According to [3], we have the exact sequence

$$0 \to T^2*(B) \xrightarrow{i} \tilde{T}^2*(B) \to \Lambda^2 T^*(B) \to 0,$$

(36)
where $A$ is a special case of the difference tensor map. Taking account of (35), the above-mentioned connection $M$ determines a splitting $\mu: T^{2\ast}(B) \to J^1\bar{T}^{2\ast}(B)$. We say that $M$ is symmetric if $\mu(T^{2\ast}(B)) \subset J^1\bar{T}^{2\ast}(B)$. We have a commutative exact diagram

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & j^1_0 & \downarrow \\
0 & T^{2\ast}(B) & \otimes & T^\ast(B) & J^1T^{2\ast}(B) & \leftarrow & J^1\bar{T}^{2\ast}(B) & \rightarrow & T^{2\ast}(B) & \rightarrow & 0 \\
j^1_0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \mu & j^1_0 & \downarrow & \downarrow & \downarrow \\
0 & \bar{T}^{2\ast}(B) & \otimes & T^\ast(B) & J^1\bar{T}^{2\ast}(B) & \leftarrow & J^1\bar{T}^{2\ast}(B) & \rightarrow & T^{2\ast}(B) & \rightarrow & 0 \\
j^1_0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & j^1_0 & \downarrow & \downarrow & \downarrow \\
0 & \lambda^2T^{2\ast}(B) & \otimes & T^\ast(B) & J^1(\lambda^2T^\ast(B)) & \rightarrow & \lambda^2T^\ast(B) & \rightarrow & 0 \\
\end{array}
\]

(37)

where $j^1A$ means the first jet prolongation of the morphism $A: \bar{T}^{2\ast}(B) \rightarrow \lambda^2T^\ast(B)$. By Lemma 2, $M$ is symmetric if and only if

\[(j^1A)\mu i = 0 \in \text{Hom}(T^{2\ast}(B), \lambda^2T^\ast(B) \otimes T^\ast(B)).\]

Applying the relation $L_1 = L_2$, we find the following coordinate form of $\mu$:

\[
y_{ij} = -I_{ij}^k y_k,
\]

(38)

\[
y_{ijk} = -I_{ijk}^l y_l - I_{ikj}^l y_l - I_{jki}^l y_l.
\]

Using the coordinate expression of the difference tensor map (see [3], p. 139), we infer that $M$ is symmetric if and only if

\[
0 = y_{[ijkl]} = -I_{[ijkl]}^l y_l.
\]

(39)

Hence we have

**Proposition 5.** A connection $C$ on $\bar{\Phi}^2(E)$ is reducible to

(a) $\bar{\Phi}^2_h(E)$ if and only if $L_1 = L_2$ and $M$ is symmetric,

(b) $\bar{\Phi}^2_h(E)$ if and only if $C_1 = C_2$ and $M$ is symmetric.

13. It remains to treat the reducibility of $C$ to $\Phi^2(E)$. Assume that $C$ is reducible to $\Phi^2(E)$. By (22), $C$ is further reducible to $\Phi^2(E)$ if and only if $I_{[ijkl]}^l = 0$ and $I_{ijkl}^l = 0$. Denote by $\gamma$ the restriction of $\gamma$ to $J^2E$. Since $C$ is reducible to $\Phi^2(E)$, the values of $\gamma$ lie in $J^1J^2E$. Consider the injection $i: J^2E \rightarrow J^2\bar{E}$. By [3], we have the exact sequence

\[
0 \rightarrow J^2E \xrightarrow{i} J^2\bar{E} \xrightarrow{\Delta} \bar{E} \otimes \lambda^2T^\ast(B) \rightarrow 0,
\]

(40)

where $\Delta$ is the difference tensor map. The morphism $\Delta$ is prolonged to a morphism $j^1\Delta: J^1J^2E \rightarrow J^1(\bar{E} \otimes \lambda^2T^\ast(B))$ and we get a commutative exact diagram
If \( C \) is reducible to \( \Phi^2(E) \), then there exists a splitting in the top row compatible with the diagram. By Lemma 2, the obstruction is an element \( \omega(\gamma) \) of \( \text{Hom}(J^2E, E \otimes \Lambda^2 T^* (B) \otimes T^* (B)) \). Using \( C_1 = C_2 \) and \( y^a_i = y^a_{bi} \), we simplify the last row of (22) to the form

\[
y^a_{ijk} = \Gamma^a_{bij} y^b_i + \Gamma^a_{bjk} y^j_i + \Gamma^a_{bik} y^b_i - \Gamma^a_{ijk} y^i + \Gamma^a_{bik} y^i - \Gamma^a_{ijk} y^i - \Gamma^a_{ijk} y^b_i.
\]

By direct evaluation, we infer that \( \omega(\gamma) \) vanishes if and only if

\[
y^a_{ijkl} = \Gamma^a_{ijkl} y^b_i - \Gamma^a_{ijkl} y^b_i.
\]

Thus, finally, we obtain

**Proposition 6.** A connection \( C \) on \( \tilde{\Phi}^2(E) \) is reducible to \( \Phi^2(E) \) if and only if \( C_1 = C_2 \) and \( \omega(\gamma) \) vanishes.

**References**


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