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Periodic solutions of $x''+f(x)x'^{2n}+g(x) = 0$ with arbitrarily large period*

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1. Recently in this journal Sędziwy [4] has found sufficient conditions for the existence of periodic solutions of period ω of

$$x''+f(x)x'^{2n}+g(x) = \mu p(t), \tag{1.1}$$

where f, g, and p are continuous for all x or t, $p(t) = p(t+\omega)$, $\omega > 0$, μ is sufficiently small, and $n \geqslant 1$. His approach is to consider the autonomous equation

$$x''+f(x)x'^{2n}+g(x) = 0 \tag{1.2}$$

and use the following lemma of Berstein and Halanay [1]:

LEMMA. *If* (1.2) *has a periodic solution with period* ω_0 *and if* $\omega \neq \omega_0$, *then for* $|\mu|$ *sufficiently small* (1.1) *has a periodic solution of period* ω.

Using this lemma the problem is reduced to showing the existence of periodic solutions of (1.2) with different periods. For $n = 1$ Sędziwy proceeds by using a first integral of (1.2) and obtaining an explicit expression for the period of a periodic solution. For $n > 1$, there is no first integral and Sędziwy's approach is to find conditions for the existence of periodic solutions with arbitrarily large, and hence different, periods.

This note considers the problem of finding in both cases $n > 1$ and $n = 1$, conditions for the existence of periodic solutions of (1.2) with arbitrarily large periods. Our result for $n > 1$ greatly improves the corresponding result of Sędziwy [4], Theorem 1. For $n = 1$ our result is independent of Sędziwy's [4], Theorem 2. The technique used here is to reduce (1.2) to a first order generalized Riccati equation and then use Ważewski's topological method.

2. (1.2) is equivalent to the system

$$x' = y, \quad y' = -f(x)y^{2n}-g(x). \tag{2.1}$$

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THEOREM 1. *Suppose that* $n > 1$ *and that*
(a) f, g *are continuous for* $-\infty < x < \infty$,
(b) *initial value problems for* (2.1) *are unique,*
(c) $xg(x) > 0$ *for* $x \neq 0, f(x) > 0$ *for all* x, *and*
(d) *there exists a function* $h(x) \in C'(-\infty, 0]$ *such that*

$$h(x) > 0,\ x \leqslant 0 \quad \textit{and} \quad \frac{dh}{dx} > -2f(x)h^n - 2g(x),\ x \leqslant 0.$$

Then equation (2.1) *has periodic solutions with arbitrarily large periods.*

Proof. First of all, note that the trajectories of (2.1) in the $x-y$ plane are symmetric with respect to the x-axis. In fact, if $(x(t), y(t))$ is a solution of (2.1), then $(x(-t), -y(-t))$ is also a solution.

In the sequel let $x(t, t_0, x_0, y_0)$, $y(t, t_0, x_0, y_0)$ be the solution of (2.1) satisfying $x(t_0, t_0, x_0, y_0) = x_0$ and $y(t_0, t_0, x_0, y_0) = y_0$.

Let $x_0 = 0$ and $y_0 > 0$. We assert that if y_0 is sufficiently small, then $x(t, 0, 0, y_0), y(t, 0, 0, y_0)$ will intersect the positive x-axis for some $t > 0$. Let $y_0 > 0$ be arbitrary. If $x(t, 0, 0, y_0), y(t, 0, 0, y_0)$ intersects the positive x-axis for some $t > 0$, it follows from uniqueness of initial value problems that $x(t, 0, 0, y), y(t, 0, 0, y)$, $0 < y < y_0$ has the same property. On the other hand, suppose that $y(t, 0, 0, y_0) > 0$ for $t > 0$. Then by (c) $y'(t, 0, 0, y_0) < 0$ and so $\lim_{t\to\infty} x(t, 0, 0, y_0) = \infty$. Let $x_0 > 0$ and consider the solution $x(t, 0, x_0, 0), y(t, 0, x_0, 0)$. By uniqueness of initial value problems and since $y' < 0$, if $y > 0$ and $x > 0$, it follows that there is a $t_1 < 0$ such that $x(t_1, 0, x_0, 0) = 0$ and $0 < y(t_1, 0, x_0, 0) < y_0$. This proves the assertion.

The trajectories of (2.1) are given by

$$y\frac{dy}{dx} = -f(x)y^{2n} - g(x). \tag{2.2}$$

Letting $z(x) = (y(x))^2$, (2.2) becomes

$$\frac{dz}{dx} = -2f(x)z^n - 2g(x). \tag{2.3}$$

Note that initial value problems for (2.3) and (2.4) below are unique since (2.1) has this property. We now want to consider solutions of (2.3) in the $x-z$ plane which enter the third quadrant from the positive z-axis. It is convenient to make the transformation $z_1(\tau) = z(x)$ and $\tau = -2x \geqslant 0$. Letting $f_1(\tau) = f(x)$ and $g_1(\tau) = -g(x)$ yields

$$\frac{dz_1}{d\tau} = f_1(\tau)(z_1)^n - g_1(\tau), \qquad \tau \geqslant 0, \tag{2.4}$$

where $f_1(\tau) > 0$, $\tau \geqslant 0$ and $g_1(\tau) > 0$, $\tau > 0$.

We now assert that if $z_1(\tau)$ is a solution of (2.4) and $z_1(0)$ is positive and sufficiently small, then z_1 has a positive zero. In fact, let $\tau_0 > 0$ and let z_{11} be the solution of (2.4) such that $z_{11}(\tau_0) = 0$. Since $g_1(\tau) > 0$ for $\tau > 0$, $z_{11}(\tau) > 0$ as far to the left on $(0, \tau_0]$ as z_{11} exists. It will now be shown that z_{11} exists on $[0, \tau_0]$. Letting $h_1(\tau) = h(x)$, where $\tau = -2x \geqslant 0$ is the above transformation, condition (d) of the hypothesis becomes

$$\frac{dh_1}{d\tau} < f_1(\tau)(h_1)^n - g_1(\tau), \qquad \tau \geqslant 0. \tag{2.5}$$

Suppose there is a τ_1, $0 \leqslant \tau_1 < \tau_0$ such that $z_{11}(\tau_1) = h_1(\tau_1)$ and let τ_1 be the first such point to the left of τ_0. Then by (2.4) and (2.5)

$$\frac{dh_1}{d\tau}(\tau_1) < \frac{dz_{11}}{\delta\tau}(\tau_1)$$

which is a contradiction to the fact that $z_{11}(\tau_0) = 0 < h_1(\tau_0)$. Therefore z_{11} exists on $[0, \tau_0]$ and $0 < z_{11}(\tau) < h_1(\tau)$ for $0 < \tau \leqslant \tau_0$. If $z_{11}(0) > 0$, then any other solution $z_1(\tau)$ such that $0 < z_1(0) < z_{11}(0)$ will have a positive zero by uniqueness of initial value problems. If $z_{11}(0) = 0$ (which is possible since $g_1(0) = 0$), let $z_{12}(\tau)$ be defined by $z_{12}(\tau_0+1) = 0$. Then z_{12} exists and is non-negative on $[0, \tau_0+1]$ by the above reasoning and $0 = z_{11}(0) < z_{12}(0)$ by uniqueness of initial value problems. This proves the assertion.

Finally it will be shown that (2.4) has a solution $z_1(\tau)$ such that z_1 exists and is positive on $[0, \infty)$. The technique is Ważewski's method (for definitions and results see [2], p. 179). Let $W = E^1 \times E^1$ and let the open set T be given by

$$T = \{(\tau, z) : 0 < \tau \text{ and } 0 < z < h_1(\tau)\}.$$

Then $\operatorname{Fr}(T) = A \cup B \cup C$ (the boundary of T in W), where

$$A = \{(\tau, h_1(\tau)) : \tau > 0\}, \qquad B = \{(\tau, 0) : \tau > 0\},$$
$$C = \{(0, z) : 0 \leqslant z \leqslant h_1(0)\}.$$

Then every point of $S = A \cup B$ is a point of strict egress since $g_1(\tau) > 0$, $\tau > 0$ and h_1 satisfies (2.5). Also no point of C is a point of egress. Let Z be given by

$$Z = \{(1, z): 0 \leqslant z \leqslant h(1)\}.$$

Then $Z \subset T \cup S$, $Z \cap S$ is a retract of S but $Z \cap S$ is no retract of Z. Therefore by Ważewski's Theorem [2], p. 181, there is a solution $z_{13}(\tau)$ such that $z_{13}(\tau) \in T$ for all $\tau \geqslant 1$ for which z_{13} exists. If $z_{13}(\tau)$ fails to exist at some finite τ_1, then $\lim_{\tau\uparrow\tau_1} z_{13}(\tau) = \pm\infty$. This is impossible by definition of T. Therefore z_{13} exists for all $\tau \geqslant 1$. But by reasoning the same way as above $z_{13}(\tau)$ exists on $[0, 1]$ and is positive on $(0, 1]$. And by uniqueness of initial value problems $0 \leqslant z_{11}(0) < z_{13}(0)$. Therefore $z_{13} > 0$ for $\tau \geqslant 0$.

It has thus been shown that (2.4) has a solution which exists and is positive for $0 \leqslant \tau < \infty$ and another solution which is positive at $\tau = 0$ and has a positive zero. Transforming back to equation (2.1) there is a solution $x(t, 0, 0, y_1), y(t, 0, 0, y_1)$ with $y_1 > 0$ which exists for all $t \leqslant 0$ and satisfies $y(t, 0, 0, y_1) > 0$ for $t \leqslant 0$. Also if $x_0 = 0$ and y_0 is sufficiently small then there exist t_1 and t_2 such that $t_1 < 0 < t_2$ and $y(t_i, 0, 0, y_0) = 0$ for $i = 1, 2$. Hence, by symmetry, $x(t, 0, 0, y_0)$, $y(t, 0, 0, y_0)$ is a periodic solution of (2.1).

From this it will follow easily that (2.1) has periodic solutions with arbitrarily large period. First suppose that $y(t, 0, 0, y_0)$, where y_0 is an arbitrary positive number, satisfies $y(t, 0, 0, y_0) = 0$ for some $t > 0$. Let $\lambda = \inf\{a :$ the solution $x(t, 0, 0, a), y(t, 0, 0, a)$ of (2.1) exists for all $t \leqslant 0$ and satisfies $y(t, 0, 0, a) > 0$ for $t \leqslant 0\}$. Then it is clear that $\lambda > 0$. Also the solution $x(t, 0, 0, \lambda), y(t, 0, 0, \lambda)$ of (2.1) exists for all $t \leqslant 0$, and satisfies $y(t, 0, 0, \lambda) > 0$ for $t \leqslant 0$ by continuous dependence on initial conditions. Let $0 < \lambda_n \uparrow \lambda$. Then $x(t, 0, 0, \lambda_n), y(t, 0, 0, \lambda_n)$ is a periodic solution of (2.1) and by continuous dependence the period of $x(t, 0, 0, \lambda_n), y(t, 0, 0, \lambda_n)$ becomes arbitrarily large as $n \to \infty$.

Now suppose there is a solution $x(t, 0, 0, y_0), y(t, 0, 0, y_0), y_0 > 0$, of (2.1) such that $y(t, 0, 0, y_0) > 0$ for $t \geqslant 0$. This solution exists for all $t \geqslant 0$ since $y'(t, 0, 0, y_0) \leqslant 0$, $t \geqslant 0$. Let $a = \inf\{b :$ the solution $x(t, 0, 0, b), y(t, 0, 0, b)$ of (2.1) exists for all $t \geqslant 0$ and satisfies $y(t, 0, 0, b) > 0$ for $t \leqslant 0\}$. Then $a > 0$ and the solution $x(t, 0, 0, a), y(t, 0, 0, a)$ exists for all $t \geqslant 0$ and satisfies $y(t, 0, 0, a) > 0, t \geqslant 0$ as above. If $\lambda \leqslant a$ proceed as in the last paragraph. If $a < \lambda$, then let $0 < a_n \uparrow a$ and proceed as in the last paragraph.

Remark. In the above theorem condition (d) in the hypothesis warrants further discussion. In fact it will now be shown that if

$$-g(x) < f(x)\Big[C_1\Big(\int_0^x g(s)\,ds\Big)^n + C_2\Big], \quad x \leqslant 0 \tag{2.5}$$

for some positive constants C_1 and C_2, then there exists a function $h(x)$ as required in the theorem. To show this let K and L be positive constants such that $C_1 < 2K^n/(K+2)$ and $C_2 < 2L^n/(K+2)$. Let

$$h(x) = K\int_0^x g(s)\,ds + L.$$

Then

$$1/(K+2)\left[\frac{dh}{dx}+2f(x)h^n+2g(x)\right]$$

$$= 1/(K+2)\left[(K+2)g(x)+2f(x)\left(K\int_0^x g(s)ds+L\right)^n\right]$$

$$\geqslant g(x)+2f(x)\left[K^n/(K+2)\left(\int_0^x g(s)ds\right)^n+L^n/(K+2)\right]$$

$$\geqslant g(x)+f(x)\left[C_1\left(\int_0^x g(s)ds\right)^n+C_2\right] > 0.$$

Therefore

$$\frac{dh}{dx} > -2f(x)h^n-2g(x), \qquad x \leqslant 0.$$

It is clear that (2.5) is generally weaker than the corresponding conditions used by Sędziwy [4] which are $\lim_{|x|\to\infty}\int_0^x g(u)du = \infty$, $\limsup_{x\to-\infty} g(x)/x = a$, and $0 < b \leqslant f(x) \leqslant c < \infty$, $\quad -\infty < x < \infty$.

THEOREM 2. *Assume that* $n = 1$ *and conditions* (a), (b), *and* (c) *of Theorem* 1 *hold. Suppose also that*

(d′) $\qquad \displaystyle\int_{-\infty}^0 \left(\exp\int_0^x 2f(u)du\right)g(x)dx > -\infty.$

Then equation (2.1) *has periodic solutions with arbitrarily large period.*

Proof. The proof is almost entirely the same as the proof of Theorem 1 except that (d′) is used instead of (d) above to show the existence of a solution $x(t,0,0,y_0), y(t,0,0,y_0), y_0 > 0$ of (2.1) which exists for all $t \leqslant 0$ and satisfies $y(t,0,0,y_0) > 0$ for $t \leqslant 0$. This is easy to establish since the transformed equation (2.3) is linear. Therefore the general solution of (2.3) is

$$z(x) = \exp\left[-\int_0^x 2f(u)du\left[c-2\int_0^x\left(\exp\int_0^u 2f(s)ds\right)g(u)du\right]\right],$$

where c is an arbitrary constant. Thus (2.3) has a solution $z(x) > 0, x \leqslant 0$ if and only if (d′) holds.

Remark. Theorem 2 establishes the existence of periodic solutions of (2.1) for $n = 1$ with distinct periods. Condition (d′) is clearly independent of the corresponding hypotheses of Sędziwy's theorem [4], Theorem 2.

Remark. It is clear that the technique of Theorem 1 will apply to the more general equation

$$x''+f(x)k(x')+g(x) = 0$$

with very mild restrictions on k in addition to the analogue of condition (d).

Remark. It is perhaps of interest to mention that the same reduction to a first order generalized Ricatti equation has been used by the author [3] to study global asymptotic stability of

$$x''+f(x)|x'|^a x'+g(x) = 0.$$

References

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