ANNALES
POLONICI MATHEMATICI
XXXV (1977)

Cluster sets of set-mappings

by Shinji Yamashita (Tokyo, Japan)

Abstract. A set-mapping S from a set $A \neq \emptyset$ into a set B is a mapping from A into the set of all subsets of B. We introduce the cluster sets of S along the classical lines, and propose some properties of them. Among other things we prove that some of the known results on the cluster sets of the usual mappings follow under far general settings; especially those of Collingwood on the boundary cluster sets of "arbitrary functions" are contained in the Young-type theorem on set-mappings defined on the circle. We further introduce the notion of the "boundary normality condition", which reveals that many results on the cluster sets of meromorphic functions in the disk depend on a simple topological property, whence do not depend on the "analyticity" heavily. The notion of set-mappings will be of use in the study of many-valued functions, for example, algebroid functions.

Introduction. A set-mapping S from a non-empty set A into a set B, in notation, $S\colon A \to P(B)$, is a mapping from A into the totality $P(B)$ of subsets of B (see [20]); thus, $S(p)$ may be empty for a $p \in A$. The notion is an extended one of a (single-valued) map f from A into B in the sense that we obtain a set-mapping $S_f\colon A \to P(B)$ on setting $S_f(p) = \{f(p)\}$ for each $p \in A$. Assume that A lies in a topological space T, and B is a topological space. We use the notation $S(A_1) = \bigcup_{p \in A_1} S(p)$ for each subset A_1 of A with $S(\emptyset) = \emptyset$. Let $p \in T$ be a point of the closure $\bar{E}$ of $E \subset A$. We define the *cluster set* $C_E(S, p)$ of $S\colon A \to P(B)$ at p relative to E by

$$C_E(S, p) = \bigcap_V \overline{S\bigl(V \cap (E - \{p\})\bigr)},$$

where V ranges over all neighbourhoods of p in T and the closure is taken in B; $C_E(S, p)$ is a (possibly empty) closed set in B.

In the present paper we shall study systematically the cluster sets of $S\colon A \to P(B)$ in the case where B is a metric space Ω, and A is the disk $D = \{z;\ |z| < 1\}$ or the circle $\Gamma = \{z;\ |z| = 1\}$ in the complex plane. (See [14] and [4] for the basic properties of cluster sets of single-valued functions. See also [19] for an abstract study.) In Section 1 we extend the symmetry theorem of W. H. Young [26] to set-mappings $S\colon \Gamma \to P(\Omega)$, Ω being K_σ, that is, Ω admits a countable covering by compact sets.

We emphasize that the extension (Theorem 1.1) would not be a mere analogy in the sense that the results of E. F. Collingwood (see [4], p. 82, Theorem 4.11) and J. T. Gresser [9], Theorem 2, follow from Theorem 1.1 under far general settings (see Corollary 1.2 and Corollary 1.3). In Section 2 we prove the two theorems (Theorem 2.1 and Theorem 2.3) for $S: D \to P(\Omega)$, Ω being K_σ. The results contain Collingwood's maximality theorem on arbitrary functions (see [4], p. 80, Theorem 4.10) and the analogous result due to P. Lappan [11], Theorem 1. In Section 3 we consider the continuous set-mappings from D into Ω. The Collingwood maximality theorems (see [4], p. 76, Theorem 4.8 and [3], p. 387, Theorem 8 and p. 388, Corollary 4) for continuous or meromorphic functions follow from Theorem 3.1 and Theorem 3.2. In Section 4 we assume on $S: D \to P(\Omega)$ a certain smoothness, called the *boundary normality condition*. Under the assumption we obtain the extensions of the results for meromorphic functions. The results of J. M. Anderson [1], Theorem, and K. Noshiro [15], Theorem 8, are contained in Theorem 4.1, Corollary 4.2 and Theorem 4.7. Further, as a special case of Corollary 4.4, we obtain an extension of the Meier theorem (see [4], p. 154, Theorem 8.8) to algebroid, proved in [23], Theorem 2. The notion of the set-mappings thus serves for the study of boundary behaviors of multiple-valued functions in D. Since there are some definitions of normal points for meromorphic functions in D, we propose Theorem 4.7, which reveals the equivalence of them. Finally in Section 5 we shall study the horocyclic versions, in the sense of F. Bagemihl [2], of the results in the preceding sections. The versions follow from some geometrical considerations in D.

1. A Young-type theorem. We first consider a set-mapping from Γ into Ω; to avoid confusions, we denote the mapping by Σ: $\Gamma \to P(\Omega)$. For $\zeta \in \Gamma$ we denote $C_\Gamma(\Sigma, \zeta) \equiv C(\Sigma, \zeta)$. Furthermore we set

$$C^R(\Sigma, \zeta) = \bigcap_V \overline{\Sigma(\Gamma_R(\zeta) \cap V)},$$

and

$$C^L(\Sigma, \zeta) = \bigcap_V \overline{\Sigma(\Gamma_L(\zeta) \cap V)},$$

where V ranges over all open disks of center ζ and

$$\Gamma_R(\zeta) = \{z \in \Gamma; \operatorname{Im}(z\zeta^{-1}) < 0\},$$

$$\Gamma_L(\zeta) = \{z \in \Gamma; \operatorname{Im}(z\zeta^{-1}) > 0\}.$$

We note that both $C^L(\Sigma, \zeta)$ and $C^R(\Sigma, \zeta)$ are closed, and

$$C^L(\Sigma, \zeta) \cup C^R(\Sigma, \zeta) = C(\Sigma, \zeta) \tag{1.1}$$

for each $\zeta \in \Gamma$. In 1928 Young [26], p. 270–271, showed that an arbitrary real function of one real variable has the symmetry property which we describe for a real function f on Γ. Namely,

$$C^L(\Sigma_f, \zeta) = C(\Sigma_f, \zeta) = C^R(\Sigma_f, \zeta)$$

for each $\zeta \in \Gamma$ except possibly for a countable subset on Γ, where $\Sigma_f(\zeta) = \{f(\zeta)\}$ at each $\zeta \in \Gamma$. We shall show that an analogous result (indeed, an extension) is true for arbitrary Σ: $\Gamma \to P(\Omega)$ if Ω is K_σ.

THEOREM 1.1. *Let Ω be K_σ, and consider Σ: $\Gamma \to P(\Omega)$. Then,*

$$C^L(\Sigma, \zeta) = C(\Sigma, \zeta) = C^R(\Sigma, \zeta) \tag{1.2}$$

and

$$C(\Sigma, \zeta) \cup \overline{\Sigma(\zeta)} = C(\Sigma, \zeta) \tag{1.3}$$

for each $\zeta \in \Gamma$ except perhaps for a countable set on Γ.

Proof. We first remark that given a natural number n, each compact set in Ω, being K_σ, can be covered by a finite number of open balls with radii less than $1/(3n)$. Therefore, Ω admits a countable covering of (non-empty) compact sets

$$K_1^n, K_2^n, \ldots, K_m^n, \ldots \tag{1.4}$$

with diameters strictly less than $1/n$. We set

$$C^*(\Sigma, \zeta) = C(\Sigma, \zeta) \cup \overline{\Sigma(\zeta)} \qquad (\zeta \in \Gamma).$$

The set

$$E^R = \{\zeta \in \Gamma;\, C^*(\Sigma, \zeta) - C^R(\Sigma, \zeta) \neq \emptyset\}$$

is decomposed as

$$E^R = \bigcup_{n,m} E_{nm} \tag{1.5}$$

with

$$E_{nm} = \{\zeta \in \Gamma;\, K_m^n \cap C^*(\Sigma, \zeta) \neq \emptyset \text{ and } K_m^n \cap C^R(\Sigma, \zeta) = \emptyset\}$$

for $n, m = 1, 2, \ldots$ First, the inclusion $\supset$ in (1.5) is obvious. To prove $\subset$ in (1.5), we let $\zeta \in E^R$. If $C^R(\Sigma, \zeta) = \emptyset$, we have $\zeta \in E_{nm}$ for K_m^n with $K_m^n \cap C^*(\Sigma, \zeta) \neq \emptyset$. If $C^R(\Sigma, \zeta) \neq \emptyset$, we may find $a \in C^*(\Sigma, \zeta) - C^R(\Sigma, \zeta)$ and n such that the distance of a and the closed set $C^R(\Sigma, \zeta)$ is more than $2/n$. We then find m such that $a \in K_m^n$. It follows from $K_m^n \cap C^R(\Sigma, \zeta) = \emptyset$ that $\zeta \in E_{nm}$. Now, we show that if $\zeta \in E_{nm}$, then ζ is the left-hand end-point, viewed from an observer at the origin, of the open arc $I(\zeta)$ on Γ such that for each $\xi \in I(\zeta)$, $C^*(\Sigma, \xi) \cap K_m^n = \emptyset$. For, if otherwise, there exists a sequence $\xi_k \in \Gamma$, with $0 < \arg(\zeta\xi_k^{-1}) \to 0$ as $k \to \infty$, such that

$C^*(\Sigma, \xi_k) \cap K_m^n \neq \emptyset$ $(k \geqslant 1)$. Then we may choose ξ_k' in a neighbourhood of each ξ_k on Γ such that $\overline{\Sigma(\xi_k')} \cap K_m^n \neq \emptyset$ $(k \geqslant 1)$ with $0 < \arg(\zeta \xi_k'^{-1}) \to 0$ as $k \to \infty$. To see this we assume

$$\overline{\Sigma(\eta)} \cap K_m^n = \emptyset$$

for each η in a neighbourhood I_k on Γ of a certain ξ_k. Since $\Sigma(\eta) \cap K_m^n = \emptyset$ for each η,

$$\Sigma(I_k) \cap K_m^n = \emptyset,$$

whence

$$\overline{\Sigma(I_k)} \cap K_m^n = \overline{\Sigma(I_k) \cap K_m^n} = \emptyset.$$

We then have a contradiction that $C^*(\Sigma, \xi_k) \cap K_m^n = \emptyset$, because

$$C^*(\Sigma, \xi_k) = \bigcap_V \overline{\Sigma(V \cap \Gamma)} \subset \overline{\Sigma(I_k)},$$

where V ranges over all open disks of center ξ_k. Now, we can choose a subsequence $\{\xi_k''\}$ of $\{\xi_k\}$ and a point β on the compact set K_m^n such that there exists $\beta_k \in \Sigma(\xi_k'')$ $(k \geqslant 1)$ with $\beta_k \to \beta$ as $k \to \infty$. This implies that $\beta \in C^R(\Sigma, \zeta)$, a contradiction to $\zeta \in E_{nm}$. Now that the existence of $I(\zeta)$ is proved, we remark $I(\zeta) \cap E_{nm} = \emptyset$ for each $\zeta \in E_{nm}$. The set E_{nm} thus consists of the left-hand end-point of non-overlapping open arcs on Γ, whence E_{nm} is countable. Thus, E^R is countable by (1.5). Similarly we can prove that the set E^L corresponding to $C^L(\Sigma, \cdot)$ is countable. Therefore, for each $\zeta \in \Gamma - (E^L \cup E^R)$,

$$C^L(\Sigma, \zeta) = C^*(\Sigma, \zeta) = C^R(\Sigma, \zeta),$$

which, combined with (1.1) and $C(\Sigma, \zeta) \subset C^*(\Sigma, \zeta)$, implies (1.2) and (1.3).

Let S: $D \to P(\Omega)$, and consider $\Sigma_S : \Gamma \to P(\Omega)$ defined by

$$\Sigma_S(\zeta) = C(S, \zeta) \equiv C_D(S, \zeta)$$

for each $\zeta \in \Gamma$. Then the sets $C^R(\Sigma_S, \zeta) \equiv C^{BR}(S, \zeta)$ and $C^L(\Sigma_S, \zeta) \equiv C^{BL}(S, \zeta)$ $(\zeta \in \Gamma)$ are called the ***right-hand*** and the ***left-hand boundary cluster sets*** of S at ζ, respectively; the definitions are consistent with those of a single-valued map f from D into Ω on considering S_f: $D \to P(\Omega)$ (see [4], p. 81–82). The set $C^B(S, \zeta) \equiv C(\Sigma_S, \zeta)$ is called the ***boundary cluster set***, a naturally extended notion of that of single-valued maps (see [4], p. 81). It is not difficult to see that $C^B(S, \zeta) \subset C(S, \zeta)$ $(\zeta \in \Gamma)$, whence it follows that

$$C(\Sigma_S, \zeta) \cup \overline{\Sigma_S(\zeta)} = C(S, \zeta) \quad (\zeta \in \Gamma). \tag{1.6}$$

Applying Theorem 1.1 to our Σ_S, and considering (1.6) we obtain

COROLLARY 1.2. *Let S: $D \to P(\Omega)$, Ω being K_σ. Then*

$$C^{BL}(S, \zeta) = C^B(S, \zeta) = C^{BR}(S, \zeta)$$

and

$$C(S,\zeta) = C^B(S,\zeta)$$

for each $\zeta \in \Gamma$ except perhaps for a countable set on Γ.

For each $\zeta \in \Gamma$ we now consider a collection $\mathfrak{L}(\zeta)$ of subsets $G(\zeta) \subset D$ such that $\zeta \in \overline{G(\zeta)}$. Given $S\colon D \to P(\Omega)$, we set

$$\Pi_{\mathfrak{L}}(S,\zeta) = \bigcap_{G(\zeta)\in\mathfrak{L}(\zeta)} C_{G(\zeta)}(S,\zeta); \tag{1.7}$$

we call $\Pi_{\mathfrak{L}}(S,\zeta)$ the $\mathfrak{L}$-*principal cluster set* of S *at* ζ. Consider $\Sigma_1\colon \Gamma \to P(\Omega)$, defined by

$$\Sigma_1(\zeta) = \Pi_{\mathfrak{L}}(S,\zeta) \quad (\zeta \in \Gamma).$$

The boundary $\mathfrak{L}$-principal cluster set of S at $\zeta \in \Gamma$ is the set

$$B\Pi_{\mathfrak{L}}(S,\zeta) \equiv C(\Sigma_1,\zeta).$$

The terminology is consistent with that of Gresser [9], p. 324, in the case $S_f\colon D \to P(\Omega)$ induced by a map $f\colon D \to \Omega$. Applying (1.3) of Theorem 1.1 to our Σ_1, and noting that $\overline{\Sigma_1(\zeta)} = \Pi_{\mathfrak{L}}(S,\zeta)$, we have

COROLLARY 1.3. *Let $S\colon D \to P(\Omega)$, Ω being K_σ. Then*

$$\Pi_{\mathfrak{L}}(S,\zeta) \subset B\Pi_{\mathfrak{L}}(S,\zeta)$$

for each $\zeta \in \Gamma$ except perhaps for a countable set on Γ.

2. Maximality theorems of Collingwood-type. By an angular domain Δ at $\zeta \in \Gamma$ we mean a triangular domain whose vertices are ζ and two points of D. Our topics in this section are concerned with the *angular cluster set* $C_\Delta(S,\zeta)$ and the *full cluster set* $C(S,\zeta) \equiv C_D(S,\zeta)$ of $S\colon D \to P(\Omega)$. The *outer angular cluster set* of S at ζ is defined by

$$C^A(S,\zeta) = \bigcup_{\Delta} C_\Delta(S,\zeta), \tag{2.1}$$

where Δ ranges over all angular domains at ζ. Then we set

$$K(S) = \{\zeta \in \Gamma;\ \bigcap_{\Delta} C_\Delta(S,\zeta) = C^A(S,\zeta)\},$$
$$J(S) = \{\zeta \in K(S);\ C(S,\zeta) = C^A(S,\zeta)\}.$$

Thus, $\zeta \in K(S)$ if and only if $C_{\Delta_1}(S,\zeta) = C_{\Delta_2}(S,\zeta)$ for each pair Δ_1, Δ_2 of angular domains at ζ, while $\zeta \in J(S)$ if and only if $C_\Delta(S,\zeta) = C(S,\zeta)$ for each angular domain Δ at ζ. A set A on Γ is called *residual* on Γ if $\Gamma - A$ is of first Baire category on Γ, while A is called a.e. (almost everywhere) on Γ if $\Gamma - A$ is of linear Lebesgue measure zero. The intersection of a countable number of open sets on Γ is called G_δ on Γ; the meaning of F_σ, $G_{\delta\sigma}$, etc., on Γ, is thus obvious.

THEOREM 2.1. *Let $S: D \to P(\Omega)$, Ω being K_σ. Then $J(S)$ is G_δ and residual on Γ.*

Since $J(S) \subset K(S)$, we obtain

COROLLARY 2.2. *The set $K(S)$ is residual on Γ for S of Theorem 2.1.*

THEOREM 2.3. *Let $S: D \to P(\Omega)$, Ω being K_σ. Then $K(S)$ is $F_{\sigma\delta}$ and a.e. on Γ.*

For a subset Ω_1 of Ω and $\varepsilon > 0$, the ε-neighborhood of Ω_1 is the set $\mathfrak{N}(\Omega_1, \varepsilon) = \{\omega \in \Omega; d(\Omega_1, \omega) < \varepsilon\}$ with the convention that $\mathfrak{N}(\emptyset, \varepsilon) = \emptyset$ and $\mathfrak{N}(\{\omega\}, \varepsilon) \equiv \mathfrak{N}(\omega, \varepsilon)$, where $d(\cdot, \cdot)$ denotes the distance in Ω. Assume that Ω is K_σ, and let $G(\zeta)$ and $\Omega(\zeta)$ be the subsets of D and Ω, respectively, depending on $\zeta \in \Gamma$ with $\zeta \in \overline{G(\zeta)}$. For $S: D \to P(\Omega)$ we set

$$E = \{\zeta \in \Gamma; \Omega(\zeta) - C_{G(\zeta)}(S, \zeta) \neq \emptyset\}. \tag{2.2}$$

We shall decompose E in a countable sum:

$$E = \bigcup_{n,m,q,\eta} E_{nmq\eta} \tag{2.3}$$

in five steps, where n, m, q, η range over all natural numbers. If $E = \emptyset$, we let each $E_{nmq\eta} = \emptyset$. We consider the case $E \neq \emptyset$.

First step. Set

$$\Omega(\zeta, n) = \Omega(\zeta) - \mathfrak{N}(C_{G(\zeta)}(S, \zeta), 1/n) \qquad (\zeta \in \Gamma)$$

and

$$E_n = \{\zeta \in \Gamma;\ \Omega(\zeta, n) \neq \emptyset\}.$$

Then we have

$$E \subset \bigcup_n E_n.$$

In effect, if $\zeta \in E$ and $C_{G(\zeta)}(S, \zeta) \neq \emptyset$, we may find $\alpha \in \Omega(\zeta) - C_{G(\zeta)}(S, \zeta)$ and n such that $2/n < d(C_{G(\zeta)}(S, \zeta), \alpha)$.

Second step. Consider K_m^n of (1.4). Since $\Omega = \bigcup_m K_m^n$, we have

$$E_n \subset \bigcup_m E_{nm},$$

where

$$E_{nm} = \{\zeta \in \Gamma;\ K_m^n \cap \Omega(\zeta, n) \neq \emptyset\}.$$

Third step. First of all we show that $C_{G(\zeta)}(S, \zeta) \cap K_m^n = \emptyset$ for each $\zeta \in E_{nm}$. Remember that

$$C_{G(\zeta)}(S, \zeta) = \bigcap_q \overline{S(G_q(\zeta))},$$

where $G_q(\zeta) = G(\zeta) \cap \{z; |z - \zeta| < 1/q\}$ $(\zeta \in \Gamma)$. Assume that there exists $\beta \in C_{G(\zeta)}(S, \zeta) \cap K_m^n$ for $\zeta \in E_{nm}$. Choose $\alpha \in K_m^n \cap \Omega(\zeta, n)$. Then, $d(\beta, \alpha) < 1/n$, whence $\alpha \in \mathfrak{N}(C_{G(\zeta)}(S, \zeta), 1/n)$ because of $\beta \in C_{G(\zeta)}(S, \zeta)$. This contradicts $\alpha \in \Omega(\zeta, n)$. Now, since K_m^n is compact, and since $G_q(\zeta)$ decreases as q increases, we obtain q for $\zeta \in E_{nm}$ such that $\overline{S(G_q(\zeta))} \cap K_m^n = \emptyset$. Setting

$$E_{nmq} = \{\zeta \in \Gamma;\ \Omega(\zeta) \cap K_m^n \neq \emptyset \text{ and } \overline{S(G_q(\zeta))} \cap K_m^n = \emptyset\},$$

we have

$$E_{nm} \subset \bigcup_q E_{nmq}.$$

Fourth step. Setting

$$E_{nmq\eta} = \{\zeta \in \Gamma;\ \Omega(\zeta) \cap K_m^n \neq \emptyset \text{ and either } S(G_q(\zeta)) = \emptyset \text{ or } d(K_m^n, \overline{S(G_q(\zeta))}) \geqslant 1/\eta\},$$

we have

$$E_{nmq} \subset \bigcup_\eta E_{nmq\eta}. \tag{2.4}$$

Fifth step. By the inclusion formula (2.4) we have $\subset$ in (2.3); it is easy to see $\supset$ in (2.3).

Proof of Theorem 2.1. Let $\Delta(1)$ be an angular domain at 1, and let $\Delta(\zeta)$ be the domain at $\zeta \in \Gamma$ obtained by rotation of $\Delta(1)$ around the origin. We consider E of (2.2) by setting $\Omega(\zeta) = C(S, \zeta)$ and $G(\zeta) = \Delta(\zeta)$; we denote $E = E(\Delta(1))$ to indicate $\Delta(1)$. Then, by (2.3),

$$E(\Delta(1)) = \bigcup_{n,m,q,\eta} E_{nmq\eta}$$

with

$$E_{nmq\eta} = \{\zeta \in \Gamma;\ C(S, \zeta) \cap K_m^n \neq \emptyset \text{ and either } S(R_q(\zeta)) = \emptyset \text{ or } d(K_m^n, \overline{S(R_q(\zeta))}) \geqslant 1/\eta\},$$

where $R_q(\zeta) = \Delta(\zeta) \cap \{z;\ |z-\zeta| < 1/q\}$. To prove that $E(\Delta(1))$ is F_σ and of first category it suffices to show that each $E_{nmq\eta}$ is closed and of first category on Γ.

Closedness. Consider a sequence $\zeta_j \in E_{nmq\eta}$ with $\zeta_j \to \zeta$. Since $C(S, \zeta_j) \cap \cap K_m^n \neq \emptyset$ for each j, we may find $\alpha_j \in K_m^n$, $z_j \in D$ and $\beta_j \in S(z_j)$ such that $d(\alpha_j, \beta_j) \to 0$ and $|z_j - \zeta_j| \to 0$ as $j \to \infty$. Since K_m^n is compact, we have a subsequence $\{z_j'\}$ of $\{z_j\}$ and a point $\alpha \in K_m^n$ such that $d(\alpha, \beta_j') \to 0$ and $|z_j' - \zeta| \to 0$, where $\beta_j' \in S(z_j')$ $(j \geqslant 1)$, whence $C(S, \zeta) \cap K_m^n \neq \emptyset$. On the other hand, each point of $R_q(\zeta)$ is contained in a certain $R_q(\zeta_j)$ because $\zeta_j \to \zeta$. Since $\zeta_j \in E_{nmq\eta}$ $(j \geqslant 1)$ it follows that either $S(R_q(\zeta)) = \emptyset$ or $d(K_m^n, \overline{S(R_q(\zeta))}) \geqslant 1/\eta$. We have thus proved $\zeta \in E_{nmq\eta}$, which shows that $E_{nmq\eta}$ is closed.

Category. Assume that a certain $E_{nmq\eta}$ is of second category on Γ. We then find an open arc A on Γ, where the closed set $E_{nmq\eta}$ is dense, or $A \subset E_{nmq\eta}$. Consider the open set

$$\mathfrak{M} = \bigcup_{\zeta \in A} R_q(\zeta).$$

Then each $z \in \mathfrak{M}$ belongs to some $R_q(\zeta)$, $\zeta \in E_{nmq\eta}$, whence, either $S(z)=\emptyset$ or else $S(z) \neq \emptyset$ with

$$d\left(K_m^n, \overline{S(z)}\right) \geqslant d\left(K_m^n, \overline{S\left(R_q(\zeta)\right)}\right) \geqslant 1/\eta.$$

On the other hand, $\mathfrak{M}$ contains a neighborhood $V \cap D$ of $\zeta_0 \in \Delta$, where V is an open disk containing ζ_0. Therefore

$$C(S, \zeta_0) \cap K_m^n = \emptyset;$$

a contradiction to $\zeta_0 \in E_{nmq\eta}$. Now that $E\left(\Delta(1)\right)$ is known to be F_σ and of first category, we prepare a countable number of angular domains $\Delta_j(1)^*$ $(j \geqslant 1)$ at 1 such that each angular domain at 1 contains one of $\Delta_j(1)$, and is contained in one of $\Delta_j(1)$. In effect, we choose all the angular domains at 1 whose other vertices are all rational points of D, that is, $z \in D$ with both $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ rational. Let

$$\Delta_j(\zeta) \quad (j \geqslant 1, \zeta \in \Gamma) \tag{2.5}$$

be the angular domain at $\zeta \in \Gamma$ obtained by rotation of $\Delta_j(1)$ around the origin. Our Theorem 2.1 now follows from the decomposition:

$$\Gamma - J(S) = \bigcup_{j=1}^{\infty} E\left(\Delta_j(1)\right).$$

Actually, to prove $\subset$, we find $\Delta(\zeta)$ at $\zeta \in \Gamma - J(S)$ such that $C_{\Delta(\zeta)}(S, \zeta) \subsetneqq C(S, \zeta)$. We have only to choose $\Delta_j(\zeta) \subset \Delta(\zeta)$.

Proof of Theorem 2.3. Let $\{\Delta_j(\zeta)\}$ be the sequence of domains of (2.5). We choose an arbitrary pair $\Delta_j(\zeta)$, $\Delta_p(\zeta)$, $j \neq p$, and we consider E of (2.2) by setting $\Omega(\zeta) = C_{\Delta_j(\zeta)}(S, \zeta)$ and $G(\zeta) = \Delta_p(\zeta)$; we denote $E = E(j, p)$ to indicate j and p. We now show that

$$\Gamma - K(S) = \bigcup_{j \neq p} E(j, p). \tag{2.6}$$

For, we may find a pair $\Delta(\zeta)$ and $\Delta'(\zeta)$ at $\zeta \in \Gamma - K(S)$ such that $C_{\Delta(\zeta)}(S, \zeta) - C_{\Delta'(\zeta)}(S, \zeta) \neq \emptyset$. We then choose j and p such that $\Delta_j(\zeta) \supset \Delta(\zeta)$ and $\Delta_p(\zeta) \subset \Delta'(\zeta)$. Then, $\zeta \in E(j, p)$. Now, from (2.3), applied to $E(j, p)$, it follows that

$$E(j, p) = \bigcup_{n, m, q, \eta} E_{nmq\eta}, \tag{2.7}$$

with

$$E_{nmq\eta} = \Big\{\zeta \in \Gamma;\, C_{\Delta_j(\zeta)}(S, \zeta) \cap K_m^n \neq \emptyset \text{ and either } S\left(R_{pq}(\zeta)\right) = \emptyset \text{ or } d\left(K_m^n, \overline{S\left(R_{pq}(\zeta)\right)}\right) \geqslant 1/\eta\Big\},$$

where we may suppose, without loss of generality, that

$$R_{pq}(\zeta) = \Delta_p(\zeta) \cap \{z;\, 1 - q^{-1} < |z| < 1\}.$$

We then set

$$Q_{js}(\zeta) = \Delta_j(\zeta)\cap\{z; 1-s^{-1} < |z| \leqslant 1-(s+1)^{-1}\}$$

for $s = 1, 2, \dots$ and $\zeta \in \Gamma$, and we set

$$F_{nmvs} = \{\zeta \in \Gamma; S(Q_{js}(\zeta))\cap\mathfrak{N}(K_m^n, 1/\nu) \neq \emptyset\}$$

for natural numbers n, m, ν and s. We now set

$$F_{nm\nu} = \overline{\lim_{s\to\infty}} F_{nm\nu s} = \bigcap_{l=1}^{\infty}\bigcup_{s=l}^{\infty} F_{nm\nu s},$$

and

$$E_{nmq\eta}^{*} = \{\zeta \in \Gamma; \text{either } S(R_{pq}(\zeta)) \text{ is empty or } d(K_m^n, \overline{S(R_{pq}(\zeta))}) \geqslant 1/\eta\}.$$

We show

$$E_{nmq\eta} = \bigcap_{\nu=1}^{\infty} (E_{nmq\eta}^{*}\cap F_{nm\nu}). \tag{2.8}$$

The inclusion $\subset$ is easy to prove. To see $\supset$, we let ζ be a point of the right-hand side set of (2.8). Since $\zeta \in F_{nm\nu}$ for each ν it follows that $C_{\Delta_j(\zeta)}(S, \zeta)\cap \cap K_m^n \neq \emptyset$. Combined with $\zeta \in E_{nmq\eta}^{*}$, this implies $\zeta \in E_{nmq\eta}$. We shall show that each $E_{nmq\eta}^{*}$ is closed (hence G_δ) and each $F_{nm\nu s}$ is open. Then each $E_{nmq\eta}$ is G_δ by (2.8), whence each $E(j, p)$ is $G_{\delta\sigma}$ by (2.7). By (2.6) we know that $K(S)$ is $F_{\sigma\delta}$. Now, it is easy to see that $E_{nmq\eta}^{*}$ is closed. To prove the openness of $F_{nm\nu s}$, let $\zeta \in F_{nm\nu s}$, Let $z_0 \in Q_{js}(\zeta)$ such that $S(z_0)\cap \cap\mathfrak{N}(K_m^n, 1/\nu) \neq \emptyset$. Then, ζ is contained in the open set

$$V(\zeta) = \{\zeta' \in \Gamma; Q_{js}(\zeta') \text{ contains } z_0\},$$

and $V(\zeta)$ is contained in $F_{nm\nu s}$. The rest we have to prove is that each Borel set $E_{nmq\eta}$ is of (outer) measure zero, which, combined with (2.6) and (2.7), implies that $K(S)$ is a.e. on Γ. Since by (2.8),

$$E_{nmq\eta} \subset A \equiv E_{nmq\eta}^{*}\cap F_{nm(\eta+1)},$$

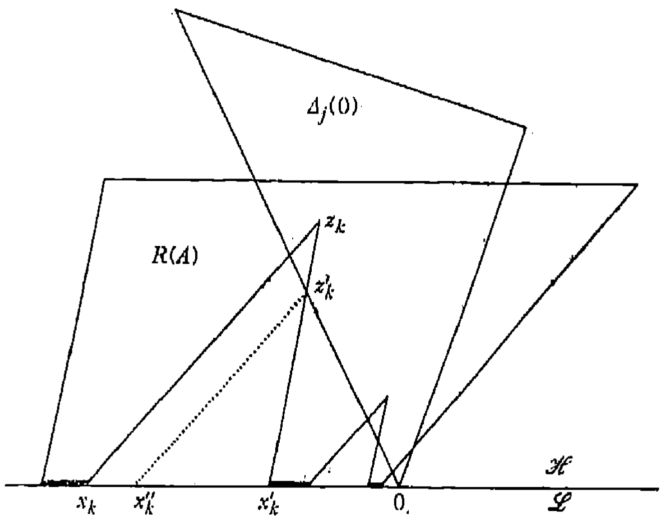
it suffices to show that A is of outer measure zero. Assume that a certain A is of positive outer measure. By the well-known density theorem (see [16], p. 129, Density theorem) almost all points of A are points of outer density for A. Let ζ_0 be a point of outer density for A. Then

$$[\text{measure of } (A\cap I_k(\zeta_0))]/(2\varepsilon_k) \to 1$$

for each sequence $\varepsilon_k > 0$, $\varepsilon_k \searrow 0$, where $I_k(\zeta_0)$ is the open arc on Γ of center ζ_0 and of arc length $2\varepsilon_k$. Then a terminal part of $\Delta_j(\zeta_0)$ is covered by

$$R(A) = \bigcup_{\zeta\in A} R_{pq}(\zeta) \tag{2.9}$$

in the sense that there is no sequence $z_k \in \Delta_j(\zeta_0)$, $z_k \to \zeta_0$, with $z_k \notin R(A)$ $(k \geqslant 1)$. Assume otherwise. To avoid the complexity, we consider, without loss of generality, the angular domains in the upper half-plane $\mathscr{H}$ with vertices on the real axis $\mathscr{L}$ and $\zeta_0 = 0$ (see the figure); in effect, we may assume $A \neq \Gamma$, and we have only to consider a conformal homeomor-



phism from D onto $\mathscr{H}$, sending a point of $\Gamma - A$ to ∞. Without saying the details we may consider the case that there exists a sequence $\{z_k\} \subset \Delta_j(0)$, $z_k \to 0$, $\mathrm{Re}\, z_k < 0$ $(k \geqslant 1)$. The line segments $x_k z_k$ and $x'_k z_k$ are parallel to the sides of $\Delta_p(0)$, respectively. We may assume that the triangular domain with vertices z_k, x_k and x'_k does not intersect $R(A)$ for each k. Suppose that $z'_k x''_k$ is parallel to $z_k x_k$. Then all triangular domains bounded by triangles $z'_k x''_k x'_k$ $(k \geqslant 1)$ are similar to one another. Therefore there exists a constant c, $0 < c < 1$, such that $x'_k = c x''_k$ for each k. Since the open interval (x''_k, x'_k) on $\mathscr{L}$ contains no point of A we have a contradiction on putting $\varepsilon_k = -x''_k$ $(k \geqslant 1)$. Now that we know $R(A)$ of (2.9) covers a terminal part of $\Delta_j(\zeta_0)$, we may find t_0 such that $Q_{js}(\zeta_0) \subset R(A)$ for each $s > t_0$. Since $A \subset E^*_{nmq\eta}$ we know that $S(z) = \emptyset$ or $d\left(K^n_m, \overline{S(z)}\right) \geqslant 1/\eta$ for each $z \in Q_{js}(\zeta_0)$. On the other hand, there is a point

$$z_0 \in \bigcup_{s > t_0} Q_{js}(\zeta_0)$$

with

$$S(z_0) \cap \mathfrak{N}\left(K^n_m, 1/(\eta+1)\right) \neq \emptyset$$

because of $\zeta_0 \in F_{nm(\eta+1)}$. This contradicts $d\left(K^n_m, \overline{S(z_0)}\right) \geqslant 1/\eta > 1/(\eta+1)$. The proof is now complete.

Remark 2.4. In the standard proof (see [11], p. 1061) of a special case of Theorem 2.3, without referring to $F_{\sigma\delta}$ property (this makes the proof shorter than ours), one uses the familiar method in the proof of the Plessner theorem, that is, using of the conformal map from D onto a Jordan subdomain of D with the rectifiable boundary. However, it is well known that the Riemann mapping theorem is not true in the Euclidean space R^n $(n \geqslant 3)$ except for the special cases. Our present proofs of Theorem 2.1 and Theorem 2.3 are available to the proofs of the versions in the open unit ball in R^n. Angular domains are interpreted as cones in the space R^n $(n \geqslant 3)$. We note that there is a conformal map from the ball onto the half-space. It is not difficult to see that $\Gamma - K(S)$ for S in Theorem 2.3 is σ-porous in the sense of E. P. Dolženko [5], p. 5. However, it appears to be open to find a subset e of Γ such that e is of first category, of measure zero, not σ-porous on Γ and that there is no map f (or set-mapping S) from D into Ω satisfying $K(S_f) = \Gamma - e$ (or $K(S) = \Gamma - e$). If there exists such an e, Dolženko's result [5], Theorem 1, extends strictly Lappan's [11], Theorem 1.

3. Continuous set-mappings. Let $\mathscr{M}$ be a subset of D, and $z_0 \in \mathscr{M}$. We say that a set-mapping $S\colon D \to P(\Omega)$ is *continuous at* z_0 *relative to* $\mathscr{M}$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $S(z_0) \subset \mathfrak{N}(S(z), \varepsilon)$ for each $z \in \mathscr{M}$ with $|z - z_0| < \delta$. We say that S is *continuous on* $\mathscr{M}$ if S is continuous at each point of $\mathscr{M}$ relative to $\mathscr{M}$. For the set-mapping S_f induced by a map $f\colon D \to \Omega$, the continuity of f means that of S_f. However, there is a pathological situation in the general case. Let $\Omega = D$, and let $S\colon D \to P(D)$ be defined by $S(0) = \emptyset$ and $S(z) = D$ $(z \neq 0)$. Then each $\mathfrak{N}(S(0), \varepsilon) = \emptyset$ does not contain $S(z)$ for $z \neq 0$, while S is continuous at 0 in our sense.

THEOREM 3.1. *Let* $S\colon D \to P(\Omega)$, Ω *being* K_σ, *and assume that there exists* r_0, $0 < r_0 < 1$, *such that* S *is continuous on each circle* $|z| = r$, $r_0 < r < 1$. *Let* $\mathfrak{K}(1)$ *be a non-degenerate continuum in* $\overline{D}$ *with* $\mathfrak{K}(1) \cap \Gamma = 1$, *and let* $\mathfrak{K}(\zeta)$ *be the set obtained by rotation of* $\mathfrak{K}(1)$ *by* $\arg\zeta$ *around* 0. *Then the set*

$$\{\zeta \in \Gamma;\ C(S, \zeta) = C_{\mathfrak{K}(\zeta)}(S, \zeta)\} \tag{3.1}$$

is G_δ *and residual on* Γ.

Proof. We consider E of (2.2) on setting $\Omega(\zeta) = C(S, \zeta)$ and $G(\zeta) = \mathfrak{K}(\zeta)$; we denote the resulting set by $E(\mathfrak{K}(1))$. By (2.3) we have

$$E(\mathfrak{K}(1)) = \bigcup_{n,m,q,\eta} E_{nmq\eta}, \tag{3.2}$$

where

$$E_{nmq\eta} = \{\zeta \in \Gamma;\ C(S, \zeta) \cap K_m^n \neq \emptyset \text{ and either } S(\mathfrak{K}_q(\zeta)) = \emptyset$$
$$\text{or } d(K_m^n, \overline{S(\mathfrak{K}_q(\zeta))}) \geqslant 1/\eta\}$$

with $\mathfrak{K}_q(\zeta) = \mathfrak{K}(\zeta) \cap \{z; |z-\zeta| < 1/q\}$. Since $E_{nmq_1\eta} \subset E_{nmq_2\eta}$ for $q_1 < q_2$, we may restrict q so that $r_0 < 1-q^{-1}$. We shall prove that each $E_{nmq\eta}$ is closed and of first category on Γ, whence by (3.2), the set of (3.1), being equal to $\Gamma - E(\mathfrak{K}(1))$, is G_δ and residual on Γ.

Closedness. Let $\zeta_j \in E_{nmq\eta}$ with $\zeta_j \to \zeta$. It is easy to see that $C(S, \zeta) \cap \cap K_m^n \neq \emptyset$. Assume that $S(\mathfrak{K}_q(\zeta)) \neq \emptyset$ with $d\left(K_m^n, \overline{S(\mathfrak{K}_q(\zeta))}\right) < 1/\eta$. Then there exist $z_0 \in \mathfrak{K}_q(\zeta)$ and $a \in S(z_0)$ with $d(K_m^n, a) < 1/\eta$. By the continuity of S on the circle $|z| = |z_0|$ we have $\zeta_j, z_j \in \mathfrak{K}_q(\zeta_j)$, $|z_j| = |z_0|$, and $a_j \in S(z_j)$ such that $d(K_m^n, a_j) < 1/\eta$. This contradicts $\zeta_j \in E_{nmq\eta}$.

Category. The proof follows on considering

$$\mathfrak{M} = \bigcup_{\zeta \in \Delta} \mathfrak{K}_q(\zeta)$$

as in the proof of Theorem 2.1.

We next investigate local properties of S at points of $J(S)$ and $K(S)$ under a certain continuity condition on $S\colon D \to P(\Omega)$. Let $X(\zeta, \varphi)$ be a line segment at $\zeta \in \Gamma$ making the directed angle φ, $|\varphi| < \pi/2$, with the radius $\varrho(\zeta)$ of D at ζ. One terminal point of $X(\zeta, \varphi)$ is ζ and the other is a point of D. By the directed angle we mean the angle at ζ made by $X(\zeta, \varphi)$ and $\varrho(\zeta)$ such that $\varphi \geqslant 0$ if $X(\zeta, \varphi)$ lies to the right of $\varrho(\zeta)$ viewed from an observer at the origin.

THEOREM 3.2. *Let $S\colon D \to P(\Omega)$, Ω being K_σ, and let $\zeta \in K(S)$. Assume that there exists r_0, $0 < r_0 < 1$, such that S is continuous on each circular arc $\{z \in D;\ |z-\zeta| = r\}$, $0 < r < r_0$. Then the set*

$$\{\varphi;\ |\varphi| < \pi/2,\ C_{X(\zeta,\varphi)}(S, \zeta) = C^\Delta(S, \zeta)\} \tag{3.3}$$

is G_δ and residual on $(-\pi/2, \pi/2)$.

COROLLARY 3.3. *Furthermore, if $\zeta \in J(S)$, then the set*

$$\{\varphi;\ |\varphi| < \pi/2,\ C_{X(\zeta,\varphi)}(S, \zeta) = C(S, \zeta)\} \tag{3.4}$$

is G_δ and residual on $(-\pi/2, \pi/2)$.

In effect, we obtain (3.4), because $C^\Delta(S, \zeta) = C(S, \zeta)$ at $\zeta \in J(S)$. As is easily seen in the following proof, the continuity described in Theorem 3.2 may be exchanged by that of S on $\{z;\ |z| = r,\ |z - \varrho\zeta| < 1-\varrho\}$, where ϱ, $1/2 < \varrho < 1$, is a number depending on $\zeta \in K(S)$ and r ranges as $2\varrho - 1 < r < 1$.

Proof of Theorem 3.2. We may assume that $\zeta = 1 \in K(S)$ and that $X(\varphi) \equiv X(1, \varphi)$ is the chord of a fixed circle touching internally to Γ at 1. The proof is similar to that of Theorem 2.1 with a minor change. We remark that $C^\Delta(S, 1) \supset C_{X(\varphi)}(S, 1)$ for each $\varphi \in (-\pi/2, \pi/2)$. Set

$$E = \{\varphi;\ |\varphi| < \pi/2,\ C^\Delta(S, 1) - C_{X(\varphi)}(S, 1) \neq \emptyset\}.$$

We obtain a decomposition

$$E = \bigcup_{n,m,q,\eta} E_{nmq\eta},$$

being similar to (2.3), where

$$E_{nmq\eta} = \{\varphi;\ |\varphi| < \pi/2,\ C^{\Delta}(S, 1) \cap K^n_m \neq \emptyset \text{ and either } S(X_q(\varphi)) = \emptyset \text{ or } d(K^n_m, \overline{S(X_q(\varphi))}) \geqslant 1/\eta\}$$

with $X_q(\varphi) = X(\varphi) \cap \{z;\ |z-1| < 1/q\}$. It is now easy to see that each $E_{nmq\eta}$ is closed and of first category on $(-\pi/2, \pi/2)$. We must note that $C^{\Delta}(S, 1)$ is closed because $C^{\Delta}(S, 1) = C_{\Delta}(S, 1)$ for each angular domain Δ at $1 \in K(S)$.

4. Boundary normality condition. As is well known, Plessner's theorem (see [4], p. 147, Theorem 8.2) depends on the "analyticity" of a function meromorphic in D in the sense that the extension to quasi-conformal functions is false; in effect, the Fatou theorem is false for a bounded quasi-conformal function in D (see [22]). On the other hand, Meier's (topological analogue of Plessner's) theorem is a consequence of a topological property of meromorphic functions in the sense that the result may be extended to quasi-conformal functions [21]. The decisive property which we shall extract from the above-mentioned will yield some results on $S\colon D \to P(\Omega)$, one of which contains Meier's theorem (see Corollary 4.4).

The disk D has the non-Euclidean distance

$$\gamma_D(z_1, z_2) = \frac{1}{2}\log\frac{|1-\bar{z}_1 z_2| + |z_1 - z_2|}{|1-\bar{z}_1 z_2| - |z_1 - z_2|} \qquad (z_1, z_2 \in D). \tag{4.1}$$

A simply connected domain in the plane, conformally homeomorphic to D, is called a *hyperbolic domain*; a hyperbolic domain $\mathscr{D}$ has the non-Euclidean distance $\gamma_{\mathscr{D}}(\cdot, \cdot)$ obtained by the obvious manner. A set-mapping $S\colon \mathscr{D} \to P(\Omega)$ is called *normal* in $\mathscr{D}$ if, for each $\varepsilon > 0$ we have $\delta > 0$ such that $\mathfrak{N}(S(z_1), \varepsilon) \supset S(z_2)$ for each pair $z_1, z_2 \in \mathscr{D}$ with $\gamma_{\mathscr{D}}(z_1, z_2) < \delta$. Since z_1 and z_2 may be exchanged, we have $\mathfrak{N}(S(z_2), \varepsilon) \supset S(z_1)$ also. The boundary normality condition for $S\colon D \to P(\Omega)$ reads as follows:

For each triangular domain Δ in D such that $\bar{\Delta} \cap \Gamma$ is one point and that $\overline{S(\Delta)} \neq \Omega$, the restriction of S to Δ, that is, $S\colon \Delta \to P(\Omega)$, is normal in Δ.

If S_f is the set-mapping from D into the Riemann sphere $\mathfrak{R}$, induced by a meromorphic function f in D, then S_f satisfies the boundary normality condition; the result follows from the theorem of O. Lehto and K. I. Virtanen [12], p. 47, that a meromorphic function in a hyperbolic $\mathscr{D}$ omitting three points of $\mathfrak{R}$ is normal in $\mathscr{D}$. It is easy to see that a meromorphic function g in $\mathscr{D}$ is normal in $\mathscr{D}$ in the sense of Lehto and Virtanen if and only if g is uniformly continuous with respect to $\gamma_{\mathscr{D}}$ in $\mathscr{D}$; to prove the "if" part we note that $\mathfrak{R}$ is compact and the family $\{g \circ T\}$, T ranging over all conformal self-homeomorphisms of $\mathscr{D}$, is equi-continuous in the Euclidean distance in $\mathscr{D}$. In the forthcoming Remark 4.8 we shall show the existence of a multiple-valued function in D which induces the set-mapping satisfying the boundary normality condition.

Consider $S\colon D \to P(\Omega)$. A point $\zeta \in \Gamma$ is called a *Plessner point of* S if $\zeta \in K(S)$ and $C^{\varDelta}(S, \zeta) = \Omega$; the set of all Plessner points of S is denoted by $I(S)$. It is easy to show $I(S) \subset J(S)$. By an admissible arc Λ at $\zeta \in \Gamma$ we mean a continuous curve $\Lambda\colon z = z(t) \in D$ $(0 \leqslant t < 1)$, $\lim_{t \to 1} z(t) = \zeta$, tangent at ζ to a chord of Γ at ζ. We denote by $\Pi_T(S, \zeta)$ the $\mathfrak{L}$-principal cluster set of S at ζ (see (1.7)), where $\mathfrak{L}(\zeta)$ is the collection of all admissible arcs Λ at ζ, namely, $\Pi_T(S, \zeta) = \bigcap_{\Lambda} C_{\Lambda}(S, \zeta)$. We have $\Pi_T(S, \zeta) \subset C^{\varDelta}(S, \zeta)$ at each $\zeta \in \Gamma$. A point $\zeta \in \Gamma$ is called a *Lindelöf point of* S if $\Pi_T(S, \zeta) = C^{\varDelta}(S, \zeta)$; the set of all Lindelöf points of S is denoted by $L(S)$. Since each angular domain at $\zeta \in \Gamma$ contains a terminal part of a chord at ζ, we can easily show $L(S) \subset K(S)$. We denote by $\Pi_{\mathfrak{X}}(S, \zeta)$ the $\mathfrak{L}$-principal cluster set of S at ζ, where $\mathfrak{L}(\zeta)$ is the collection of all chords of Γ at ζ; plainly, $\Pi_{\mathfrak{X}}(S, \zeta) \supset \Pi_T(S, \zeta)$. A point $\zeta \in \Gamma$ is called a *Meier point of* S if $\Pi_{\mathfrak{X}}(S, \zeta) = C(S, \zeta) \neq \Omega$; the set of all Meier points of S is denoted by $M(S)$. It is easy to see $M(S) \subset J(S)$. A point $\zeta \in \Gamma$ is called a *normal point of* S if S is normal in each angular domain at ζ; the set of all normal points of S is denoted by $N(S)$.

THEOREM 4.1. *Let $S\colon D \to P(\Omega)$, and assume that S satisfies the boundary normality condition. Then*

$$L(S) \cup \{I(S) - N(S)\} = K(S). \tag{4.2}$$

From Theorem 4.1, Corollary 2.2 and Theorem 2.3 follows

COROLLARY 4.2. *Furthermore, if Ω is K_{σ}, then $L(S) \cup \{I(S) - N(S)\}$ is $F_{\sigma\delta}$, residual and a.e. on Γ.*

THEOREM 4.3. *Let $S\colon D \to P(\Omega)$, and assume that S satisfies the boundary normality condition. Then*

$$M(S) \cup I(S) = J(S).$$

From Theorem 4.3 and Theorem 2.1 follows

COROLLARY 4.4. *Furthermore, if Ω is K_{σ}, then $M(S) \cup I(S)$ is G_{δ} and residual on Γ.*

We begin with a comparison of non-Euclidean distances. We denote $d\gamma_D(z) = |dz|/(1 - |z|^2)$, $z \in D$; thus, it is obvious what is meant by $d\gamma_{\mathscr{D}}(z)$, $z \in \mathscr{D}$, $\mathscr{D}$ being a hyperbolic domain. Further, given $k > 0$ we set

$$\mathscr{D}(k) = \{z \in \mathscr{D};\, 1 - |z|^2 \leqslant k \ \mathrm{dis}(z, \partial\mathscr{D})\},$$

where $\mathrm{dis}(\cdot,\cdot)$ denotes the Euclidean distance and $\partial\mathscr{D}$ is the boundary of $\mathscr{D}$ in the plane. For each pair of points $z_1, z_2 \in \mathscr{D}$, the geodesic line segment in the sense of $\gamma_{\mathscr{D}}(\cdot,\cdot)$ connecting z_1 and z_2 is called the *$\mathscr{D}$-geodesic* from z_1 to z_2.

LEMMA 4.5. *Let $\mathscr{D}$ be a hyperbolic domain contained in D. Then for each pair $z_1, z_2 \in \mathscr{D}$, we have*

$$\gamma_D(z_1, z_2) \leqslant \gamma_{\mathscr{D}}(z_1, z_2). \tag{4.3}$$

Further, given $k > 0$, for each pair $z_1, z_2 \in \mathscr{D}(k)$ such that the D-geodesic from z_1 to z_2 is contained in $\mathscr{D}(k)$, we have

$$\gamma_{\mathscr{D}}(z_1, z_2) \leqslant k\gamma_D(z_1, z_2). \tag{4.4}$$

Proof. The first half is a consequence of the principle of hyperbolic metrics: $d\gamma_D(z) \leqslant d\gamma_{\mathscr{D}}(z)$ $(z \in \mathscr{D})$ (see [8], p. 326–327). Integrating both sides of the inequality along the $\mathscr{D}$-geodesic from z_1 to z_2 we have (4.3). To obtain (4.4) it suffices to show

$$d\gamma_{\mathscr{D}}(z) \leqslant k d\gamma_D(z) \tag{4.5}$$

for each $z \in \mathscr{D}(k)$. Integrating both sides of (4.5) along the D-geodesic from z_1 to z_2 we obtain (4.4). To prove (4.5) we apply the result due to W. Seidel and J. L. Walsh [17], p. 133, Theorem 2, (4.4), to a conformal homeomorphism $z = f(w)$ from D onto $\mathscr{D}$, and we obtain

$$\mathrm{dis}(z, \partial\mathscr{D}) \leqslant |f'(w)|(1-|w|^2) \qquad (z \in \mathscr{D}).$$

Then, letting $w = g(z)$ be the inverse of f, we have for $z \in \mathscr{D}(k)$,

$$d\gamma_{\mathscr{D}}(z) = \frac{|g'(z)|\,|dz|}{1-|g(z)|^2} \leqslant \frac{|dz|}{\mathrm{dis}(z, \partial\mathscr{D})} \leqslant k d\gamma_D(z).$$

Remark 4.6. Assume that $S\colon D \to P(\Omega)$ is normal in D. Then $N(S) = \Gamma$, the result being a consequence of (4.3) applied to each angular domain $\mathscr{D} = \Delta$. Therefore, it follows from (4.2) that $L(S) = K(S)$. For an application of this equality for normal S, we refer to [24].

Proof of Theorem 4.1. Since $L(S) \subset K(S)$ and since $I(S) - N(S) \subset K(S)$, we obtain the inclusion $\subset$ in (4.2). To prove the inverse $\supset$ we first assert

$$K(S) - L(S) \subset \Gamma - N(S). \tag{4.6}$$

Actually we have an admissible arc Λ at $\zeta \in K(S) - L(S)$ such that $C^{\Lambda}(S, \zeta) - C_{\Lambda}(S, \zeta)$ contains a point α. Let X ($\equiv X(\zeta, \varphi)$) be a line segment at ζ tangent at ζ to Λ, and let Δ be an angular domain at ζ such that $\Delta \cap X \neq \emptyset$. We then choose an angular domain Δ' at ζ and a constant $k > 0$ such that $\Delta' \cap X \neq \emptyset$ and $\Delta' \subset \Delta(k)$, where $\Delta(k) = \mathscr{D}(k)$ with $\mathscr{D} = \Delta$.

It follows form the well-known geometrical property (see [18], p. 511, Theorem XI.2) that there exists a natural number n_0 such that a terminal part of $(1/n_0)$-neighborhood of X in $\gamma_D(\cdot,\cdot)$ is contained in Δ', and further that, for each $n \geqslant n_0$, the $(1/n)$-neighborhood of X in $\gamma_D(\cdot,\cdot)$ contains an angular domain Δ_n at ζ satisfying $\Delta_n \cap X \neq \emptyset$ and $\Delta_n \subset \Delta' \cap \{z; |z-\zeta| < 1/n\}$. Since $\zeta \in K(S)$, we know that $\alpha \in C^{\Delta}(S,\zeta) = C_{\Delta_n}(S,\zeta)$ $(n \geqslant n_0)$. Therefore there exist $z_n \in \Delta_n$, $\alpha_n \in S(z_n)$, such that $d(\alpha_n, \alpha) \to 0$. Then we may find $z'_n \in \Lambda \cap \Delta_n$ such that $\gamma_D(z_n, z'_n) < 1/n$ and the D-geodesic from z_n to z'_n is contained in Δ' $(n \geqslant n_0)$. By (4.4) applied to $\mathscr{D} = \Delta$ with $\Delta' \subset \Delta(k)$ we obtain

$$\gamma_{\Delta}(z_n, z'_n) \leqslant k\gamma_D(z_n, z'_n) < k/n \qquad (n \geqslant n_0). \tag{4.7}$$

Assume that $\zeta \in N(S)$. Since S is normal in Δ, it follows from (4.7) that we have a subsequence $\{z'_{n_j}\}$ of $\{z'_n\}$ and points $\alpha'_{n_j} \in S(z'_{n_j})$ such that $d(\alpha'_{n_j}, \alpha) \to 0$ as $n_j \to \infty$. We thus have a contradiction that $\alpha \in C_{\Lambda}(S,\zeta)$. This completes the proof of (4.6). We next prove

$$K(S) - L(S) \subset I(S), \tag{4.8}$$

which, combined with (4.6), implies $K(S)-L(S) \subset I(S)-N(S)$, whence $L(S) \cup \{I(S)-N(S)\} \supset K(S)$ as desired. Assume that there is a point $\zeta \in K(S)-L(S)-I(S)$. Since $\zeta \notin L(S)$ we have an admissible arc Λ at ζ, with the line segment X at ζ as a tangent at ζ, such that $C^{\Delta}(S,\zeta) - C_{\Lambda}(S,\zeta)$ contains a point α. Since $\zeta \notin I(S)$ and since $\zeta \in K(S)$, $C_{\Delta''}(S,\zeta) \neq \Omega$ for each angular domain Δ'' at ζ with $\Delta'' \cap X \neq \emptyset$. Hence we may find an angular domain Δ at ζ with $\Delta \cap X \neq \emptyset$ and $\overline{S(\Delta)} \neq \Omega$. Consequently, S is normal in Δ. The rest of the proof is now the same as that of (4.6).

Proof of Theorem 4.3. Since $M(S) \subset J(S)$ and $I(S) \subset J(S)$ it follows that $M(S) \cup I(S) \subset J(S)$. The proof of the inverse inclusion is the same as that of Theorem 4.1 with a slight change. Assume that there is a point $\zeta \in J(S) - \{M(S) \cup I(S)\}$. It follows from $\zeta \in J(S)-I(S)$ that $C(S,\zeta) \neq \Omega$. Since $\zeta \notin M(S)$ we may find a segment X at ζ such that $C(S,\zeta) - C_X(S,\zeta)$ contains a point α. Since $\zeta \in J(S)-I(S)$ we may find an angular domain Δ at ζ such that $\Delta \cap X \neq \emptyset$ and that $\overline{S(\Delta)} \neq \Omega$, whence S is normal in Δ. By the same reasoning as in the proof of Theorem 4.1, we have a sequence $\{z'_{n_j}\} \subset X$, $z'_{n_j} \to \zeta$ and $\alpha'_{n_j} \in S(z'_{n_j})$ with $d(\alpha'_{n_j}, \alpha) \to 0$ as $n_j \to \infty$. We thus have a contradiction that $\alpha \in C_X(S,\zeta)$.

We restrict our topics to a function f meromorphic in D. For simplicity we denote $I(f) = I(S_f)$, $N(f) = N(S_f)$, etc., where S_f: $D \to P(\mathfrak{R})$ induced by f, $\mathfrak{R}$ being the Riemann sphere endowed with the chordal distance $\chi(\cdot,\cdot)$. Anderson's definition [1] of normal points of f reads as follows. A point $\zeta \in \Gamma$ is called a *normal point of* f provided that for each angular domain Δ at ζ and for each sequence $z_n \in \Delta$ converging to ζ, there exists r_0, $0 < r_0 < 1$, depending on $\{z_n\}$ such that the family of functions

$$f\left(\frac{z-z_n}{1-\bar{z}_n z}\right) \qquad (n \geqslant 1)$$

is normal in $|z| < r_0$ in the sense of P. Montel, the convergence being taken in $\chi(\cdot, \cdot)$. We denote by $N_\Delta(f)$ the set of all normal points of f in Anderson's sense. Anderson and Noshiro proved

$$K(f) \subset L(f) \cup \{I(f) - N_\Delta(f)\}. \tag{4.9}$$

There is another notion of normal points introduced by S. Dragosh [7], p. 60. According to him, a point $\zeta \in \Gamma$ is called a normal point of f if, for each angular domain Δ at ζ, and for each sequences $\{z_n\}$ and $\{z_n'\}$ in Δ with $\gamma_D(z_n, z_n') \to 0$, $\lim f(z_n) = \alpha$ implies $\lim f(z_n') = \alpha$. The set of all normal points of f in the sense of Dragosh is denoted by $N_D(f)$. We show

THEOREM 4.7. *For f meromorphic in D, we obtain*

$$N_\Delta(f) = N_D(f) = N(f). \tag{4.10}$$

Proof. (Added in proof. I found that the equality $N_\Delta(f) = N_D(f)$ in (4.10) had been pointed out by H. Yoshida at p. 60 of his paper [25]. His proof is different from ours). It is well known that f is normal in an angular domain Δ at ζ if and only if

$$\sup\left\{\frac{|f'(z)|}{1+|f(z)|^2}\cdot\frac{|dz|}{d\gamma_\Delta(z)};\ z \in \Delta\right\} < +\infty \tag{4.11}$$

(see [12], p. 56, Theorem 3). We assert that $\zeta \in N(f)$ if and only if

$$\sup\left\{\frac{(1-|z|^2)\,|f'(z)|}{1+|f(z)|^2};\ z \in \Delta\right\} < +\infty \tag{4.12}$$

for each angular domain Δ at ζ. We then obtain $N_D(f) = N(f)$ by the characterization of $N_D(f)$ due to Dragosh [7], p. 60. We first assume $\zeta \in N(f)$, whence (4.11) for each Δ at ζ. We can choose $\Delta_1 \supset \Delta$ at ζ and $k > 0$ such that $\Delta \subset \Delta_1(k)$. Then by (4.5) with $\mathscr{D} = \Delta_1$ we have

$$1-|z|^2 = \frac{|dz|}{d\gamma_D(z)} \leqslant \frac{k\,|dz|}{d\gamma_{\Delta_1}(z)}$$

for each $z \in \Delta$. Therefore (4.12) follows from (4.11). For the proof of the converse, we have only to remember $d\gamma_\Delta(z) \geqslant d\gamma_D(z)$ for all $z \in \Delta$. Since $|dz|/d\gamma_\Delta(z) \leqslant 1-|z|^2$, $z \in \Delta$, from (4.12) follows (4.11). To prove $N_\Delta(f) \subset N(f)$ we assume that there be $\zeta \in N_\Delta(f) - N(f)$. Then we have an angular domain Δ at ζ, a positive number ε, and two sequences $\{z_n\}$, $\{z_n'\}$ in Δ such that $\gamma_\Delta(z_n, z_n') \to 0$ but $\chi(f(z_n), f(z_n')) > \varepsilon$ $(n \geqslant 1)$; we may assume $z_n, z_n' \to \zeta$ since $\gamma_D(z_n, z_n') \to 0$ by (4.3). Since $\zeta \in N_\Delta(f)$ the family of functions

$$F_n(z) = f\left(\frac{z+z_n}{1+\bar{z}_n z}\right) \qquad (n \geqslant 1)$$

is normal in a disk $|z|<r_0$ (we note that the family $\{g(z)\}$ is normal in $|z|<r_0$ if and only if $\{g(-z)\}$ is normal in $|z|<r_0$). Setting $w_n=(z'_n-z_n)/(1-\bar{z}_n z'_n)$ we have $\chi(F_n(w_n), F_n(0))=\chi(f(z'_n), f(z_n))>\varepsilon$ $(n\geqslant 1)$ with $w_n\to 0$. This contradicts the continuity of the limiting function of a subsequence of $\{F_n\}$ in $|z|<r_0$. We next show $N_\Delta(f)\supset N(f)$. For each Δ at $\zeta\in N(f)$ and for each sequence $z_n\in\Delta$, converging to ζ, we may find r_0, $0<r_0<1$, such that $w=(z+z_n)/(1+\bar{z}_n z)\in\Delta_1$ for all z, $|z|<r_0$, where $\Delta_1\supset\Delta$ is an angular domain at ζ. Setting $F_n(z)=f((z+z_n)/(1+\bar{z}_n z))$, $|z|<r_0$, we have

$$\frac{|F'_n(z)|}{1+|F_n(z)|^2}=\frac{(1-|w|^2)|f'(w)|}{(1-|z|^2)(1+|f(w)|^2)}\leqslant(1-r_0^2)^{-1}\frac{(1-|w|^2)|f'(w)|}{1+|f(w)|^2}.$$

Since $\zeta\in N(f)$ and since $w\in\Delta_1$, it follows from (4.12) for Δ_1, combined with the Marty criterion (see [10], p. 158), that $\{F_n\}$ is normal in $|z|<r_0$. This completes the proof of Theorem 4.7.

Now, (4.9) follows from (4.2) and (4.10).

Remark 4.8. A function g defined in D is called *quasi-conformal* (simply, KQC, $K\geqslant 1$) if g is of composed form $G\circ\mu$, where G is a single-valued meromorphic function in D and μ is a KQC homeomorphism from D onto D (see [13], p. 250). A KQC algebroid (simply, $KQCA$) function f in D is a multiple-valued function in D defined by an algebraic equation

$$f^n(z)+a_1(z)f^{n-1}(z)+\ldots+a_n(z)=0\qquad(n\geqslant 1),\tag{4.13}$$

where $a_j=b_j\circ\mu$ $(1\leqslant j\leqslant n)$ are KQC functions in D, b_j being meromorphic in D and μ being a KQC self-homeomorphism of D common to all j. A $1QCA$ function in D is an algebroid function in the usual sense; further, if $n=1$, then the function is nothing but a single-valued meromorphic function in D. Now, a $KQCA$ function f in D defines a set-mapping S_f: $D\to P(\mathfrak{R})$, on assuming $a\in S_f(z)$ if and only if there exists $f_j(z)=a$, where $f_j(z)$ $(1\leqslant j\leqslant n)$ are n branches of f at $z\in D$. We shall show that S_f satisfies the boundary normality condition. We note that the multiple-valued function $1/(f-\beta)$ with a complex constant β is again $KQCA$. Since each angular domain at $\zeta\in\Gamma$ is conformally equivalent to D, we have only to show that a bounded $KQCA$ function f in D defined by (4.13) induces the set-mapping S_f normal in D in our sense. The word "bounded" means that there is $B>0$ such that $S_f(D)\subset\mathfrak{R}(0,B)$. The function f is "of composed form", $g\circ\mu$, where g is a bounded algebroid function with $S_g(D)\subset\mathfrak{R}(0,B)$ defined by

$$g^n(z)+b_1(z)g^{n-1}(z)+\ldots+b_n(z)=0.$$

By the Schwarz lemma [23], p. 284, Lemma 2, applied to g in each disk $D(z_0)=\{z;\ |z-z_0|<1-|z_0|\}$, $z_0\in D$, we have

$$S_g(z)\subset[\mathfrak{R}(S_g(z_0), c(1-|z_0|)^{-1/n}|z-z_0|^{1/n})]^-$$

for $z \in D(z_0)$, where the bar means the closure and $c > 0$ is a constant depending only on n and B (we note that the irreducibility of the equation in [23], Lemma 2, is indeed not used in the proof of the lemma). Now, given ϱ, $0 < \varrho < 1/3$, the disk $U(z_0, \varrho) = \{z;\ |z-z_0|/|1-\bar{z}_0 z| \leqslant \varrho\}$ is contained in $V(z_0) = \{z;\ |z-z_0| \leqslant \varrho(1-|z_0|^2)/(1-\varrho|z_0|)\}$ (see [18], p. 511, (i)), and $V(z_0)$ is contained in $D(z_0)$. Further, for $z \in (z_0, \varrho)$, $|z-z_0|/(1-|z_0|) \leqslant \varrho(1+|z_0|)/(1-\varrho|z_0|) < 3\varrho$, whence

$$S_g(z) \subset [\mathfrak{N}/S_g(z_0), 3^{1/n} c \varrho^{1/n}/]^- .$$

Therefore S_g is normal in D. On the other hand, μ satisfies

$$|\mu(z)-\mu(z_0)|/|1-\overline{\mu(z_0)}\mu(z)| \leqslant 4|z-z_0|^{1/K}|1-\bar{z}_0 z|^{-1/K}$$

(see [13], p. 68), whence μ is uniformly continuous with respect to $\gamma_D(\cdot, \cdot)$. Therefore, it is not difficult to see that the "composed" set-mapping $S_f = S_g \circ \mu$ is normal in D.

5. Horocyclic versions. A right horocyclic arc (RH-arc) at $\zeta \in \Gamma$ is a continuous curve $h^+(\zeta): z = z(t) \in D$ $(0 \leqslant t < 1)$, $\lim_{t\to 1} z(t) = \zeta$, such that there exists ϱ, $0 < \varrho < 1$, with $\operatorname{Im}\{\zeta^{-1}z(t)\} \leqslant 0$ and $|z(t)-(1-\varrho)\zeta| = \varrho$ for all $t \in [0, 1)$. A right horocyclic angular domain (RH-angular domain) at $\zeta \in \Gamma$ is a Jordan domain $H^+(\zeta) \subset D$ whose boundary consists of the point ζ, two RH-arcs at ζ, $h_j^+(\zeta)$: $z_j = z_j(t)$ $(0 \leqslant t < 1)$, $j = 1, 2$, and the D-geodesic from $z_1(0)$ to $z_2(0)$. A left horocyclic arc and a left horocyclic angular domain at $\zeta \in \Gamma$ are defined dually. If we do not specify whether a right or a left horocyclic arc (angular domain) at ζ is in question, we call it simply H-arc (H-angular domain) at ζ, and denote it by $h(\zeta)$ ($H(\zeta)$). An admissible tangential arc at $\zeta \in \Gamma$ is a continuous curve Λ_*: $z = z(t) \in D$ $(0 \leqslant t < 1)$, $\lim_{t\to 1} z(t) = \zeta$, tangent at ζ to an H-arc at ζ.

If we replace the term "angular domain" by "H-angular domain" in the definitions of $K(S)$ and $J(S)$, we have the horocyclic versions of Theorem 2.1, Corollary 2.2 and Theorem 2.3. Theorem 3.2 and Corollary 3.3 admit their horocyclic versions, if we further replace "chord" by "H-arc". For the proof of the horocyclic Theorem 2.3 we follow the proof of Theorem 2.3 up to the stage of the contradictory assumption that A is of positive measure. Let P be a perfect set of positive measure in A. If P is dense on an open arc of Γ, the proof is easy, while if P is nowhere dense on Γ, then thanks to the result of Dragosh [6], p. 64, Lemma 5, the domain

$$R(P) = \bigcup_{\zeta \in P} R_{pq}(\zeta)$$

with $R_{pq}(\zeta) = H_p(\zeta) \cap \{z;\ 1-2^{-q} < |z| < 1\}$, contains a terminal part of each open disk internally tangent to Γ at ζ for a.e. $\zeta \in \Gamma$. This contradicts $P \subset F_{nm(\eta+1)}$.

We now remember the definitions of "the boundary normality condition", and of the sets $I(S)$, $L(S)$, $M(S)$ and $N(S)$. If, in addition to the horocyclic versions of $J(S)$ and $K(S)$, we further replace the terms "angular domain", "chord of Γ at $\zeta \in \Gamma$" and "admissible arc" by the terms "H-angular domain", "H-arc at ζ" and "admissible tangential arc", respectively, we obtain the horocyclic Theorem 4.1, Corollary 4.2, Theorem 4.3 and Corollary 4.4. Thus, for example, ζ is a horocyclic Lindelöf point of S if

$$\bigcap_{\Lambda_*} C_{\Lambda_*}(S, \zeta) = \bigcup_{H(\zeta)} C_{H(\zeta)}(S, \zeta),$$

where Λ_* ranges over all admissible tangential arcs at ζ and $H(\zeta)$ ranges over all H-angular domains at ζ. We note that the essentials of the proofs of the versions lie in the comparison (4.4) of the non-Euclidean distances in the horocyclic form, that is, in

LEMMA 5.1. (1) *Let $H^+(\zeta)$ be an RH-angular domain and $h^+(\zeta)$ be an RH-arc at ζ such that $H^+(\zeta) \cap h^+(\zeta) = \emptyset$. Then we may find an RH-angular domain $H_1^+(\zeta)$ at ζ and a constant $k > 0$ such that $H_1^+(\zeta) \subset H^+(\zeta)$, $H_1^+(\zeta) \cap h^+(\zeta) \neq \emptyset$ and*

$$\gamma_{H^+(\zeta)}(z_1, z_2) \leqslant k\gamma_D(z_1, z_2) \tag{5.1}$$

for each pair $z_1, z_2 \in H_1^+(\zeta)$, provided D-geodesic from z_1 to z_2 is contained in $H_1^+(\zeta)$. (2) *Let $H_1^+(\zeta)$ be an RH-angular domain at $\zeta \in \Gamma$. Then we may find an RH-angular domain $H(\zeta)$ at ζ and a constant $k > 0$ such that $H(\zeta) \supset H_1^+(\zeta)$ and* (5.1) *is valid.* (3) *Given an RH-arc $h^+(\zeta)$ at $\zeta \in \Gamma$, each ε-neighborhood of $h^+(\zeta)$ in the distance $\gamma_D(\cdot, \cdot)$ $(\varepsilon > 0)$ contains an RH-angular domain at ζ. Conversely, given $H^+(\zeta)$ and $h^+(\zeta)$ at ζ with $H^+(\zeta) \cap h^+(\zeta) \neq \emptyset$, we may find $\varepsilon > 0$ such that the ε-neighborhood of a terminal part of $h^+(\zeta)$ in the distance $\gamma_D(\cdot, \cdot)$ is contained in $H^+(\zeta)$.*

We remark that although we describe the lemma in the *RH* form, the same is true for the *LH* case. For the proof we may consider the case $\zeta = 1$. (1) The image H of $H^+(1)$ by the map $w = i(1+z)/(1-z)$ contains the half-strip $H' = \{w = x+iy;\, a < x,\, b < y < b+p\}$ in the upper half-plane $\mathscr{H}$. By a short calculation we have

$$d\gamma_{H'}(w) = \frac{\pi|1-\xi^2||\xi||dw|}{2p(\operatorname{Im}\xi)(1-|\xi|^2)},$$

where $\xi = \xi(w) = -\exp\{(\pi/p)(\alpha - w)\}$, $\alpha = a+ib$. Therefore, for $w = x+ +iy \in H'$,

$$d\gamma_{H'}(w)/d\gamma_{\mathscr{H}}(w) = \frac{\pi|1-\xi^2||\xi|}{p(1-|\xi|^2)}(\operatorname{Im}w/\operatorname{Im}\xi)$$

$$= \frac{\pi|1-\xi^2|}{p(1-|\xi|^2)}\cdot\frac{y}{\sin\{(\pi/p)(b-y)+\pi\}}.$$

It is now easy to choose a half-strip $H_1 \subset H'$ with $H_1 \cap \lambda \neq \emptyset$, λ being the image of $h^+(1)$, such that $d\gamma_{H'}(w)/d\gamma_{\mathscr{H}}(w) < +\infty$ $(w \in H_1)$, which, combined with the principle of hyperbolic metrics, means that $d\gamma_H(w) \leqslant k d\gamma_{\mathscr{H}}(w)$ for each $w \in H_1$, where $k > 0$ is a constant. The inverse image $H_1^+(1)$ of H_1 is the desired. (2) The proof is similar to the above. (3) The first half. The image of the disk

$$\{z;\ |z-a|/|1-\bar{a}z| < R\}, \qquad 0 < R < 1,\ a \in h^+(1),$$

by the map $w = i(1+z)/(1-z)$ contains the disk $\delta(a) = \{w;\ |w-i(1+a)/(1-a)| < [2R/(1+R)](1-\varrho)/\varrho\}$, where $h^+(1)$ lies on $|z-(1-\varrho)| = \varrho$. As $a \to 1$ along $h^+(1)$, the disks $\delta(a)$ sweep a half-strip of with $[4R/(1+R)](1-\varrho)/\varrho$ in $\mathscr{H}$, whose inverse image is the desired RH-angular domain. The proof of the second half is similar, with a few modifications.

References

[1] J. M. Anderson, *Boundary properties of meromorphic functions*, Quart. J. Math. Oxford (2), 18 (1967), p. 103–107.

[2] F. Bagemihl, *Horocyclic boundary properties of meromorphic functions*, Ann. Acad. Sci. Fenn. Ser. A, I. Math. 385 (1966), p. 1–18.

[3] E. F. Collingwood, *On sets of maximum indetermination of analytic functions*, Math. Z. 67 (1957), p. 377–396; addendum, ibidem 68 (1958), p. 498–499.

[4] — and A. J. Lohwater, *The theory of cluster sets*, Cambridge University Press, London 1966.

[5] Е. П. Долженко, *Граничные свойства произвольных функций*, Изв. Акад. Наук СССР, сер. мат. 31 (1967), p. 3–14

[6] S. Dragosh, *Horocyclic cluster sets of functions defined in the unit disc*, Nagoya Math. J. 35 (1969), p. 53–82.

[7] — *The spherical derivative of meromorphic functions*, J. Reine Angew. Math. 252 (1972), p. 51–67.

[8] Г. М. Голузин, *Геометрическая теория функций комплексного переменного*, Изд. Наука, Москва 1966.

[9] J. T. Gresser, *The local behavior of principal and chordal principal cluster sets*, Trans. Amer. Math. Soc. 165 (1972), p. 323–332.

[10] W. K. Hayman, *Meromorphic functions*, Oxford University Press, London 1964.

[11] P. Lappan, *A property of angular cluster sets*, Proc. Amer. Math. Soc. 19 (1968), p. 1060–1062.

[12] O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. 97 (1957), p. 47–65.

[13] — — *Quasikonforme Abbildungen*, Springer-Verlag, Berlin–Heidelberg–New York 1965.

[14] K. Noshiro, *Cluster sets*, Springer-Verlag, Berlin–Göttingen–Heidelberg 1960.

[15] — *Some remarks on cluster sets*, J. Analyse Math. 14 (1967), p. 283–294.

[16] S. Saks, *Theory of the integral*, Hafner Publi. Co., New York, the reprint of the second edition in Warszawa 1937.

[17] W. Seidel and J. L. Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence*, Trans. Amer. Math. Soc. 52 (1942), p. 128–216.
[18] M. Tsuji, *Potential theory in modern function theory*, Maruzen Co., Tokyo 1959.
[19] J. D. Weston, *Some theorems on cluster sets*, J. London Math. Soc. 33 (1958), p. 435–441.
[20] S. Yamashita, *Some theorems on cluster sets of set-mappings*, Proc. Japan Acad. 46 (1970), p. 30–32.
[21] – *Quasi-conformal extension of Meier's theorem*, ibidem 46 (1970), p. 323–325.
[22] – An abstract of the paper: *Complex-harmonic Meier's theorem*, Zentralblatt für Math. und ihre Grenzgebiete. 199 (1971), p. 130–131.
[23] – *Cluster sets of algebroid functions*, Tôhoku Math. J. 22 (1970), p. 273–289.
[24] – *The derivative of a holomorphic function in the disk*, Michigan Math. J. 21 (1974), p. 129–132.
[25] H. Yoshida, *On the boundary properties and the spherical derivatives of meromorphic functions in the unit disc*, Math. Z. 132 (1973), p. 51–68.
[26] W. H. Young, *La symmétrie de structure des fonctions de variable réelles*, Bull. Sci. Math. (2) 52 (1928), p. 265–280.

DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
TOKYO, JAPAN

Reçu par la Rédaction le 20. 6. 1975