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Asymptotic behavior of solutions of non-linear differential equations with deviating arguments via non-standard analysis

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Abstract. We discuss the asymptotic behavior of solutions of the following n-th order differential equation with deviating arguments ($n \geqslant 2$)

$$E(\delta) \qquad x^{(n)}(t)+\delta b(t)f(x[g_1(t)],\ldots,x[g_m(t)]) = h(t), \quad \delta = \pm 1$$

by using non-standard techniques. Under some assumptions on b, f and h, we proved some theorems about oscillatory, non-oscillatory and unboundedness.

1. Introduction. In the last few years, there has been an increasing interest in the study of the asymptotic behavior of solutions of differential equations with retarded arguments. For example, we refer to Chen and Yeh [1], Kartsatos [2], Kusano and Onose [5], Staiko and Sficas [8].

Non-standard analysis was introduced in oscillatory theory by Komkov and Waid [4] and Komkov [3]. They considered the following differential equations

$$\big(a(t)\,p(x)\,x'\big)'+b(t)\,F(x) = h(t)$$

and

$$(*) \qquad x^{(n)}(t)+F\big(t,\,x(t),\,x'(t),\ldots,x^{(n-1)}(t)\big) = h(t).$$

In this paper we improve their results and give some new criteria for the asymptotic behavior of solutions of the following n-th order differential equation with deviating arguments ($n \geqslant 2$)

$$E(\delta) \qquad x^{(n)}(t)+\delta b(t)f\big(x[g_1(t)],\ldots,x[g_m(t)]\big) = h(t), \qquad \delta = \pm 1$$

by using non-standard technique, in the frame-work of Robinson's theory [6], [7]. Let $\boldsymbol{R}^*$ denote the non-standard extension of the real line $\boldsymbol{R}$, which

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has the property that sentences formulated in language $\mathscr{L}$ are true in $\boldsymbol{R}^*$ if and only if they are true in $\boldsymbol{R}$ (see [6]). $\boldsymbol{R}$ is a subset of $\boldsymbol{R}^*$. $\boldsymbol{R}^*$ also contains infinitesimal numbers and infinite numbers which are not in $\boldsymbol{R}$. An infinite positive (negative) number is a non-standard number which is greater (smaller) than any real number. We shall denote by $\boldsymbol{R}^*_{+\infty}$ and $\boldsymbol{R}^*_{-\infty}$, respectively, the set of the infinite positive and negative numbers. The reciprocal of an infinite number is called an *infinitesimal number*. If x is a real number, then we call x a *standard number* of $\boldsymbol{R}^*$, otherwise it is called a *non-standard number*. $\boldsymbol{R}^*_{bd}$ denotes the set of the elements of $\boldsymbol{R}^*$ which are bounded in absolute value by a standard number. If x, y are elements of $\boldsymbol{R}^*$ such that $x-y$ is an infinitesimal, we shall say that x is *infinitely close to* y, and denote this by $x =_1 y$.

Let $I \equiv [t_0, \infty)$ for some fixed $t_0 > 0$. Throughout this paper, we assume that the following two conditions always hold:

(a) $b, h, g_i \in C(I, \boldsymbol{R})$, $b(t) > 0$ and $\lim_{t\to\infty} g_i(t) = \infty$ for $i = 1,\dots, m$,

(b) $f \in C(\boldsymbol{R}^m, \boldsymbol{R})$.

The following lemmas are due to Komkov and Waid [4].

LEMMA 1. *A standard function $x(t)$, $t\in I$, is oscillatory if and only if $x(t)$, $t\in \boldsymbol{R}^*$, vanishes for some $t\in \boldsymbol{R}^*_{+\infty}$.*

LEMMA 2. *A standard function $x(t)$ is unbounded if and only if $|x(t)| \in \boldsymbol{R}^*_{+\infty}$ for some $t\in \boldsymbol{R}^*_{+\infty}$.*

LEMMA 3. *$\int_{t_0}^{\infty} g(t)\,dt$ converges if and only if $\int_{t_1}^{t_2} g(t)\,dt =_1 0$ for any t_1, $t_2 \in \boldsymbol{R}^*_{+}$,* ([7], p. 75).

LEMMA 4. *If $x^{(n)}(t) < 0$ (> 0) and is bounded away from zero on I for some $n > 1$, then $\lim_{t\to\infty} x(t) = -\infty$ $(+\infty)$ for any $x(t) \in C^n(I, \boldsymbol{R})$.*

LEMMA 5. *Let $\lim_{t\to\infty} \int_{t_0}^{t} g(s)\,ds = +\infty$ $(-\infty)$. Then for any $A \in \boldsymbol{R}^*$, $A > 0$ (< 0), and any $t_1 > t_0$, $t_1 \in \boldsymbol{R}^*$, there exists $t_2 \in \boldsymbol{R}^*$, $t_2 > t_1$, such that $\int_{t_1}^{t_2} g(t)\,dt > A$ $(< A)$. Moreover, for any $t_3 \in \boldsymbol{R}^*_{bd}$, $t_4 \in \boldsymbol{R}^*_{+\infty}$ $(\boldsymbol{R}^*_{-\infty})$, we have $\int_{t_3}^{t_4} g(t)\,dt \in \boldsymbol{R}^*_{+\infty}$ $(\boldsymbol{R}^*_{-\infty})$.*

2. Main results.

THEOREM 1. *Let*

(C_1) *$f(y_1,\dots, y_m)$ be a non-decreasing function with respect to $y_1,\dots, y_m$*

and

$$y_i > 0 \ (i = 1, 2, \ldots, m) \Rightarrow f(y_1, y_2, \ldots, y_m) > 0,$$
$$y_i < 0 \ (i = 1, 2, \ldots, m) \Rightarrow f(y_1, y_2, \ldots, y_m) < 0,$$

(C_2) $\int_{t_0}^{\infty} h(t)\,dt$ *converge,*

and

(C_3) $\int_{t_0}^{\infty} b(t)\,dt = \infty$.

Then every solution of $E(1)$ *cannot be bounded away from zero.*

Proof. Assume to the contrary that there exists a solution $x(t)$ of $E(1)$ such that $x(t)$ is bounded away from zero on I. Without loss of generality, we assume that $x(t) > K > 0$ for some standard number K. Condition (a) implies that there exists a $t_1 > t_0$ such that

$$x[g_i(t)] > K$$

for $t \geqslant t_1$ and $i = 1, \ldots, m$. Hence, by (b) and (C_1), we have

$$f(x[g_1(t)], \ldots, x[g_m(t)]) \geqslant f(K, \ldots, K) \equiv k > 0 \tag{1}$$

for $t \geqslant t_1$ and particularly for all $t \in R^*_{+\infty}$. It follows from (C_2) and Lemma 3 that

$$\int_{\xi}^{\eta} h(t)\,dt =_1 0$$

for any $\xi, \eta \in R^*_{+\infty}$. Hence

$$\int_{\xi}^{\eta} h(t)\,dt < 1. \tag{2}$$

By (1), (2) and the fundamental theorem of calculus

$$\begin{aligned} x^{(n-1)}(\eta) &= x^{(n-1)}(\xi) + \int_{\xi}^{\eta} [h(t) - b(t) f(x[g_1(t)], \ldots, x[g_m(t)])]\,dt \\ &< x^{(n-1)}(\xi) + 1 - k \int_{\xi}^{\eta} b(t)\,dt. \end{aligned} \tag{3}$$

Regarding ξ as fixed, by (C_3) and Lemma 5, we can choose η so that

$$\int_{\xi}^{\eta} b(t)\,dt > k^{-1}[2 + x^{(n-1)}(\xi)]. \tag{4}$$

From (3) and (4), we have

$$x^{(n-1)}(\eta) < -1 \tag{5}$$

for all η satisfying (4). Since $x(t)$ is positive, (5) implies that $x(t)$ changes sign for some $t \in R^*_{+\infty}$. Therefore, by Lemma 1, $x(t)$ is oscillatory, a contradiction. This contradiction proves our theorem.

Remark 1. In Theorem 2.1 of [3], Komkov proved that if (C_2) holds and

$$y_1 F(t, y_1, \dots, y_m) > 0 \quad \text{if } y_1 \neq 0$$

for t large enough, then every solution of $(*)$ cannot be bounded away from zero.

This is wrong. The following is a counterexample.

EXAMPLE 1. The differential equation

$$x''(t) + t^{-2} x(t) \{1 + [x'(t)]^2\} = \tfrac{3}{4} t^{-3/2} + \tfrac{1}{4} t^{-5/2} \tag{6}$$

satisfies all the conditions of Komkov's theorem. But the solution $x(t) = t^{1/2}$ of (6) is bounded away from zero.

COROLLARY. *Under the assumptions of Theorem 1, every solution of $E(1)$ is oscillatory or such that* $\liminf_{t\to\infty} |x(t)| = 0$.

EXAMPLE 2. The differential equation

$$x''(t) + 2x(t) = 2(\sin t - \cos t) e^{-t} \tag{7}$$

satisfies all the conditions of Theorem 1. Hence every solution $x(t)$ of (7) is oscillatory or such that $\liminf_{t\to\infty} |x(t)| = 0$. In fact, $x(t) = e^{-t} \sin t$ is an oscillatory solution of (7).

THEOREM 2. *Let*

$$\liminf_{t\to\infty} b(t) = c > 0, \tag{C_4}$$

$$\lim_{t\to\infty} \frac{h(t)}{b(t)} = +\infty \tag{C_5}$$

hold. Then every solution of $E(\delta)$ is unbounded.

Proof. Assume to the contrary that there exists a solution $x(t)$ of $E(\delta)$ which is bounded. Then $x[g_i(t)]$ is bounded for $i = 1, \dots, m$. From (C_4), we have

$$\frac{2x^{(n)}(t)}{c} > \frac{x^{(n)}(t)}{b(t)} = \frac{h(t)}{b(t)} - \delta f(x[g_1(t)], \dots, x[g_m(t)]).$$

This and (C_5) imply that $x^{(n)}(t) \in R^*_{+\infty}$ for all $t \in R^*_{+\infty}$. By Lemma 4, $x(t)$ is an infinite positive number for all $t \in R^*_{+\infty}$, a contradiction. This contradiction completes our proof.

EXAMPLE 3. The differential equations

$$x'''(t)-2^{-1}e^{\pi}x(t-\pi) = 2^{-1}e^{t}, \tag{8}$$

$$y''(t)+y(t) = t^{1/2}-4^{-1}t^{-3/2} \tag{9}$$

satisfy all the conditions of Theorem 3. $x(t) = e^t$ and $y(t) = t^{1/2}$ are solutions of (8) and (9), respectively, which are unbounded.

THEOREM 3. *Let* (C_4) *and*

$$\liminf \frac{h(t)}{b(t)} \geqslant r > 0, \tag{C_6}$$

$$f(0, \ldots, 0) = 0 \tag{C_7}$$

hold. Then no nonoscillatory positive (negative) solution of $E(1)$ $(E(-1))$ *approaches zero.*

Proof. We only prove the case $E(1)$. Let $x(t)$ be a nonoscillatory positive solution of $E(1)$ which approaches zero. Then there exists a $t_1 > t_0$ such that for all $t \geqslant t_1$

$$f(x[g_1(t)],\ldots,x[g_m(t)]) < 4^{-1}r.$$

Since

$$2c^{-1}x^{(n)}(t) > \frac{x^{(n)}(t)}{b(t)} = -(x[g_1(t)],\ldots,x[g_m(t)])+\frac{h(t)}{b(t)}$$
$$> -4^{-1}r+2^{-1}r = 4^{-1}r > 0$$

for $t \geqslant t_1$. This and Lemma 4 imply that $x(t)$ is an infinite positive number for $t \in R^{*}_{+,}$, a contradiction. This contradiction completes our proof.

EXAMPLE 4. The differential equations

$$x''(t)+e^{\pi}x(t-\pi) = 2e^{-t}, \qquad x''(t)+x(t) = 2e^{-t}$$

satisfy conditions (C_4) and (C_7), but do not satisfy (C_6). These equations have $x(t) = e^{-t}$ as a solution which approaches zero.

EXAMPLE 5. The differential equation

$$x''(t)+t^{-4}[x(t)]^2 = 2t^{-3}+t^{-6}$$

satisfies conditions (C_6) and (C_7), but does not satisfy (C_4). The equation has a non-oscillatory solution $x(t) = t^{-1}$ which approaches zero as $t \to \infty$.

EXAMPLE 6. The differential equations

$$x'''(t)+x(t) = 1, \tag{10}$$

$$y'''(t)-y(t) = 1 \tag{11}$$

satisfy all the conditions of Theorem 3. $x(t) = 1+e^{-t}$ and $y(t) = -1-e^{-t}$ are non-oscillatory solutions of (10) and (11) respectively, which satisfy $\lim_{t\to\infty} x(t) = 1 \neq 0$ and $\lim_{t\to\infty} y(t) = -1 \neq 0$.

References

[1] Lu-San Chen and Cheh-Chih Yeh, *Nonoscillation generating delay terms in odd order differential equations*, Rend. Accad. Sci. Fis. Mat. Napoli 43 (1976), p. 189–198.

[2] A. G. Kartsakos, *Oscillation and nonoscillation for perturbed differential equations*, Hiroshima Math. J. 8 (1978), p. 1–10.

[3] V. Komkov, *Asymptotic behavior of non-linear inhomogeneous differential equations via non-standard analysis*, Ann. Polon. Math. 30 (1974), p. 205–219.

[4] –, and C. Waid, *Asymptotic behavior of non-linear inhomogeneous equations via non-standard analysis*, ibidem 28 (1973), p. 67–87.

[5] T. Kusano and H. Onose, *On the oscillation of solutions of nonlinear functional differential equations*, Hiroshima Math. J. 6 (1976), p. 635–645.

[6] A. Robinson, *Nonstandard analysis*, North Holland, Amsterdam 1966.

[7] –, *Introduction to model theory and to the mathematics of algebra*, North Holland, Amsterdam 1966.

[8] V. A. Staikos and Y. G. Sficas, *Forced oscillations for differential equations of arbitrary order*, J. Diff. Eq. 17 (1975), p. 1–11.

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