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# General finite difference approximation to the Cauchy problem for non-linear parabolic differential-functional equations

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**Abstract.** An explicit finite difference scheme is used to approximate the solution of the initial value problem for parabolic non-linear differential-functional equations. General difference operators used in this scheme satisfy assumptions which imply stability and convergence of this scheme.

**1. Introduction.** The convergence theorems for the finite difference approximation to initial boundary problems for partial differential equations were proved by many authors with the help of recurrence inequalities methods: in [9], [10] for first order partial differential equations and in [2], [3], [6], [7] for the Dirichlet problem for parabolic equations using the methods of [4].

It is easy to see that one can use the methods of these papers to the Cauchy problem for parabolic equations when we consider the class of bounded functions only. In [1] the convergence theorem for the Cauchy problem was established under the assumption that the approximated solution is allowed to grow like $\exp(K|y|^2)$. In [8] the result of [1] was generalized to the class of differential-functional equations with right-hand sides which satisfy the Volterra condition. In [1] and [8] typical discrete operators were used.

We give some conditions for general discrete operators and general difference scheme, and we prove the convergence theorem for parabolic differential-functional equations using simple recurrence inequalities and estimation theorems. The method used in this paper is a generalization of the method applied by Besala in [1] and by Malec in [8]. The main result is formulated in Theorem 1.

**2. Some notation and two lemmas.** For arbitrary sets $X_1$, $X_2$ let $\mathcal{F}(X_1, X_2)$ denote the class of all functions from $X_1$ taking values in $X_2$. Assume that $X$ is a fixed non-empty set. For all integers $i, j$, $i \leqslant j$, we define

$$Z_{i,j} = \{i, i+1, \ldots, j\}.$$

Let $n_0 \geqslant 0$, $n^* \geqslant 1$ be fixed integers. Denote by $\mathcal{F}$ a linear, partially ordered subspace of $\mathcal{F}(Z_{-n_0,n^*} \times X, \boldsymbol{R})$. Moreover, let (see [5])

$$\mathcal{F}^+ = \{w \in \mathcal{F} \mid w \geqslant 0\}.$$

LEMMA 1. *Suppose that:*

($\alpha$) $F\colon Z_{0,n^*-1}\times\mathscr{F}\to\mathscr{F}$ *is non-decreasing with respect to the functional variable and satisfies the Volterra condition, i.e.*

$$\big(w,\bar{w}\in\mathscr{F},\ i\in Z_{0,n^*-1};\ w(j,\cdot)=\bar{w}(j,\cdot)\text{ for }j\in Z_{-n_0,i}\big)\Rightarrow F(i,w)=F(i,\bar{w}),$$

($\beta$) $w, z\in\mathscr{F}$ *satisfy*

$$w(i+1,t)\leqslant F(i,w)(t),\qquad (i,t)\in Z_{0,n^*-1}\times X,$$

$$z(i+1,t)\geqslant F(i,z)(t),\qquad (i,t)\in Z_{0,n^*-1}\times X,$$

$$z(i,t)\geqslant w(i,t),\qquad (i,t)\in Z_{-n_0,0}\times X.$$

*Then*

$$z(i,t)\geqslant w(i,t),\qquad (i,t)\in Z_{-n_0,n^*}\times X.$$

The proof of Lemma 1 can be found in [3] or [5]. Lemma 2 below is a simple consequence of Lemma 1. The estimate given in Lemma 2 arises from [3], [5] or [6].

LEMMA 2. *Suppose that:*

(a) $F\colon Z_{0,n^*-1}\times\mathscr{F}\to\mathscr{F}$ *satisfies the Volterra condition,*

(b) $Q\colon Z_{0,n^*-1}\times\mathscr{F}^+\to\mathscr{F}^+$ *is non-decreasing with respect to the functional variable and satisfies the Volterra condition,*

(c) *for* $w,\bar{w}\in\mathscr{F}$ *and* $(i,t)\in Z_{0,n^*-1}\times X$ *we have*

$$|F(i,w+\bar{w})(t)-F(i,w)(t)|\leqslant Q(i,|\bar{w}|)(t), \tag{1}$$

*where* $|\bar{w}|(i,t)=|\bar{w}(i,t)|$ *for* $(i,t)\in Z_{-n_0,n^*}\times X$,

(d) $u,v\in\mathscr{F}$ *satisfy*

$$|u(i+1;t)-F(i,u)(t)|\leqslant\gamma(i,t)\qquad (i,t)\in Z_{0,n^*-1}\times X, \tag{2}$$

*where* $\gamma\in\mathscr{F}$, *and*

$$v(i+1,t)=F(i,v)(t),\qquad (i,t)\in Z_{0,n^*-1}\times X, \tag{3}$$

(e) *there is* $z\in\mathscr{F}^+$ *such that*

$$z(i,t)\geqslant|v(i,t)-u(i,t)|,\qquad (i,t)\in Z_{-n_0,0}\times X, \tag{4}$$

$$z(i+1,t)\geqslant Q(i,z)(t)+\gamma(i,t),\qquad (i,t)\in Z_{0,n^*-1}\times X, \tag{5}$$

*Then*

$$z(i,t)\geqslant|v(i,t)-u(i,t)|,\qquad (i,t)\in Z_{-n_0,n^*}\times X. \tag{6}$$

Proof. The proof is by induction on $i\in Z_{-n_0,n^*}$. For $i\in Z_{-n_0,0}$ (6) is just (4). Assume that (6) holds for $(i,t)\in Z_{0,n^*-1}$; we shall prove it for $i+1$. We get from (2), (3)

$$|v(i+1,t)-u(i+1,t)|\leqslant|F(i,v)(t)-F(i,u)(t)|+\gamma(i,t),\qquad (i,t)\in Z_{0,n^*-1}\times X,$$

and by (1) and the inductive assumption it follows that

$$|v(i+1, t)-u(i+1,t)| \leqslant Q(i, z)(t)+\gamma(i, t) \leqslant z(i+1, t), \quad (i, t)\in Z_{0,n^*-1}.$$

Using (5) we obtain (6) for $i:=i+1$. This completes the proof of Lemma 2.

**3. Formulation of a parabolic differential-functional problem, some assumptions and further notation.** Now we consider the parabolic differential-functional problem

$$\text{(7)}\quad \begin{aligned} &D_x w(x, y) = f\big(x, y, w(x, y), w, D_y w(x, y), D_{yy} w(x, y)\big), \\ &\qquad\qquad (x, y)\in E = (0, a]\times \boldsymbol{R}^n, \\ &w(x, y) = \varphi(x, y), \quad (x, y)\in E_0 = [-\tau_0, 0]\times \boldsymbol{R}^n, \end{aligned}$$

where $\tau_0 \geqslant 0$, $a \geqslant 0$.

We assume that (7) has a solution $u(x, y)$ which is of class $C^2$ on $\bar{E}$, has continuous third order derivatives with respect to the variables $y_i$ $(i = 1, \ldots, n)$ in $\bar{E}$, and satisfies the growth condition

$$\text{(8)}\quad |u(x, y)|, |D_x u(x, y)|, |D_{xx} u(x, y)|, |D_{y_i y_j y_\tau} u(x, y)|, |D_{y_i y_j} u(x, y)| \leqslant H(y, M, K):= M\sum_{\nu=1}^{n} \exp[K y_\nu^2]$$

for all $(x, y)\in \bar{E}$, $i, j, \tau = 1, \ldots, n$ and some constants $M, K > 0$.

We also assume that the function $f(x, y, p, w, q, r)$ is continuous for $(x, y, p, w, q, r)\in \Sigma := \bar{E}\times \boldsymbol{R}\times C(E_0\cup E)\times \boldsymbol{R}^{n+n^2}$ and of class $C^1$ with respect to $p, q, r$. Moreover, we assume that $D_{r_{ij}} f = D_{r_{ji}} f$ for $i, j = 1, \ldots, n$ $(i \neq j)$, and

$$\text{(9)}\quad |f(x, y, 0, 0, 0, 0)| \leqslant H(y, M, K), \quad (x, y)\in E,$$

and that there exist constants $L_0, L_1, L_2, L_3$ such that

$$\text{(10)}\quad |D_p f| \leqslant L_0, \quad |D_{q_i} f| \leqslant L_1, \quad |D_{r_{ij}} f| \leqslant L_2$$

on $E$ for $i, j = 1, \ldots, n$, and

$$\text{(11)}\quad |f(x, y, p, w+\bar{w}, q, r)-f(x, y, p, w, q, r)| \leqslant L_3 H(y, M, N)\, \|\bar{w}\|(x),$$

where

$$\text{(12)}\quad \|\bar{w}\|(x) = \sup_{-\tau_0\leqslant \bar{x}\leqslant x, \bar{y}\in \boldsymbol{R}^n} |\bar{w}(\bar{x}, \bar{y})|\, H^{-1}(\bar{y}, Me^{S\bar{x}}, N),$$

$N > K$ and $S > 0$ are fixed constants.

The parabolicity of (7) is meant in the following sense:

$$D_{r_{ii}} f - \sum_{j=1, j\neq i}^{n} |D_{r_{ij}} f| > 0 \quad (\text{on } E \text{ for } i = 1, \ldots, n).$$

Now we define a mesh. For constants $g$, $G$, $h_0$ such that $0 < g \leqslant G$, $h_0 > 0$ we introduce a set of discretization parameters by

$$I_{\mathrm{disc}} = \{(k, h) \in \boldsymbol{R}^2 |\ k/h^2 = \bar{g}(h),\ 0 < h \leqslant h_0\},$$

where $\bar{g}\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is such that $g \leqslant \bar{g}(h) \leqslant G$ for $0 < h \leqslant h_0$.

Next, for $(k, h) \in I_{\mathrm{disc}}$ we define a set of multiindices by

$$\mathcal{M}_h = \{m = (\tau, m') \in \boldsymbol{Z}^{1+n} |\ \tau \in Z_{-n_0, n^*},\ m' = (m_1, \ldots, m_n)\}.$$

If $(k, h) \in I_{\mathrm{disc}}$ there exist natural numbers $n_0$, $n^*$ such that

$$n^* k \leqslant a < (n^*+1)k, \qquad n_0 k \leqslant \tau < (n_0+1)k.$$

For $m = (\tau, m') \in \mathcal{M}_h$ let $x^{(m)} = \tau k$, $y^{(m)} = (m_1 h, \ldots, m_n h)$. Moreover, let

$$\mathcal{M}_h^* = \{m = (\tau, m') \in \mathcal{M}_h |\ \tau \in Z_{0, n^*-1}\},$$

$$\bar{\mathcal{M}}_h = \{m = (\tau, m') \in \mathcal{M}_h |\ (x^{(m)}, y^{(m)}) \in E\},$$

$$\mathcal{M}_{0,h} = \{m = \{\tau, m') \in \mathcal{M}_h |\ \tau \in Z_{n_0, 0}\},$$

$$E_h^* = \{(x, y) \in \boldsymbol{R}^{1+n} |\ x = x^{(m)},\ y = y^{(m)} \text{ for some } m \in \mathcal{M}_h^*\},$$

$$E_h = \{(x, y) \in \boldsymbol{R}^{1+n} |\ x = x^{(m)},\ y = y^{(m)} \text{ for some } m \in \bar{\mathcal{M}}_h\},$$

$$E_{0,h} = \{(x, y) \in \boldsymbol{R}^{1+n} |\ x = x^{(m)},\ y = y^{(m)} \text{ for some } m \in \mathcal{M}_{0,h}\},$$

Assume $\lambda \geqslant 1$ is a fixed natural number. Define

$$S_\lambda = \{s = (s_1, \ldots, s_n) \in \boldsymbol{Z}^n |\ |s_i| \leqslant \lambda,\ i = 1, \ldots, n\}.$$

For $w \in C(E_0 \cup E, \boldsymbol{R})$ we denote by $w_h \in \mathcal{F}(E_{0,h} \cup E_h, \boldsymbol{R})$ the function such that $w_h^{(m)} = w_h(x^{(m)}, y^{(m)}) := w(x^{(m)}, y^{(m)})$ and for $w_h \in \mathcal{F}(E_{0,h} \cup E_h, \boldsymbol{R})$ we denote by $[w_h]_h$ a fixed function which is an extension of $w_h$ on $E_0 \cup E$.

Now we define discrete operators $A$, $B$, $C$, $D$. For $(k, h) \in I_{\mathrm{disc}}$, $w_h \in \mathcal{F}(E_{0,h} \cup E_h, \boldsymbol{R})$, $m = (\tau, m') \in \mathcal{M}_h^*$ we define

$$Aw_h^{(m)} = \sum_{s \in S_\lambda} a_s(m) w_h^{(\tau, m'+s)}, \qquad Bw_h^{(m)} = \sum_{s \in S_\lambda} b_s(m) w_h^{(\tau, m'+s)},$$

$$C_i w_h^{(m)} = \sum_{s \in S_\lambda} \frac{1}{h} c_s^{(i)}(m) w_h^{(\tau, m'+s)}, \tag{13}$$

$$D_{ij} w_h^{(m)} = \sum_{s \in S_\lambda} \frac{1}{h^2} d_s^{(i,j)}(m) w_h^{(\tau, m'+s)},$$

where $a_s(m), b_s(m), c_s^{(i)}(m), d_s^{(i,j)}(m) \in \boldsymbol{R}$ for $s \in S_\lambda$, $m \in \mathcal{M}_h^*$, $i, j = 1, \ldots, n$.

Let

$$C = (C_1, \ldots, C_n), \qquad D = [D_{ij}]_{i,j=1,\ldots,n},$$

$$\Sigma_h = E_h^* \times \boldsymbol{R} \times \mathcal{F}_{N,S}(E_{0,h} \cup E_h, \boldsymbol{R}) \times \boldsymbol{R}^{n+n^2},$$

where

$$\mathscr{F}_{N,S}(E_{0,h}\cup E_h, \mathbf{R}) = \{z_h \in \mathscr{F}(E_{0,h}\cup E_h, \mathbf{R})|\ |z_h(x^{(m)}, y^{(m)})| \leqslant H(y^{(m)}, \bar{M}, Ne^{Sx^{(m)}}),\ m\in\mathscr{M}_h \text{ for some } \bar{M}\geqslant 0\}.$$

Moreover, let $\Phi_h \in \mathscr{F}(\Sigma_h, \mathbf{R})$, $\delta_h \in \mathscr{F}(E_{0,h}, \mathbf{R})$, with

(14) $$|\delta_h^{(m)}| \leqslant \tilde{\gamma}_h H(y^{(m)}, M, N), \quad m\in\mathscr{M}_{0,h},$$

where $\tilde{\gamma}_h \in \mathbf{R}^+$.

**4. Formulation of a difference scheme and Assumptions** $\mathrm{H}_1$, $\mathrm{H}_2$. For problem (7) we consider the following explicit difference scheme:

(15) $$w_h^{(\tau+1,m')} = Aw_h^{*(m)} + k\Phi_h(x)(x^{(m)}, y^{(m)}, Bw_h^{(m)}, w_h, C\dot{w}_h^{(m)}, Dw_h^{(m)}), \quad m=(\tau, m')\in\mathscr{M}_k^*,$$

$$w_h^{(m)} = \varphi_h^{(m)} + \delta_h^{(m)}, \quad m\in\mathscr{M}_{0,h}.$$

We give sufficient conditions for convergence of the solution of scheme (15) to the solution of problem (7).

We introduce first the following:

Assumption $\mathrm{H}_1$. Suppose that:

(i) $\Phi_h \in \mathscr{F}(\Sigma_h, \mathbf{R})$ satisfies the Volterra condition,

(ii) $\Phi_h(x^{(m)}, y^{(m)}, p, w_h, q, r)$ is differentiable with respect to $(p, q, r)$, and $|D_p\Phi_h| \leqslant L_0$, $|D_{q_i}\Phi_h| \leqslant L_1$, $|D_{r_{ij}}\Phi_h| \leqslant L_2$ on $\Sigma_h$,

(iii) $D_{r_{ij}}\Phi_h = D_{r_{ji}}\Phi_h$ on $\Sigma_h$,

(iv) the constant $L_3$ is such that

(16) $$|\Phi_h(x^{(m)}, y^{(m)}, p, w_h+\bar{w}_h, q, r) - \Phi_h(x^{(m)}, y^{(m)}, p, w_h, q, r)| \leqslant L_3 H(y^{(m)}, M, N)\,\|[\bar{w}_h]_h\|(x^{(m)})$$

for $(x^{(m)}, y^{(m)}, p, w_h, q, r)$, $(x^{(m)}, y^{(m)}, p, w_h+\bar{w}_h, q, r)\in\Sigma_h$,

(v) $\Phi_h$ satisfies the following growth condition:

(17) $$|\Phi_h(x^{(m)}, y^{(m)}, 0, 0, 0, 0)| \leqslant H(y^{(m)}, M, K), \quad m\in\mathscr{M}_h^*,$$

(vi) for $(k, h)\in I_{\mathrm{disc}}$, $m=(\tau, m')\in\mathscr{M}_h^*$, $s\in S_\lambda$, $P_m \doteq (x^{(m)}, y^{(m)}, p, w_h, q, r)\in\Sigma_h$ we have

(18) $$a_s(m) + kb_s(m)D_p\Phi_h(P_m) + \sum_{i=1}^{n}\frac{k}{h}c_s^{(i)}(m)D_{q_i}\Phi_h(P_m) + \sum_{i,j=1}^{n}\frac{k}{h^2}d_s^{(i,j)}(m)D_{r_{ij}}\Phi_h(P_m) \geqslant 0.$$

Remark. $\Phi_h$ is usually defined from $f$. In [1], $\Phi_h = f$. Using the Taylor expansion of $f$ and some extension operators $[\cdot]_h$ we can obtain $\Phi_h \neq f$ too.

Conditions (i)–(v) of Assumption $H_1$ often follow from (9)–(12). Without loss of generality we can assume that the constants $L_0, L_1, L_2, L_3$ are the same as in (9)–(12). Using (8), (16)–(18) one can verify that a solution of problem (15) belongs to $\mathscr{F}_{N,S}(E_{0,h}\cup E_h, \mathbf{R})$.

Now we define $F_h\colon Z_{0,n^*-1}\times\mathscr{F}_{N,S}(E_{0,h}\cup E_h, \mathbf{R})\to\mathscr{F}_{N,S}(E_{0,h}\cup E_h, \mathbf{R})$ by

$$(19)\qquad F_h(\tau, w_h)(m') = Aw_h^{(m)}+k\Phi_h(x^{(m)}, y^{(m)}, Bw_h^{(m)}, w_h, Cw_h^{(m)}, Dw_h^{(m)})$$

for $m=(\tau, m')\in\mathscr{M}_h^*$ and $w_h\in\mathscr{F}_{N,S}(E_{0,h}\cup E_h, \mathbf{R})$. (Cf. $F$ in Lemma 2.)

We search for a function

$$Q_h\colon Z_{0,n^*-1}\times\mathscr{F}_{N,S}(E_{0,h}\cup E_h, \mathbf{R}^+)\to\mathscr{F}_{N,S}(E_{0,h}\cup E_h, \mathbf{R}^+)$$

which estimates the difference

$$\Delta F_h^{(m)} = |F_h(\tau, w_h+\bar{w}_h)(m')-F_h(\tau, w_h)(m')|$$

for $m=(\tau, m')\in\mathscr{M}_h^*$ and $w_h, \bar{w}_h\in\mathscr{F}_{N,S}(E_{0,h}\cup E_h, \mathbf{R})$.

From the definition of $\Delta F_h^{(m)}$ we have

$$\begin{aligned}(20)\qquad \Delta F_h^{(m)} \leqslant\ & \big|A\bar{w}_h^{(m)}+k\big[\Phi_h(x^{(m)}, y^{(m)}, B(w_h+\bar{w}_h)^{(m)}, w_h+\bar{w}_h, C(w_h+\bar{w}_h)^{(m)},\\ & D(w_h+\bar{w}_h)^{(m)}) - \Phi_h(x^{(m)}, y^{(m)}, Bw_h^{(m)}, w_h+\bar{w}_h, Cw_h^{(m)}, Dw_h^{(m)})\big]\big|\\ & +k|\Phi_h(x^{(m)}, y^{(m)}, Bw_h, w_h+\bar{w}_h, Cw_h, Dw_h)\\ & -\Phi_h(x^{(m)}, y^{(m)}, Bw_h^{(m)}, w_h, Cw_h^{(m)}, Dw_h^{(m)})|.\end{aligned}$$

Using the mean value theorem and Assumption $H_1$ we obtain from (20)

$$\begin{aligned}(21)\qquad \Delta F_h^{(m)} \leqslant\ & \sum_{s\in S_\lambda}|\bar{w}_h^{(\tau,m'+s)}|\bigg(a_s(m)+kb_s(m)D_p\Phi_h(P_m)\\ & +\sum_{i=1}^n\frac{k}{h}c_s^{(i)}(m)D_{q_i}\Phi_h(P_m)+\sum_{i,j=1}^n\frac{k}{h^2}d_s^{(i,j)}(m)D_{r_{ij}}\Phi_h(P_m)\bigg)\\ & +kL_3H(y^{(m)}, M, N)\|[\bar{w}_h]_h\|(x^{(m)}),\end{aligned}$$

where $P_m=(x^{(m)}, y^{(m)}, p, w_h+\bar{w}_h, q, r)$ is an intermediate point.

We define $Q_h(\tau, |\bar{w}_h|)(m')$ as the right-hand side of formula (21). Condition (18) implies that $Q_h$ is non-decreasing with respect to the functional variable. Obviously $Q_h$ satisfies the Volterra condition.

Now we formulate some conditions which imply the consistency of scheme (15) with problem (7). We shall need the following:

ASSUMPTION $H_2$. Suppose that:

(i) $|\Phi_h(x^{(m)}, y^{(m)}, Bw_h^{(m)}, w_h, Cw_h^{(m)}, Dw_h^{(m)})-f(x^{(m)}, y^{(m)}, Bw_h^{(m)}, w_h, Cw_h^{(m)}, Dw_h^{(m)})|\leqslant\bar{\gamma}_hH(y^{(m)}, M, N)$ for $m\in\mathscr{M}_h^*$ and for $w\in C(E_0\cup E, \mathbf{R})$ such that $w_h\in\mathscr{F}_{N,S}(E_{0,h}\cup E_h, \mathbf{R})$, and $\lim_{h\to 0}\bar{\gamma}_h=0$,

(ii) the operators $A$, $B$ defined by (13) satisfy

$$\sum_{s\in S_\lambda} a_s(m) = \sum_{s\in S_\lambda} b_s(m) = 1,$$
(22)
$$\sum_{s\in S_\lambda} a_s(m)s_i = \sum_{s\in S_\lambda} a_s(m)s_i s_j = \sum_{s\in S_\lambda} a_s(m)s_i s_j s_l = \sum_{s\in S_\lambda} b_s(m)s_i = 0$$

for $i, j, l = 1, \ldots, n$, $m \in \mathcal{M}_h^*$,

(iii) the operators $C$, $D$ satisfy

(23)
$$\sum_{s\in S_\lambda} c_s^{(i)}(m) = \sum_{s\in S_\lambda} c_s^{(i)}(m)s_j s_l = \sum_{s\in S_\lambda} d_s^{(i,j)}(m) = \sum_{s\in S_\lambda} d_s^{(i,j)}(m)s_l = \sum_{s\in S_\lambda} d_s^{(i,j)}(m)s_{i'}s_{j'}s_l = 0$$

for $i, j, l, i', j' = 1, \ldots, n$,

(24)
$$\sum_{s\in S_\lambda} c_s^{(i)}(m)s_j = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n,$$

(25)
$$\sum_{s\in S_\lambda} d_s^{(i,j)}(m)s_l s_v = \begin{cases}\delta_{il}\delta_{jv}, & i \neq j,\\ 2\delta_{il}\delta_{jv}, & i = j,\end{cases}$$

for $i, j, l, v = 1, \ldots, n$, $m \in \mathcal{M}_h^*$,

(iv) there is a constant $c_0 \geqslant 0$ (independent of $(k, h) \in I_{\mathrm{disc}}$ and $m \in \mathcal{M}_h^*$) such that

(26)
$$|a_s(m)|,\ |b_s(m)|,\ |c_s^{(i)}(m)|,\ |d_s^{(i,j)}(m)| \leqslant c_0$$

for $s \in S_\lambda$, $m \in \mathcal{M}_h^*$, $i, j = 1, \ldots, n$,

(v) the interpolation operator $[\cdot]_h$ satisfies

$$\|[w_h]_h - w\|(x) \leqslant \gamma_h^* \|w\|(x), \quad \text{and} \quad \gamma_h^* \to 0 \text{ as } h \to 0,$$

for $w \in C(E_0 \cup E, \boldsymbol{R})$ such that $w_h \in \mathcal{F}_{N,S}(E_{0,h} \cup E_h, \boldsymbol{R})$.

**5. Lemmas on the order of approximation, on consistency and on inequalities.** We can now give a lemma on the order of approximation of a function and its derivatives by the discrete operators $A$, $B$, $C$, $D$.

LEMMA 3. *Suppose that the solution $u$ of problem* (7) *satisfies condition* (8), *and Assumption* $\mathrm{H}_2$ *is satisfied. Then for $m = (\tau, m') \in \mathcal{M}_h^*$, $i, j = 1, \ldots, n$, we have*

$$|u_h^{(\tau+1,m')} - Au_h^{(m)} - kD_x u_h^{(m)}|$$
$(27_1)$
$$\leqslant \left[\frac{k^2}{2} + \frac{h^3}{6}\lambda^3 n^3 \exp\left(\frac{NK}{N-K}\lambda^2 h^2\right) \sum_{s\in S_{\lambda,0}} |a_s(m)|\right] H(y^{(m)}, M, N),$$

$$|Bu_h^{(m)} - u_h^{(m)}| \leqslant \frac{h^2}{2}\lambda^2 n^2 \exp\left(\frac{NK}{N-K}\lambda^2 h^2\right) \sum_{s\in S_{\lambda,0}} |b_s(m)|\, H(y^{(m)}, M, N),$$

$$(27_2)\quad |C_i u_h^{(m)} - D_{y_i} u_h^{(m)}| \leqslant \frac{h^2}{6}\lambda^3 n^3 \exp\left(\frac{NK}{N-K}\lambda^2 h^2\right) \sum_{s\in S_{\lambda,0}} |c_s^{(i)}(m)|\, H(y^{(m)}, M, N),$$

$$|D_{ij} u_h^{(m)} - D_{y_i y_j} u_h^{(m)}| \leqslant \frac{h}{6}\lambda^3 n^3 \exp\left(\frac{NK}{N-K}\lambda^2 h^2\right) \sum_{s\in S_{\lambda,0}} |d_s^{(i,j)}(m)|\, H(y^{(m)}, M, N),$$

*where* $S_{\lambda,0} = S_\lambda \setminus \{0\}$.

Proof. The estimates in (27) are easily obtained from Assumption $H_2$ using the Taylor theorem.

The above lemma is similar to Lemma 2 from [1].

Let $u$ be a solution of problem (7) which satisfies (8). Now we prove a lemma on consistency.

LEMMA 4. *Suppose that Assumptions* $H_1$, $H_2$ *are satisfied. Then there is* $\gamma_h$ *such that* $\lim_{h\to 0}\gamma_h = 0$ *and*

$$(28)\quad |u_h^{(\tau+1,m')} - F_h(\tau, u_h)(m')| \leqslant k\gamma_h H(y^{(m)}, M, N), \qquad m = (\tau, m') \in \mathscr{M}_h^*.$$

Proof. Take $m = (\tau, m') \in \mathscr{M}_h^*$. We have

$$\begin{aligned}
&|u_h^{(\tau+1,m')} - F_h(\tau, u_h)(m')| \\
&\qquad \leqslant |u_h^{(\tau+1,m')} - Au_h^{(m)} - kf(x^{(m)}, y^{(m)}, Bu_h^{(m)}, u, Cu_h^{(m)}, Du_h^{(m)}) \\
&\qquad\quad - k[D_x u_h^{(m)} - f(x^{(m)}, y^{(m)}, u_h^{(m)}, u, D_y u_h^{(m)}, D_{yy} u_h^{(m)})]| \\
&\qquad\quad + k|f(x^{(m)}, y^{(m)}, Bu_h^{(m)}, u, Cu_h^{(m)}, Du_h^{(m)}) \\
&\qquad\quad - \Phi_h(x^{(m)}, y^{(m)}, Bu_h^{(m)}, u_h, Cu_h^{(m)}, Du_h^{(m)})| \\
&\qquad \leqslant |u_h^{(\tau+1,m')} - Au_h^{(m)} - kD_x u_h^{(m)}| + kL_0 |Bu_h^{(m)} - u_h^{(m)}| \\
&\qquad\quad + kL_1 \sum_{i=1}^{n} |C_i u_h^{(m)} - D_{y_i} u_h^{(m)}| + kL_2 \sum_{i,j=1}^{n} |D_{ij} u_h^{(m)} - D_{y_i y_j} u_h^{(m)}| \\
&\qquad\quad + k\bar{\gamma}_h H(y^{(m)}, M, N).
\end{aligned}$$

This by Lemma 3 implies (28) with $\gamma_h$ given by

$$\begin{aligned}
(29)\quad \gamma_h = \frac{k}{2} + h\lambda^2 n^2 \exp\left(\frac{NK}{N-K}\lambda^2 n^2\right) \sum_{s\in S_{\lambda,0}} \Bigg(&\frac{\lambda n}{6g}|a_s(m)| + \frac{h}{2}L_0 |b_s(m)| \\
&+ \frac{h}{6}L_1 \lambda n \sum_{i=1}^{n} |c_s^{(i)}(m)| + \tfrac{1}{6}L_2 \lambda n \sum_{i,j=1}^{n} |d_s^{(i,j)}(m)|\Bigg) + \bar{\gamma}_h,
\end{aligned}$$

and the proof of Lemma 4 is complete.

Now we define $z_h \in \mathscr{F}_{N,S}(E_{0,h} \cup E_h, R^+)$ by

$$(30)\qquad z_h^{(m)} = \mu_h H(y^{(m)}, M, N), \qquad m \in \mathscr{M}_{0,h},$$

$$(31)\qquad z_h^{(m)} = \mu_h M \sum_{\nu=1}^{n} \exp[Nh^2 m_\nu^2 e^{Sk\tau} + Lk\tau], \qquad m = (\tau, m') \in \bar{\mathscr{M}}_h,$$

where $L > 0$ is some constant, $\mu_h = \max\{\gamma_h, \tilde{\gamma}_h\}$, $\gamma_h$ is defined by (29) and $\tilde{\gamma}_h$ satisfies (14). The constants $M$, $N$, $S$ are fixed from Section 3 on.

We shall give conditions which imply that $z_h$ satisfies

$$z_h^{(\tau+1,m')} \geqslant Q_h(\tau, z_h)(m') + k\gamma_h H(y^{(m)}, M, N), \quad m = (\tau, m') \in \mathcal{M}_h^*, \tag{32}$$

$$z_h^{(m)} \geqslant |\delta_h^{(m)}|, \quad m \in \mathcal{M}_{0,h}. \tag{33}$$

Compare (32) and (33) with (4) and (5).

LEMMA 5. *Suppose that:*

(A) *Assumptions* $\mathrm{H}_1$, $\mathrm{H}_2$ *are satisfied and* $z_h$ *is defined by* (30)–(31) *where* $\tilde{\gamma}_h$ *satisfies* (14), *and* $\lim_{h\to 0} \tilde{\gamma}_h = 0$,

(B) *the constants* $a$, $S$, $L$, $N$ *satisfy for* $(k, h) \in I_{\mathrm{disc}}$ *the conditions*

$$NS \geqslant \max\{1+\xi_1,\ 1+\sqrt{\zeta_4},\ \ln(1+\xi+\tfrac{1}{2}\zeta\exp\mathcal{D}),\ \ln(\xi+\tfrac{1}{4}\zeta(\mathcal{D}+4)\exp\mathcal{D}),$$
$$\ln(\tfrac{1}{8}\zeta(\mathcal{D}+4)(\mathcal{D}+3)\exp\mathcal{D}),\ \tfrac{1}{4}\mathcal{D}^2+\tfrac{1}{32}\zeta(\mathcal{D}+4)(\mathcal{D}+3)\}, \tag{34}$$

$$L \geqslant \max\{\tfrac{1}{4}\xi_1^2+\xi_0,\ \tfrac{1}{2}\zeta_2+3P^2,\ (\zeta_0+P^4)^{1/2}\}, \tag{35}$$

*where*

$$\mathcal{D} \doteq \frac{2}{\sqrt{g}} Ne^{Sa}; \quad P = \max\{\tfrac{1}{4}\zeta_3, \sqrt[3]{\tfrac{1}{4}\zeta_1}\}; \quad \xi = \sum_{i=0}^{2} \xi_i; \quad \zeta = \sum_{i=0}^{4} \zeta_i;$$

$$\xi_0 = L_0 + L_3 + 2L_2 Ne^{Sa}; \quad \xi_1 = 2L_1 Ne^{Sa}; \quad \xi_2 = 4L_2 N^2 e^{2Sa};$$

$$\zeta_0 = \frac{\lambda^2}{g}\exp[Ne^{Sa}h_0\lambda]\Big\{N^2e^{2Sa}\Big(\bar{a}+\frac{1}{g}L_2\lambda\bar{d}\Big)+2L_0\bar{b}Ne^{Sa}+h_0\lambda 4L_1\lambda\bar{c}N^2e^{2Sa}$$
$$+h_0^2\lambda^2\Big[4N^3e^{3Sa}\Big(\bar{a}+\frac{1}{g}L_2\lambda\bar{d}\Big)+4L_0\bar{b}N^2e^{2Sa}\Big]+h_0^3\lambda^3\tfrac{8}{3}L_1\lambda\bar{c}N^3e^{3Sa}$$
$$+h_0^4\lambda^4\tfrac{4}{3}N^4e^{4Sa}\Big(\bar{a}+\frac{1}{g}L_2\lambda\bar{d}\Big)\Big\};$$

$$\zeta_1 = 4L_1\lambda\bar{c}N^2e^{2Sa}+2h_0\lambda\Big[4N^3e^{3Sa}\Big(\bar{a}+\frac{1}{g}L_2\lambda\bar{d}\Big)+4L_0\bar{b}N^2e^{2Sa}\Big]$$
$$+3h_0^2\lambda^2\tfrac{8}{3}L_1\lambda\bar{c}N^3e^{3Sa}+4h_0^3\lambda^3\tfrac{4}{3}N^4e^{4Sa}\Big(\bar{a}+\frac{1}{g}L_2\lambda\bar{d}\Big);$$

$$\zeta_2 = 4N^3e^{3Sa}\Big(\bar{a}+\frac{1}{g}L_2\lambda\bar{d}\Big)+4L_0\bar{b}N^2e^{2Sa}+3h_0\lambda\tfrac{8}{3}L_1\lambda\bar{c}N^3e^{3Sa}$$
$$+6h_0^2\lambda^2\tfrac{4}{3}N^4e^{4Sa}\Big(\bar{a}+\frac{1}{g}L_2\lambda\bar{d}\Big);$$

$$\zeta_3 = \tfrac{8}{3}L_1\lambda\bar{c}N^3e^{3Sa}+4h_0\lambda\tfrac{4}{3}N^4e^{4Sa}\Big(\bar{a}+\frac{1}{g}L_2\lambda\bar{d}\Big); \quad \zeta_4 = \tfrac{4}{3}N^4e^{4Sa}\Big(\bar{a}+\frac{1}{g}L_2\lambda\bar{d}\Big);$$

$$\bar{a} = \sup_m \sum_{s \in S_{\lambda,0}} |a_s(m)|; \qquad \bar{b} = \sup_m \sum_{s \in S_{\lambda,0}} |b_s(m)|;$$

$$\bar{c} = \sup_m \sum_{s \in S_{\lambda,0}} \sum_{i=1}^{n} |c_s^{(i)}(m)|; \qquad \bar{d} = \sup_m \sum_{s \in S_{\lambda,0}} \sum_{i,j=1}^{n} |d_s^{(i,j)}(m)|.$$

*Then* $z_h$ *satisfies* (32) *and* (33).

Remark. Notice that (34), (35) are satisfied when $a$ is sufficiently small. $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ exist by (26).

Proof of Lemma 5. By the definition of $\mu_h$, $\mu_h \geqslant \tilde{\gamma}_h$. Hence

$$z_h^{(m)} \geqslant \tilde{\gamma}_h H(y^{(m)}, M, N) \geqslant |\delta_h^{(m)}|,$$

and we see that (33) is satisfied.

Next, we prove (32). Let $m = (\tau, m') \in \mathcal{M}_h^*$. First we prove that

$$\exp[k(NSy^2+L)] \geqslant 1 + k(\xi_0+\xi_1 y+\xi_2 y^2) + \tfrac{1}{2}k^2(\zeta_0+\ldots+\zeta_4 y^4)\exp(\mathcal{D}y) \tag{36}$$

for $(k, h) \in I_{\text{disc}}$ and $0 \leqslant y$.

Let $0 \leqslant y \leqslant 1/\sqrt{k}$. Then from (34), (35) it follows that

$$(N^2S^2-\tilde{\zeta}_4)y^4 - \tilde{\zeta}_3 y^3 + (2NSL-\tilde{\zeta}_2)y^2 - \tilde{\zeta}_1 y + L^2 - \tilde{\zeta}_0 \geqslant y^4 - 4Py^3 + 6P^2y^2 - 4P^3y + P^4 \geqslant 0,$$

where $\tilde{\zeta}_i = \zeta_i \exp \mathcal{D}$, $i = 0, \ldots, 4$, and we obtain

$$(NSy^2+L)^2 \geqslant \tilde{\zeta}_0 + \ldots + \tilde{\zeta}_4 y^4. \tag{37}$$

(34) and (35) also imply

$$NSy^2 + L \geqslant \xi_0 + \xi_1 y + \xi_2 y^2. \tag{38}$$

Indeed, it is sufficient to verify that for $T(y) = (NS-\xi_2)y^2 - \xi_1 y + L - \xi_0$ the coefficient of $y^2$ is positive and the discriminant is non-positive.

Applying (37), (38) and the inequality $\exp(k\varkappa) \geqslant 1 + k\varkappa + \frac{1}{2}\varkappa^2$ for $\varkappa \geqslant 0$ we obtain

$$\exp[k(NSy^2+L)] \geqslant 1 + k(\xi_0+\xi_1 y+\xi_2 y^2) + \tfrac{1}{2}k^2(\tilde{\zeta}_0+\ldots+\tilde{\zeta}_4 y^4). \tag{39}$$

For $0 \leqslant y \leqslant 1/\sqrt{k}$, (36) follows easily from (39). Now, let $1/\sqrt{k} \leqslant y$ or $\eta := \sqrt{k}y \geqslant 1$. One can prove that the function

$$\vartheta(\eta) := \exp(NS\eta) - 1 - \eta^2\zeta - \tfrac{1}{2}\zeta\eta^4\exp(\mathcal{D}\eta), \qquad \eta \geqslant 1,$$

satisfies (by (34), (35))

$$\vartheta(1) \geqslant 0, \qquad \vartheta'(\eta) \geqslant 0 \qquad \text{for } \eta \geqslant 1,$$

which implies $\vartheta(\eta) \geqslant 0$ for $\eta \geqslant 1$, and (36) holds for every $y \geqslant 0$.

Now let

$$m = (\tau, m') \in \mathscr{M}_h^*, \qquad P_m = (x^{(m)}, y^{(m)}, p, w_h, q, r) \in \Sigma_h,$$
$$Y_{m,\nu} = h(|m_\nu| + \lambda) \quad \text{for } \nu = 1, \ldots, n.$$

By Taylor's formula and Assumptions $H_1$, $H_2$ it is easy to see that

$$(40) \quad \sum_{s \in S_\lambda} \exp[Nh^2(m_\nu + s_\nu)^2 e^{Sk\tau}] a_s(m) \leqslant \exp[Nh^2 m_\nu^2 e^{Sk\tau}] + k^2 \frac{1}{24g} \lambda^4 \sum_{s \in S_{\lambda,0}} |a_s(m)| \exp[NY_{m,\nu}^2 e^{Sk\tau}] 4Ne^{Sk\tau}[3 + 12Ne^{Sk\tau} Y_{m,\nu}^2 + 4N^2 e^{2Sk\tau} Y_{m,\nu}^4],$$

$$(41) \quad \sum_{s \in S_\lambda} \exp[Nh^2(m_\nu + s_\nu)^2 e^{Sk\tau}] b_s(m) D_p \Phi_h(P_m) \leqslant L_0 \exp[Nh^2 m_\nu^2 e^{Sk\tau}] + kL_0 \lambda^2 \frac{1}{2g} \sum_{s \in S_{\lambda,0}} |b_s(m)| \exp[NY_{m,\nu}^2 e^{Sk\tau}] 2Ne^{Sk\tau}[1 + 2Ne^{Sk\tau} Y_{m,\nu}^2],$$

$$(42) \quad \sum_{s \in S_\lambda} \frac{1}{h} \exp[Nh^2(m_\nu + s_\nu)^2 e^{Sk\tau}] c_s^{(i)}(m) D_{q_i} \Phi_h(P_m) \leqslant \delta_{i\nu} L_1 \exp[Nh^2 m_\nu^2 e^{Sk\tau}] 2Ne^{Sk\tau} h|m_\nu| + kL_1 \lambda^3 \frac{1}{6g} \sum_{s \in S_\lambda} |c_s^{(i)}(m)| \exp[NY_{m,\nu}^2 e^{Sk\tau}] 4N^2 e^{2Sk\tau}[3Y_{m,\nu} + 2Ne^{Sk\tau} Y_{m,\nu}^3],$$

$$(43) \quad \sum_{s \in S_\lambda} \frac{1}{h^2} \exp[Nh^2 m_\nu^2 e^{Sk\tau}] d_s^{(i,j)}(m) D_{r_{ij}} \Phi_h(P_m) \leqslant \delta_{i\nu} \delta_{j\nu} L_2 \exp[Nh^2 m_\nu^2 e^{Sk\tau}] 2Ne^{Sk\tau}[1 + 2Ne^{Sk\tau} h^2 m_\nu^2] + kL_2 \frac{1}{24g^2} \lambda^4 \sum_{s \in S_{\lambda,0}} |d_s^{(i,j)}(m)| \exp[NY_{m,\nu}^2 e^{Sk\tau}] 4Ne^{Sk\tau}[3 + 12Ne^{Sk\tau} Y_{m,\nu}^2 + 4N^2 e^{2Sk\tau} Y_{m,\nu}^4]$$

for $i, j, \nu = 1, \ldots, n$.

Combining (40)–(43) with (36), where $y = hm_\nu$, $Y = |y| + h\lambda$, we get

$$(44) \quad \exp[Nh^2 m_\nu^2 e^{Sk(\tau+1)} + Lk(\tau+1)] \geqslant \sum_{s \in S_\lambda} \exp[Nh^2(m_\nu + s_\nu)^2 e^{Sk\tau}] \Big( a_s(m) + kb_s(m) D_p \Phi_h(P_m) + \sum_{i=1}^{n} \frac{k}{h} c_s^{(i)}(m) D_{q_i} \Phi_h(P_m) + \sum_{i,j=1}^{n} \frac{k}{h^2} d_s^{(i,j)}(m) D_{r_{ij}} \Phi_h(P_m) \Big) + k \exp[Nh^2 m_\nu^2] + kL_3 \exp[Nh^2 m_\nu^2 + Lk\tau]$$

for $\nu = 1, \ldots, n$, $m = (\tau, m') \in \mathscr{M}_h^*$, $P_m = (x^{(m)}, y^{(m)}, p, w_h, q, r)$.

Adding (44) for $\nu = 1, \ldots, n$ and multiplying by $\mu_h M$ we check at once that (32) holds, and this completes the proof of Lemma 5.

**6. Main result.** Now, we can formulate our main result.

THEOREM 1. *Suppose that*:

(i) *the assumptions of Lemma* 5 *are satisfied*,
(ii) $u$ *is a solution of problem* (7) *satisfying* (8),
(iii) $v_h$ *is a solution of scheme* (15).

*Then*

$$|u_h^{(m)} - v_h^{(m)}| \leqslant z_h^{(m)}, \qquad m \in \mathcal{M}_h \ (h \in I_{\text{disc}}), \tag{45}$$

*and* $z_h^{(m)}$ *tends to* 0 *almost uniformly as* $h \to 0$.

Proof. Lemmas 4 and 5 imply that the assumptions of Lemma 2 are satisfied if we replace $F$ by $F_h$, $Q$ by $Q_h$, $z$ by $z_h$ etc. (where $z_h$ is defined by (30)–(31)). By Lemma 2 we have (45).

The almost uniform convergence follows from the obvious fact that $\mu_h \to 0$ as $h \to 0$. The proof of Theorem 1 is complete.

Remark. The above result can be easily extended to the Cauchy problem for a weakly coupled system of the form

$$D_x w_\nu(x, y) = f_\nu(x, y, w(x, y), w, D_y w_\nu(x, y), D_{yy} w_\nu(x, y)),$$
$$(x, y) \in E, \ \nu = 1, \ldots, l,$$
$$w(x, y) = \varphi(x, y), \qquad (x, y) \in E_0,$$

where $w = (w_1, \ldots, w_l)$, $\varphi = (\varphi_1, \ldots, \varphi_l)$.

**7. Examples.** Now we shall give two examples illustrating the method applied in this paper.

EXAMPLE 1. Consider the Cauchy problem

$$D_x w(x, y) = -w(x, y) + D_y w(x, y) + D_{yy} w(x, y), \qquad (x, y) \in E = (0, a] \times \boldsymbol{R},$$
$$w(0, y) = \varphi(0, y), \qquad y \in \boldsymbol{R},$$

and the corresponding difference scheme

$$w_h^{(\tau+1, m')} = A w_h^{(m)} - kB w_h^{(m)} + k\Delta w_h^{(m)} + kD w_h^{(m)}, \qquad m = (\tau, m') \in \mathcal{M}_h^*,$$
$$w_h(0, y^{(m)}) = \varphi(0, y^{(m)}) + \delta_h^{(m)},$$

where

$$|\delta_h^{(m)}| \leqslant h \exp[N h^2 m'^2], \qquad |\varphi(0, y)| \leqslant \exp[y^2],$$

and

$$Aw_h^{(m)} = 4w_h^{(m)} - 2[w_h^{(\tau,m'+1)} + w_h^{(\tau,m'-1)}] + \tfrac{1}{2}[w_h^{(\tau,m'+2)} + w_h^{(\tau,m'-2)}],$$

$$Bw_h^{(m)} = \tfrac{1}{2}w_h^{(m)} + \tfrac{1}{4}[w_h^{(\tau,m'+1)} + w_h^{(\tau,m'-1)}], \qquad \Delta w_h^{(m)} = \frac{1}{2h}[w_h^{(\tau,m'+1)} - w_h^{(\tau,m'-1)}],$$

$$Dw_h^{(m)} = \frac{1}{h^2}\{-\tfrac{32}{9}w_h^{(m)} + \tfrac{25}{12}[w_h^{(\tau,m'+1)} + w_h^{(\tau,m'-1)}] - \tfrac{1}{3}[w_h^{(\tau,m'+2)} + w_h^{(\tau,m'-2)}]$$
$$+ \tfrac{1}{36}[w_h^{(\tau,m'+3)} + w_h^{(\tau,m'-3)}]\}.$$

Suppose that $k/h^2 = 1$, $h \leqslant \frac{1}{24}$ for $(k, h) \in I_{\mathrm{disc}}$. Then (18) holds and the operators $A$, $B$, $C = \Delta$ and $D$ satisfy (22)–(25).

EXAMPLE 2. To illustrate the generality of our method we shall find a difference scheme for the following simple non-linear problem:

$$D_x u(x, y) = 2D_{y_1y_1}u(x, y) + 2D_{y_2y_2}u(x, y) + \sin D_{y_1y_2}u(x, y),$$
$$(x, y) \in E = [0, a] \times \boldsymbol{R}^2,$$
$$u(x, y) = \varphi(y), \qquad y \in \boldsymbol{R}^2,$$

where $\varphi$ is such that a solution of this problem satisfies the assumptions given in the paper. The following scheme was applied in [1], [8] and other papers:

$$v_h^{(\tau+1,m')} = v_h^{(m)} + k[2\Delta_{11}^{(2)}v_h^{(m)} + 2\Delta_{22}^{(m)}v_h^{(m)} + \sin \Delta_{12}^{(2)}v_h^{(m)}],$$
$$(\tau, m') = (\tau, m_1, m_2) \in \mathcal{M}_h^*,$$
$$v_h^{(m)} = \varphi(y^{(m)}), \qquad m = (0, m') \in \mathcal{M}_{0,h},$$

where

$$\Delta_{ii}^{(2)}v_h^{(m)} = h^{-2}[v_h^{i(m)} - 2v_h^{(m)} + v_h^{-i(m)}], \qquad i = 1, 2,\ m \in \mathcal{M}_h^*,$$
$$1(m) = (\tau, m_1, m_2), \quad -1(m) = (\tau, m_1 - 1, m_2), \quad 2(m) = (\tau, m_1, m_2 + 1),$$
$$-2(m) = (\tau, m_1, m_2 - 1), \qquad m = (\tau, m_1, m_2),$$

and $\Delta_{12}^{(2)}$ is one of two operators approximating a mixed derivative of second order (see [1], [7], [8]). This usual scheme does not converge because it is not stable when $D_{r_{12}}f = \cos r_{12}$ is not always positive or always negative. But there is a convergent scheme, namely

$$v_h^{(\tau+1,m')} = Av_h^{(m)} + k[2\Delta_{11}^{(2)}v_h^{(m)} + 2\Delta_{22}^{(2)}v_h^{(m)} + \sin D_{12}v_h^{(m)}],$$
$$(\tau, m') = (\tau, m_1, m_2) \in \mathcal{M}_h^*,$$
$$v_h^{(m)} = \varphi(y^{(m)}), \qquad m = (0, m') \in \mathcal{M}_{0,h},$$

where

$$Av_h^{(m)} = \tfrac{5}{4}v_h^{(m)} - \tfrac{1}{8}[v_h^{1(m)} + v_h^{-1(m)} + v_h^{2(m)} + v_h^{-2(m)}]$$
$$+ \tfrac{1}{16}[v_h^{1(2(m))} + v_h^{-1(-2(m))} + v_h^{-1(2(m))} + v_h^{1(-2(m))}],$$
$$D_{12}v_h^{(m)} = \tfrac{1}{4}h^{-2}[v_h^{1(2(m))} + v^{-1(-2(m))h} - v_h^{-1(2(m))h} - v_h^{1(-2(m))}],$$

for $m \in \mathcal{M}_h^*$.

A solution of this scheme converges to a solution of the differential problem if we assume that $kh^{-2} = \frac{1}{16}$. This follows easily from Theorem 1.

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