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ON AFFINE EXTENSIONS OF THE HOLONOMY HOMOMORPHISMS OF FLAT PRINCIPAL BUNDLES

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0. Introduction. In two earlier papers ([7], [8]) we have dealt with the interaction between principal bundle connections and equations with total differentials. As a matter of fact, using techniques of this particular type of equations (cf., for instance, [3]), one can shed light on the structure of a given bundle and the behavior of its connections.

The purpose of the present note is to sharpen further the previous idea and to characterize the global triviality of a flat bundle (namely one with flat connections), as well as the reduction of its connections to constant ones, through an affine extension of the corresponding holonomy homomorphisms.

More explicitly, assume that ω is a flat connection on a principal bundle with structure group G and (connected) base B, the latter being equipped with a linear connection ∇. If the holonomy homomorphism $h_\omega\colon \pi_1(B)\to G$ extends to an affine morphism $H\colon \tilde{B}\to G$ (with respect to the lifting $\tilde{\nabla}$ of ∇ in the universal covering $\tilde{B}$ of B and the direct connection of G), then we prove that

(i) the bundle is trivial and

(ii) ω reduces to a connection of the trivial bundle such that its corresponding local connection form is constant with respect to ∇.

The converse is also true.

On the other hand, if we restrict our considerations to the level of a trivial bundle, flat connections simply correspond to integrable equations with total differentials in B. As a result, condition (ii) above amounts to the transformation of an equation to one with constant coefficient (with respect to ∇). Thus we obtain, as a byproduct, Jackin's generalization of the classical theorem of Ljapunov–Floquet (on ordinary linear equations with periodic coefficients) in the context of equations with total differentials (cf. [1], [3] and Corollary 2 below). With this in mind, one could legitimately call our main result a Ljapunov–Floquet type theorem within the geometric framework of arbitrary principal bundles.

The precise statement and the proof of the main result will be given in Section 4, while the preceding sections contain the necessary preliminary material towards this. The terminology of the title will be explained in the comment following the proof of the main theorem.

1. Flat principal bundles. All of our considerations are in the category of smooth manifolds and bundles.

Let $l = (P, G, B, \pi)$ be a principal fibre bundle with *connected* base B. If $\mathfrak{g}$ is the Lie algebra of the structure group G, we denote by $\Lambda^1(M, \mathfrak{g})$ the space of $\mathfrak{g}$-valued differential 1-forms on a given manifold M.

A *connection* on l will be handled by its *connection form* $\omega \in \Lambda^1(P, \mathfrak{g})$ (cf., e.g., [2], [5]). In particular, if ω is a *flat* connection on l (this means that the corresponding curvature form vanishes identically), the pair (l, ω) will be called a *flat bundle.* We recall that a flat bundle (l, ω) is completely determined by the *holonomy homomorphism* $h_\omega\colon \pi_1(B) \to G$ of ω (cf. [4]).

For fixed B and G as above, we consider the set of all possible flat bundles with base B and group G. Two such flat bundles (l, ω) and (l', ω') are said to be *equivalent* if there exists a G-B-morphism f of l onto l' such that $f^*\omega' = \omega$. The set of all equivalence classes thus obtained is denoted by $\mathscr{H}(B, G)$.

On the other hand, we denote by $\mathscr{S}(B, G)$ the set of equivalence classes of all homomorphisms $\pi_1(B) \to G$ classified by the *conjugation* (or similarity) relationship.

As a consequence, we obtain

LEMMA 1. *The sets* $\mathscr{H}(B, G)$ *and* $\mathscr{S}(B, G)$ *are equivalent, i.e.,* $\mathscr{S}(B, G) = \mathscr{H}(B, G)$ *within a bijection.*

For an elementary geometric proof we refer to [6].

2. Equations with total differentials and connections. An extensive account of the equations with total differentials is given in [3]. We recall only a few facts needed in the sequel.

Equations considered here are of the form

$$Dx = \alpha, \tag{2.1}$$

where $\alpha \in \Lambda^1(B, \mathfrak{g})$ and the (right) *total differential*

$$D\colon \mathscr{C}^\infty(B, G) \to \Lambda^1(B, \mathfrak{g})$$

is defined by

$$(Df)_x := d_{f(x)} R_{f(x)^{-1}} \circ d_x f, \qquad x \in B,$$

with R_g denoting the right translation (by $g \in G$) of G.

Equation (2.1) is said to be (completely) *integrable* if and only if $d\alpha = \frac{1}{2}[\alpha, \alpha]$. This assures the existence of local solutions $f\colon U \to G$ ($U \subset B$ open) for given initial data.

On the other hand, if $(\tilde{B}, B, \tilde{\pi})$ is the universal covering of B, lifting (2.1) to $\tilde{B}$ we obtain the equation

$$D\tilde{x} = \tilde{\pi}^* \alpha. \tag{2.2}$$

If (2.1) is integrable, so is (2.2) and it admits global solutions on $\tilde{B}$.

We fix once for all two reference points $b_0 \in B$ and $\tilde{b}_0 \in \tilde{\pi}^{-1}(b_0)$.

For an integrable equation (2.1), its *fundamental solution* is defined to be the global solution $\Phi\colon \tilde{B} \to G$ of (2.2) with initial condition $\Phi(\tilde{b}_0) = e$. Accordingly, the *monodromy homomorphism* $\alpha^{\#}\colon \pi_1(B) \to G$ of the original equation (2.1) is given by

$$\Phi(\tilde{b} \cdot [\gamma]) = \Phi(\tilde{b}) \cdot \alpha^{\#}([\gamma]), \qquad \tilde{b} \in \tilde{B},\ [\gamma] \in \pi_1(B) \cong \pi_1(B, b_0). \tag{2.3}$$

Let now $l_0 = (B \times G, G, B, \mathrm{pr}_1)$ be the trivial bundle. If ω^0 is any flat connection on l_0, the natural section $s\colon b \mapsto (b, e)$ of l_0 determines the (unique) *local connection form* $s^*\omega^0$. Setting $\alpha_0 := -s^*\omega^0$, we may consider the integrable equation with coefficient α_0 (the minus sign is a simplifying technicality). Conversely, it is quite clear that the coefficient of an integrable equation on B can be thought of as the local connection form of a flat connection on l_0.

If h_{ω^0} is the holonomy homomorphism of ω^0, then we obtain

LEMMA 2 ([7]). *$[h_{\omega^0}] = [\alpha_0^{\#}]$ in $\mathcal{S}(B, G)$.*

3. Equations and affine morphisms. Let B and B' be two manifolds equipped with the linear connections ∇ and ∇', respectively. A smooth morphism $h\colon B \to B'$ is called *affine* (with respect to ∇ and ∇') if, for every smooth curve $\beta\colon I \to B$,

$$dh \circ \tau_\beta = \tau'_{\beta'} \circ (dh \mid T_{\beta(0)}B),$$

where τ_β and $\tau'_{\beta'}$ denote the parallel translations along β and $\beta' := h \circ \beta$ (with respect to ∇ and ∇').

For (B, ∇) as before, a differential form $\alpha \in \Lambda^1(B, \mathfrak{g})$ is said to be *constant* (with respect to ∇) if $\nabla\alpha = 0$.

Note that if $\tilde{\nabla}$ denotes the linear connection defined on $\tilde{B}$ by ∇, then $\nabla\alpha = 0$ if and only if $\tilde{\nabla}(\tilde{\pi}^*\alpha) = 0$.

In this regard, the following holds:

LEMMA 3. *A smooth mapping $h\colon \tilde{B} \to G$ is an affine morphism (with respect to $\tilde{\nabla}$ and the direct connection of G) if and only if $\tilde{\nabla}(Dh) = 0$.*

This is a particular case of Theorem 2.4.1 in [3].

4. The main result. We are in a position to give the following

THEOREM. *Let ω be a flat connection on the principal bundle $l = (P, G, B, \pi)$. If B is a connected manifold equipped with a linear connection ∇, then the following conditions are equivalent:*

(i) *The flat bundle* (l, ω) *is equivalent to* (l_0, ω^0), *where* l_0 *is the trivial bundle and* ω^0 *is a flat connection such that its corresponding local connection form is constant with respect to* ∇.

(ii) *There exists an affine morphism* $H: \tilde{B} \to G$ (*with respect to* $\tilde{\nabla}$ *and the direct connection of* G) *extending* h_ω, *i.e.*,

$$H(\tilde{b}\cdot[\gamma]) = H(\tilde{b})\cdot h_\omega([\gamma]), \qquad \tilde{b}\in\tilde{B},\ [\gamma]\in\pi_1(B).$$

Proof. Let f be a G-B-morphism of l onto l_0. If $\omega^0 = (f^{-1})^*\omega$ is the flat connection on l_0 induced by ω and f, we set $\alpha_0 := -s^*\omega^0$. Thinking of α_0 as the coefficient of an integrable equation, Lemmas 1 and 2 imply that

$$(4.1) \qquad [h_\omega] = [h_{\omega^0}] = [\alpha_0^\#].$$

Hence, there exists $g\in G$ such that

$$(4.2) \qquad \alpha_0^\# = g\cdot h_\omega \cdot g^{-1}.$$

On the other hand, since $\nabla\alpha_0 = 0$, the fundamental solution Φ of the same equation is an affine morphism (cf. Section 3). Thus, setting $H := \Phi\cdot g$ and combining formulas (2.3) and (4.2), we obtain an affine morphism as in (ii).

Conversely, assume that (ii) holds and let $H(\tilde{b}_0) = g$. It is easy to show that the form $DH\in\Lambda^1(\tilde{B}, \mathfrak{g})$ is integrable and $\pi_1(B)$-invariant. As a result, there is an integrable form $\alpha_0\in\Lambda^1(B, \mathfrak{g})$ with $DH = \tilde{\pi}^*\alpha_0$. Since $\Phi := H\cdot g^{-1}$ is the fundamental solution of the equation with coefficient α_0, the assumption implies equality (4.2), thus $[\alpha_0^\#] = [h_\omega]$. On the other hand, thinking of α_0 as the local connection form of a flat connection ω^0 on l_0 with $\alpha_0 = -s^*\omega^0$, Lemma 2 yields (4.1). Therefore, (l, ω) is equivalent to (l_0, ω^0) by Lemma 1. Finally, since H is affine, $\tilde{\nabla}(DH) = 0$ in virtue of Lemma 3. Consequently, $\nabla\alpha_0 = 0$ and this completes the proof.

Note. From the preceding proof it is clear that

$$h_\omega([\gamma]) = g^{-1}\cdot H(\tilde{b}_0\cdot[\gamma]).$$

Thus, up to a left translation (by g^{-1}), the holonomy homomorphism h_ω coincides with the restriction of the affine morphism H to the fibre $\tilde{\pi}^{-1}(b_0)$ of $\tilde{B}$. In this sense H is an extension of h_ω, thus explaining the terminology of the title and the main result.

COROLLARY 1. *Let* ω *be any connection on a principal bundle*

$$l = (P, \mathrm{GL}(F), S^1, \pi),$$

where F *is a Banach space. If* l_1 *is the trivial bundle over* S^1 *with structure group* $\mathrm{GL}(F)$, *then the following conditions are equivalent*:

(i) h_ω *extends to a homomorphism* $H: \mathbf{R}\to\mathrm{GL}(F)$ *such that* $H|\mathbf{Z} = h_\omega$.

(ii) (l, ω) *is equivalent to* (l_1, ω_1), *where* ω_1 *is a connection such that*

$$(*) \qquad (s^*\omega_1)(\partial) = 0$$

if s *is the natural section of* l_1 *and* ∂ *the basic vector field of* $\boldsymbol{R}$.

(iii) *The logarithm* $\log(h_\omega(1))$ *exists.*

Proof. Let $\boldsymbol{R}$ and S^1 be equipped with their natural connections. For the equivalence (i)$\Leftrightarrow$(ii) it suffices to remark that the affine morphisms of $\boldsymbol{R}$ into $\mathrm{GL}(\boldsymbol{F})$ coincide with the (group) homomorphisms of the same spaces, whereas equality $(*)$ amounts to the fact that $s^*\omega_1$ is constant with respect to the natural connection of S^1 (for a direct proof of the same equivalence see also [8]). The equivalence (ii)$\Leftrightarrow$(iii) can be obtained as in [7].

Let us restrict again our considerations to the trivial bundle l_0. Following the comments preceding Lemma 2, we immediately check that there exists a bijection between flat connections ω on l_0 and coefficients $\alpha \in \Lambda^1(B, \mathfrak{g})$ of integrable equations $Dx = \alpha$ on B so that $\alpha = -s^*\omega$.

As a result, if ω and ω' are two flat connections on l_0 with corresponding local connection forms α and α', (l_0, ω) and (l_0, ω') are equivalent if and only if

$$(4.3) \qquad \alpha' = (\mathrm{Ad}\,Q)\alpha + DQ,$$

where $Q\colon B \to G$ is a smooth morphism.

If (4.3) holds, we say that the forms α and α' (or the corresponding equations) are *equivalent*.

On the level of l_0, the main theorem now turns into

COROLLARY 2. *Let* $Dx = \alpha$ *be a completely integrable equation with* $\alpha \in \Lambda^1(B, \mathfrak{g})$, *where* B *admits a linear connection* ∇. *Then the following conditions are equivalent*:

(i) α *is equivalent to a constant form* $\beta \in \Lambda^1(B, \mathfrak{g})$, *i.e.*, $\nabla\beta = 0$.

(ii) *The monodromy homomorphism* $\alpha^\#$ *extends to an affine morphism* $F\colon \tilde{B} \to G$ *such that*

$$F(\tilde{b}\cdot[\gamma]) = F(\tilde{b})\cdot\alpha^\#([\gamma]) \quad \textit{for every } \tilde{b} \in \tilde{B} \textit{ and } [\gamma] \in \pi^1(B).$$

Concluding remarks. Corollary 2 is a restatement of (a part of) Theorem 4.2.1 in [3] and its (equivalent) original versions given in Theorems 1 and 2 of [1]. This extends to the context of equations with total differentials the classical theorem of Ljapunov–Floquet for equations with periodic coefficients. This is indeed the case, since the periodicity of the coefficient of an ordinary linear equation naturally leads to a total equation on S^1 (in this regard cf. also [7]).

Accordingly, taking into account Corollary 2 and the previous comments, we see that the main result (as well as its Corollary 1) can be characterized as a geometric abstraction of the Ljapunov–Floquet theory in the context of arbitrary principal bundles and connections.

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