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Unitary dilations in case of ordered groups

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In this paper we investigate unitary dilations of semi-groups of contractions. Such semi-groups are representations of semi-groups of some groups. It is assumed that those semi-groups induce an order in a group in question.

If we are given the semi-group $T(\xi)$ of contractions and the order in a group is induced by a semi-group whose representation is $T(\xi)$, then $T(\xi)$ may be extended on the whole group to a certain operator-valued function. We show that this function is positive-definite and consequently has a unitary dilation. The group is assumed to be abelian.

We present the canonical decomposition of $T(\xi)$ and establish several geometrical relations in terms of some subspaces. This is done in the spirit of the prediction theory. The investigations are modeled after the method used by the author for semi-groups of powers of a single contraction in [3], [4]. We study mainly the remote pasts of the processes arising from unitary dilations.

1. Let G be an abelian group. The inner group operations are written additively. Suppose now that G_+ is a semi-group in G, i.e. that

$$\text{if } a \in G_+ \text{ and } \beta \in G_+ \text{ then } a+\beta \in G_+ . \tag{1.0}$$

We assume throughout the present paper that zero of G belongs to G_+, i.e. $0 \in G_+$. For $Z \subset G$ we define $-Z = \{a \in G \mid -a \in Z\}$. Suppose that

$$(G_+) \cup (-G_+) = G , \tag{1.1}$$

$$(G_+) \cap (-G_+) = \{0\} . \tag{1.2}$$

Then G_+ induces the ordering relation "$\leqslant$" defined by $a \leqslant \beta \equiv \beta - a \in G_+$. Conversely, if an order in G is given, then the set of elements dominating 0 forms a semi-group which satisfies (1.1) and (1.2), provided the order is compatible with the group operations. We will write for simplicity $a < \beta$ in case $a \leqslant \beta$ and $a \neq \beta$.

Suppose that H is a Hilbert space with the inner product (f, g) $(f, g \in H)$. Let us denote by $B(H)$ the totality of bounded linear operators in H. Suppose we are given the function $T(\xi) \in B(H)$ defined for $\xi \in G_+$. We say that $T(\xi)$ is a *semi-group of operators* if

$$(1.3) \qquad T(0) = I\,, \quad T(\xi+\eta) = T(\xi)T(\eta) \quad \text{for} \quad \xi, \eta \in G_+\,.$$

Although the above definition depends on the order in G, namely on G_+, there will be no difficulty since we will work with one fixed order.

For $T \in B(H)$ T^* stands for the adjoint of T. If $V(\xi)$ is an arbitrary operator-valued function defined on G_+ and such that $V(0) = V(0)^*$ we define a new function V_ξ, determined on the whole group G by the formula

$$(1.4) \qquad V_\xi = \begin{cases} V(\xi) & \text{if} \quad \xi \in G_+\,, \\ V(-\xi)^* & \text{if} \quad \xi \in (-G_+)\,. \end{cases}$$

Since (1.1), (1.2) hold, the definition (1.4) is well posed. If G is an arbitrary group and V_ξ is an arbitrary function with values in $B(H)$, we say that V_ξ is *positive difinite* [5] on G if

$$(1.5) \qquad \sum_{\eta,\xi} \big(V_{-\eta+\xi}\, g(\xi), g(\eta)\big) \geqslant 0$$

for every vector-valued function $g(a) \in H$ such that $g(a) \neq 0$ only for a finite number of a. The unitary representation U_ξ of G in a Hilbert space $K \supset H$ is called the *unitary dilation of* V_ξ if

$$(1.6) \qquad V_\xi f = PU_\xi f \quad \text{for} \quad f \in H \text{ and } \xi \in G\,.$$

P stands for the orthogonal projection of K onto H. The unitary dilation U_ξ is called *minimal* if $K = \bigvee_{\xi \in G} U_\xi H$. If G is abelian and ordered by G_+, then the equality in (1.6) satisfied merely for $\xi \in G_+$ implies that it is true for $\xi \in (-G_+)$ provided $V_{-\xi} = V_\xi^*$. This applies when V_ξ arises from a semi-group $V(\xi)$ according to formula (1.4). In this case we may call U_ξ, without any ambiguity, the unitary dilation of the semi-group $V(\xi)$.

2. Suppose that the group G is ordered and abelian. Let $V_\xi \in B(H)$ be defined for $\xi \in G$. The left-hand side in the positivity condition (1.5) may be written in the form

$$L = \sum_{\xi,\eta} \big(V_{\xi-\eta}\, g(\xi), g(\eta)\big)\,.$$

Let $a_0 < a_1 \ldots < a_n$ be the sequence of all a for which $g(a)$ is different from zero. Then

$$L = \sum_{i,j=0}^{n} \big(V_{a_i-a_j}\, g(a_i), g(a_j)\big)\,.$$

Now write $V_{ij} = V_{a_j-a_i}$ and consider the operator-valued matrix $\widetilde{V}$ with the (i,j)-entry equal to V_{ij}. The matrix $\widetilde{V}$ is an operator in the orthogonal sum H^n of $n+1$ copies of H and $(\tilde{V}\tilde{g}, \tilde{g}) = L$ for $\tilde{g} \in H^n$, $\tilde{g} = \{g(a_n)\}$. The positivity condition (1.5) is equivalent to the positive definiteness of $\tilde{V}$ in H^n for $n \geqslant 0$. Let $T(\xi)$ be the semi-group of operators and T_ξ its extension by (1.4) and let $\tilde{T}$ be the corresponding matrix. It is easy to see that $\tilde{T}$ is now Hermitian symmetric and its (i, j)-entries for $i \leqslant j$ are the values of T_ξ in G_+. I. Halperin in [1] proved that $\tilde{T}$ admits a factorization of the form $\hat{T} = W^*DW$, D being a diagonal matrix, if G is an additive group of integers. In this case $T(\xi)$ is the semi-group of powers of a single contraction. The following lemma shows that the arguments of Halperin apply in a more general situation, namely if G is an arbitrary ordered group. More precisely, the following lemma holds true:

LEMMA 2.1. *Let the order be induced in the abelian group G by G_+ and let $T(\xi)$ be a semi-group of operators. Define T_ξ by* (1.4) *and suppose that $a_0 < a_1 < \ldots < a_n$ are arbitrary elements of G. Define $\tilde{T}$ = the matrix with (i, j)-entry $(i, j = 0, 1, \ldots, n)$ equal to $T_{a_j - a_i}$. Then $\tilde{T} = W^*DW$ where the (i, j)-entry W_{ij} of W is defined by*

$$W_{ij} = \begin{cases} T_{a_j - a_i} & \text{if} \quad i \leqslant j\,, \\ 0 & \text{if} \quad i > j\,, \end{cases}$$

and the entries of D are

$$D_{00} = I\,,$$
$$D_{ij} = 0\,, \quad \text{if} \quad i \neq j\,,$$
$$D_{ii} = I - T^*_{a_i - a_{i-1}} T_{a_i - a_{i-1}} \quad \text{for} \quad i = 1, 2, \ldots, n\,.$$

Proof. Since both $\tilde{T}$ and W^*DW are Hermitian symmetric, it is sufficient to prove that their (i, j)-entries are equal for $i \leqslant j$. Let $(W^*DW)_{ij}$ be the (i, j)-entry of W^*DW. Since $(W^*)_{ik} = (W_{ki})^*$, the (i, k)-entry of W^*, is zero for $k > i$, we have

$$(W^*DW)_{ij} = \sum_{k=0}^{i} (W^*)_{ik} (DW)_{kj}\,.$$

If $i = 0$, then obviously $(W^*DW)_{0j} = (\tilde{T})_{0j} = T_{a_j - a_0}$. We can just assume that $i > 0$. Since D is diagonal, we get

$$(W^*DW)_{ij} = \sum_{k=0}^{i} (W^*)_{ik} D_{kk} W_{kj} = \sum_{k=0}^{i} T^*_{a_i - a_k} D_{kk} T_{a_j - a_k}$$

$$= T^*_{a_j - a_0} T_{a_j - a_0} + \sum_{k=1}^{i} T^*_{a_i - a_k} T_{a_j - a_k} - \sum_{k=1}^{i} T^*_{a_i - a_k} T^*_{a_k - a_{k-1}} T_{a_k - a_{k-1}} T_{a_j - a_k}\,.$$

Since

$$T^*_{a_i-a_k}T^*_{a_k-a_{k-1}} = T^*_{a_i-a_{k-1}} \quad \text{and} \quad T_{a_k-a_{k-1}}T_{a_j-a_k} = T_{a_j-a_{k-1}}$$

by the semi-group property, we get

$$(W^*DW)_{ij} = T^*_{a_i-a_0}T_{a_j-a_0} + \sum_{k=1}^{i} T^*_{a_i-a_k}T_{a_j-a_k} - \sum_{k=1}^{i} T^*_{a_i-a_{k-1}}T_{a_j-a_{k-1}}$$
$$= T^*_0 T_{a_j-a_i} = T_{a_j-a_i} = (\tilde{T})_{ij},$$

which completes the proof.

Suppose now that the semi-group $T(\xi)$ consists of contractions. Consequently $I - T(\xi)^* T(\xi) \geqslant 0$ for $\xi \in G_+$, which implies that the matrix D of lemma 2.1 is positive definite on H^n. It follows that $\tilde{T}$ is positive definite. Indeed, $(\tilde{T}\tilde{g}, \tilde{g}) = (DW\tilde{g}, W\tilde{g}) \geqslant 0$ in this case.

Lemma 2.1, the above discussion and the dilation theorem of [5] imply the following theorem:

THEOREM 2.1. *Let the abelian group G be ordered by the semi-group G_+ and let $T(\xi)$ be a contraction-valued semi-group. Then $T(\xi)$ has a unitary dilation U_ξ in a suitable dilation space $K \supset H$; U_ξ is determined by the minimality condition $K = \bigvee_{\xi \in G} U_\xi H$ uniquely up to the unitary isomorphism.*

The above theorem extends the results of Sz.-Nagy, who proved it for $G = N$ — the additive group of integers and $G = R$ — the additive group of reals, both with natural order (see [5]). In case $G = R$ the semi-group was assumed to be weakly continuous. The next theorem concerns the so called *canonical decomposition of a semi-group of contractions*. It extends the results of Langer [2] and Sz.-Nagy and Foiaş [6]. The proof of this theorem is essentially the same as in [6] and may be omited.

THEOREM 2.2. *Let the abelian group G be ordered by G_+ and let T_ξ be the function defined by* (1.4) *for the semi-group $T(\xi)$ of contractions acting in the space H. Then there exists a unique decomposition $H = H' \oplus H''$ such that:*

(2.1) *H' and H'' reduce all T_ξ for $\xi \in G$,*

(2.2) *The part of T_ξ in H' is a unitary representation of G,*

(2.3) *The only vector $f \in H''$ which satisfies $|T_\xi f| = |f|$ for all $\xi \in G$ is the zero vector — $f = 0$.*

The space H' is of the form $H' = \bigcap_{\xi \in G} U_\xi H$ and is called the *unitary part* of H with respect to T_ξ. H'' is called the *non-unitary part* of H. The part $T'(\xi)$ of $T(\xi)$ in H' is formed by unitary operators in H'. The part

$T''(\xi)$ of $T(\xi)$ in H'' is called the *completely non-unitary part* of $T(\xi)$. If $T(\xi) = T''(\xi)$, then $T(\xi)$ is called a *completely non-unitary semi-group* of contractions. In this case H' reduces to $\{0\}$. $T(\xi)$ is completely non-unitary iff for every $f \in H$ there is a $\xi \in G$ and $0 \leqslant k < 1$ such that $|T_\xi f| \leqslant k|f|$. The decomposition $H = H' \oplus H''$ is called the *canonical decomposition* of H. The decomposition $T(\xi) = T'(\xi) \oplus T''(\xi)$ is called the *canonical decomposition* of the semi-group $T(\xi)$.

3. Let U_ξ be the unitary dilation of the semi-group $T(\xi)$ of contractions. The space in which U_ξ act is denoted by K, and that in which $T(\xi)$ act — by $H \subset K$. Let S be a subspace of H and write

$$M_+(S) = \bigvee_{\xi>0} U_\xi S\,, \qquad M_-(S) = \bigvee_{\xi<0} U_\xi S\,,$$

$$R_+(S) = \bigcap_{\xi>0} U_\xi M_+(S)\,, \qquad R_-(S) = \bigcap_{\xi<0} U_\xi M_-(S)\,.$$

If S is a one-dimensional space spanned by $f \neq 0$, we write f in place of S in the above definitions. Note that $\underset{(-)}{R_+}(S)$ reduces U_ξ.

The basic lemma is the following one:

LEMMA 3.1. *Let U_ξ be a unitary dilation of the semi-group of contractions $T(\xi)$. Suppose that $h \in U_\xi M_+(S)$ for some $\xi > 0$. Then*

$$T(\xi)PU_{-\xi}h = Ph\,. \tag{3.1}$$

If $h \in U_{\xi} M_-(H)$ then $(\xi > 0)$

$$T(\xi)^* P U_\xi h = Ph\,. \tag{3.2}$$

Proof. Since $h = \lim\limits_{n\to\infty} \sum\limits_{\nu=1}^{r_n} U_{\xi_\nu^{(n)}} f_\nu^{(n)}$ with suitable $\xi_\nu^{(n)} > \xi$ and $f_\nu^{(n)} \in H$ we get

$$Ph = \lim_{n\to\infty} \sum_{\nu=1}^{r_n} T(\xi)\,T(\xi_\nu^{(n)} - \xi) f_\nu^{(n)} = \lim_{n\to\infty} T(\xi) P U_{-\xi} \sum_{\nu=1}^{r_n} U_{\xi_\nu^{(n)}} f_\nu^{(n)} = T(\xi) P U_{-\xi} h,$$

as was to be proved. (3.2) follows by symmetry.

The above lemma will be used in the proof of the following one:

LEMMA 3.2. *Suppose that the assumptions of lemma 3.1 are satisfied. Then*

$$(I-P)R_+(H) \perp U_{-\xi}H\,, \quad (I-P)R_-(H) \perp U_\xi H \quad \text{for} \quad \xi \geqslant 0\,, \tag{3.3}$$

$$(I-P)\underset{(-)}{R_+}(H) \perp \underset{(+)}{M_-}(H)\,, \tag{3.4}$$

$$R_+(H) \perp \big(U_{-\xi} - T(\xi)^*\big)H\,, \quad R_-(H) \perp \big(U_\xi - T(\xi)\big)H \quad \text{for} \quad \xi \geqslant 0\,. \tag{3.5}$$

Proof. Let $\xi > 0$. By lemma 3.1

$$T(\xi)PU_{-\xi}h - Ph = 0 \qquad (h \in R_+(H))$$

and consequently for $f \in H$

$$\big((P-I)\,U_{-\xi}h,\ U_{-\xi}f\big) = 0\,. \tag{3.6}$$

Since every element of $R_+(H)$ is of the form $U_{-\xi}h$ with suitable $h \in R_+(H)$ we conclude that the first relation of (3.3) holds true. The second one follows by symmetry. (3.4) follows immediately from (3.3). Relation (3.5) follows easily from (3.6).

We are now in a position to characterize the unitary part of the canonical decomposition in terms of $R_+(H)$ and $R_-(H)$.

THEOREM 3.1. *Let U_ξ be the minimal unitary dilation of the semigroup of contractions $T(\xi)$. Then*

$$\bigcap_{\xi \in G} U_\xi H = R_+(H) \cap R_+(H)\,.$$

Proof. It follows from (3.3) of lemma 3.2 that

$$(I-P)\big(R_+(H) \cap R_-(H)\big) \perp U_\xi H \qquad \text{for} \qquad \xi \in G\,.$$

Since U_ξ is minimal, we conclude that

$$(I-P)\big(R_+(H) \cap R_-(H)\big) = \{0\}\,,$$

i.e. $R^+(H) \cap R_-(H) \subset H$. On the other hand,

$$U_\xi\big(R_+(H) \cap R_-(H)\big) = R_+(H) \cap R_-(H)\,.$$

It follows that

$$R_+(H) \cap R_-(H) \subset \bigcap_{\xi \in G} U_\xi H\,.$$

Suppose now that $h \in \bigcap_{\xi \in G} U_\xi H$. Let $\xi > 0$ and $\eta > 0$. There is an element $\tilde{h} \in H$ such that $h = U_{\xi+\eta}\tilde{h}$. Since $U_{\xi+\eta}\tilde{h} = U_\xi U_\eta \tilde{h}$, we infer that $h = U_\xi U_\eta \tilde{h} \in U_\xi M_+(H)$. Since ξ was arbitrary, we get $h \in R_+(H)$. By similar arguments we prove that $h \in R_-(H)$. Hence $\bigcap_{\xi \in G} U_\xi H \subset R_+(H) \cap$ $\cap\, R_-(H)$ and we are done.

It follows from theorem 3.1 that $T(\xi)$ is completely non-unitary if and only if

$$R_+(H) \cap R_-(H) = \{0\}\,. \tag{3.7}$$

In general, if $S \subset H$ then

$$R_+(S) \cap R_-(S) \subset \bigcap_{\xi \in G} U_\xi H\,. \tag{3.8}$$

In particular, if S is a subspace of the non-unitary part of the canonical decomposition of the space H, then $R_+(S) \cap R_-(S) = \{0\}$.

Let S be a subspace of H and denote by P_ξ the orthogonal projection onto $U_\xi M_+(S)$. Lemma 3.1 implies

THEOREM 3.2. *Suppose that* $f \in S$. *Then*

$$|P_\xi f| \leqslant |T(\xi)^* f| \quad \text{for} \quad \xi > 0 . \tag{3.9}$$

Proof. Since $P_\xi f \in U_\xi M_+(S) \subset U_\xi M_+(H)$, we have

$$T(\xi) P U_{-\xi} P_\xi f = P P_\xi f .$$

But $Pf = f$. Hence

$$\big(P U_{-\xi} P_\xi f, T(\xi)^* f\big) = |P_\xi f|^2 . \tag{3.10}$$

Using (3.10) and the Schwartz inequality, we obtain

$$|P_\xi f|^2 \leqslant |P_\xi f|\,|T(\xi)^* f| ,$$

as was to be proved.

Remark. It follows from theorem 3.2 that if $\inf\limits_{\xi>0} |T(\xi)^* f| = 0$ for $f \in S$ then $R_+(S) = \{0\}$.

An important question is to find conditions under which $M_+(S) \ominus R_+(S) \neq \{0\}$. Suppose the contrary, i.e. $M_+(S) = R_+(S)$. Then $U_{-\eta} M_+(S) = R_+(S)$ for every $\eta > 0$. Hence $S \subset R_+(S)$, which implies that $T(\xi) P U_{-\xi} f = f$ ($S \subset H$!) for $f \in S$ and $\xi > 0$. It follows that $U_{-\xi} f = T(\xi)^* f$ for $\xi > 0$ and $f \in S$, and consequently $S \subset \bigcap\limits_{\xi>0} U_\xi H$. Using similar arguments one proves that if $M_-(S) = R_-(S)$ then $S \subset \bigcap\limits_{\xi>0} U_{-\xi} H$. We infer from theorem 3.1 that the following theorem holds true:

THEOREM 3.3. *Suppose* $S \subset H$ *and suppose that the projection of* S *onto the non-unitary part of the canonical decomposition is not a zero space. Then* $M_+(S) \ominus R_+(S)$ *or* $M_-(S) \ominus R_-(S)$ *is non-trivial.*

The above theorem is completed by the following one:

THEOREM 3.4. *If for some* $f \in S \subset H$ *and some* $\xi > 0$ *there is a constant* $k \in [0, 1)$ *such that* $|T(\xi)^* f| \leqslant k|f|$, *then the subspace* $M_+(S) \ominus U_\xi M_+(S)$ *is non-trivial and the projection* $P_\xi f$ *of* f *onto* $U_\xi M_+(S)$ *satisfies* $|(I - P_\xi) f|^2 \geqslant (1 - k^2)|f|^2$.

The proof of theorem 3.4 is based on theorem 3.2.

Suppose now that H is a complex Hilbert space and G is an ordered, abelian, locally compact group. Using the standard arguments of harmonic analysis on groups and using the spectral measure of unitary dilation one easily proves the following theorem:

THEOREM 3.5. *Suppose that the semi-group $T(\xi)$ of contractions is weakly continuous in the topology of G. Let f be a vector belonging to the non-unitary part of the canonical decomposition of H corresponding to the semi-group $T(\xi)$. Then $R_+(f) = R_-(f) = \{0\}$.*

The proof of theorem 3.5 is almost the same as in the case of a semi-group of powers of a single contraction (see [3]) and may be omitted.

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