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Note on a functional equation

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Abstract. Let f, g be given decreasing functions on $[a, b]$, let g be an involutory function and let $f(\xi) = \xi$. The general monotonic solution of the equation

$$\varphi[f(x)] = g[\varphi(x)]$$

is given by the formula

$$\varphi(x) = \begin{cases} \psi(x) & \text{for } x \in [a, \xi], \\ g(\psi[f(x)]) & \text{for } x \in [\xi, b], \end{cases}$$

where ψ is an arbitrary monotonic function such that $\psi(\xi) = g[\psi(\xi)]$ and $\psi[f^2(x)] = \psi(x)$ for $x \in [a, \xi]$.

The object of this note is a study of the existence of monotonic solutions and of continuous solutions of the equation

$$\varphi[f(x)] = g[\varphi(x)], \tag{1}$$

where:

(H_1) f is a given strictly decreasing and continuous function on $[a, b]$ such that

$$f([a, b]) \subset [a, b],$$

(H_2) g is a given decreasing and involutory function on $[a, b]$, i.e.

$$g^2(x) = x \quad \text{for every } x \in [a, b].$$

The existence of monotonic solutions and of continuous solutions of functional equations of the first order has been investigated in many paper (cf. the references in [1] and [2]).

The following lemma is obvious.

LEMMA 1. *Let f and g be defined on a set E, $f(E) \subset E$, $g(E) \subset E$, and let g be an involutory function on E. If a function φ satisfies equation* (1) *on E, then this function satisfies the equation*

$$\varphi[f^2(x)] = \varphi(x).$$

LEMMA 2. *Let f and g be defined on a set E, $f(E) \subset E$, $g(E) \subset E$, and let g be an involutory function on E. If functional equation* (1) *has an invertible solution φ in E, then f is an involutory function.*

This lemma is a simple consequence of Lemma 1.

LEMMA 3. *Let f and g be defined on a set $E \subset R$, $f(E) \subset E$, $g(E) \subset E$, and let g be a decreasing involutory function on E. If functional equation* (1) *has a strictly monotonic solution in E, then is strictly decreasing in E.*

Proof. The function g is decreasing and involutory; therefore g is strictly decreasing (cf. [1], Chapter XV, § 1). Suppose that φ is a strictly increasing solution of (1) and $x < y$, $x, y \in E$. Then $\varphi(x) < \varphi(y)$ and according to (1) we have

$$\varphi[f(y)] = g[\varphi(y)] < g[\varphi(x)] = \varphi[f(x)]$$

because g is strictly decreasing. But φ is strictly increasing; therefore

$$f(y) < f(x).$$

This shows that f is strictly decreasing.

If φ is a strictly decreasing solution of (1), then $\varphi_1 = -\varphi$ is a strictly increasing solution of the equation

$$\varphi_1[f(x)] = g_1[\varphi_1(x)],$$

where $g_1(x) = -g(-x)$. The function g_1 is decreasing and $g_1^2(x) = x$. Thus the assertion results from the first part of the proof.

DEFINITION 1. Let hypothesis (H_1) be fulfilled. For $x \in [a, b]$ we write

$$\underline{x} = \max\{y \in [a, b]: y \leqslant x, f^2(y) = y\},$$
$$\bar{x} = \min\{y \in [a, b]: y \geqslant x, f^2(y) = y\}.$$

If $f^2(x) \neq x$, then

$$\underline{x} < x < \bar{x}.$$

Since f^2 is strictly increasing, we have

$$\underline{x} = f^2(\underline{x}) < f^2(x) < f^2(\bar{x}) = \bar{x},$$

whence

$$\underline{x} < f^2(x) < \bar{x}. \tag{2}$$

In this way we have proved

LEMMA 4. *Let hypothesis* (H_1) *be fulfilled. If $f^2(x) \neq x$, then inequality* (2) *holds.*

LEMMA 5. *Let hypothesis* (H_1) *be fulfilled. If $f^2(x) \neq x$, then the sequence $f^{2n}(x)$, $n = 0, \pm 1, \pm 2, \ldots$, is monotonic and either*

$$\lim_{n\to\infty} f^{2n}(x) = \bar{x} \quad \textit{and} \quad \lim_{n\to\infty} f^{-2n}(x) = \underline{x}$$

or

$$\lim_{n\to\infty} f^{2n}(x) = \underline{x} \quad \textit{and} \quad \lim_{n\to\infty} f^{-2n}(x) = \bar{x}.$$

This lemma is a simple corollary of Theorem 0.4 in [1].

It is easy to see that Lemmas 1, 4 and 5 imply

LEMMA 6. *Let hypotheses* (H_1) *and* (H_2) *be fulfilled. If* φ *is a monotonic solutions of* (1), *then for every* $x \in [a, b]$ *such that* $f^2(x) \neq x$ *the function* φ *is constant on the interval* $(\underline{x}, \overline{x})$.

If hypothesis (H_1) *is fulfilled, then there exists a point* $\xi \in (a, b)$ *such that*

$$f(\xi) = \xi.$$

Then

$$f([a, \xi]) \subset [\xi, b] \quad \textit{and} \quad f([\xi, b]) \subset [a, \xi].$$

If φ *is a solution of* (1), *we have*

$$g(\eta) = \eta$$

for $\eta = \varphi(\xi)$.

DEFINITION 2. Ψ is the class of real-valued monotonic functions ψ on $[a, \xi]$ such that

$$g[\psi(\xi)] = \psi(\xi),$$

where $f(\xi) = \xi$, and ψ is constant on $(\underline{x}, \overline{x})$ whenever $\underline{x} < \overline{x}$.

THEOREM 1. *Suppose that hypotheses* (H_1) *and* (H_2) *are fulfilled. Then the formula*

$$\varphi(x) = \begin{cases} \psi(x) & \textit{for } x \in [a, \xi], \\ g(\psi[f(x)]) & \textit{for } x \in [\xi, b], \end{cases} \tag{3}$$

where ψ *is an arbitrary function from the class* Ψ, *determine the general monotonic solution of* (1) *on* $[a, b]$.

Proof. Suppose that $\psi \in \Psi$ is decreasing. The function $\varphi(x) = g(\psi[f(x)])$ is a superposition of three decreasing functions; therefore φ is decreasing on $[\xi, b]$. But ψ is decreasing and $\psi(\xi) = g(\psi[f(\xi)])$; hence φ is decreasing on $[a, b]$.

Let $x \in [a, \xi]$; then $f(x) \in [\xi, b]$. According to formula (3) we have

$$\varphi(x) = \psi(x) \quad \text{and} \quad \varphi[f(x)] = g(\psi[f^2(x)]). \tag{4}$$

If $f^2(x) = x$, then by (4) we obtain (1). If $f^2(x) \neq x$, then we have

$$\psi[f^2(x)] = \psi(x) \tag{5}$$

because $\psi \in \Psi$. Formulas (4) and (5) imply (1).

Let $x \in [\xi, b]$; then $f(x) \in [a, \xi]$. According to formula (3) we have

$$\varphi(x) = g(\psi[f(x)]) \tag{6}$$

and

$$\varphi[f(x)] = \psi[f(x)]. \tag{7}$$

From (6) and (7) follows (1). This completes the proof of the theorem when ψ is decreasing.

On the other hand, if the function φ is a decreasing solution of (1), then

$$\psi = \varphi|_{[a,\xi]}$$

is decreasing,

$$\psi(\xi) = g[\psi(\xi)]$$

and, according to Lemma 6, ψ is constant on $(\underline{x}, \overline{x})$ if $f^2(x) \neq x$. Therefore ψ belong to Ψ.

Let $x \in [\xi, b]$; then $f^2(x) \in [\xi, b]$ and $f(x) \in [a, \xi]$. From (1) we obtain

$$\varphi[f^2(x)] = g(\varphi[f(x)]) = g(\psi[f(x)]).$$

According to Lemma 1 we have

$$\varphi(x) = \varphi[f^2(x)] = g(\psi[f(x)]).$$

Hence formula (3) holds. This prove that formula (3) defines the general decreasing solution of (1) on $[a, b]$.

For increasing solutions of (1) the proof is similar.

Similarly as Theorem 1 we can prove the following

THEOREM 2. *Suppose that hypotheses* (H_1) *and* (H_2) *are fulfilled and* f *is an involutory function. Then formula* (3), *where* ψ *is an arbitrary invertible function from the class* Ψ, *determine the general strictly monotonic solution of* (1) *on* $[a, b]$.

LEMMA 7. *Let hypotheses* (H_1) *and* (H_2) *be fulfilled. If* φ *is a continuous solution of* (1), *then for every* $x \in [a, b]$ *such that* $f^2(x) \neq x$ *the function* φ *is constant on* $[\underline{x}, \overline{x}]$.

Proof. Let $\underline{x} < \overline{x}$; then according to Lemmas 1 and 4 we have

$$\varphi(x) = \varphi[f^{2n}(x)] = \varphi[f^{-2n}(x)]. \tag{8}$$

The continuity of φ, formula (8) and Lemma 5 imply

$$\varphi(x) = \varphi(\overline{x}) = \varphi(\underline{x}).$$

COROLLARY. *Suppose that hypotheses* (H_1) *and* (H_2) *are fulfilled and the set*

$$A = \{x \in [a, b] : f^2(x) = x\}$$

has a finite derived set A^d. *If* φ *is a continuous solution of* (1), *then* φ *is constant.*

Proof. We denote by B the set

$$\{x \in [a, b] : f^2(x) \neq x\}.$$

The set A has a finite derived set; therefore the set B is non-empty. Let φ be a continuous solution of (1) in $[a, b]$. The set B is open; therefore according to Lemma 7 there exists an interval (c, d) such that φ is constant on (c, d). The function φ is constant on the closed interval $[c, d]$, because φ is a continuous function.

We suppose that $d < b$. If $e = \min\{x \in A : x > d\}$ exists, then according to Lemma 7 the function φ is constant on $[d, e]$; therefore φ is constant on $[c, e]$.

If e does not exist, then there exists a positive number δ such that the set $(d, d+\delta) \cap A^d$ is empty, because the set A^d is finite. It is easy to prove that there exists a decreasing sequence $\{d_n\}$ such that

$$A \cap (d, d+\delta) = \{d_1, d_2, \ldots\}$$

and

$$\lim_{n\to\infty} d_n = d.$$

According to Lemma 7 the function φ is constant on the intervals $[d_2, d_1]$, $[d_3, d_2]$, ... Hence φ is constant on $(d, d_1]$. The function φ is continuous, whence φ is constant on $[d, d_1]$; therefore φ is constant on $[c, d_1]$. This proves that if $d < b$ there exists a $d_1 > d$ such that φ is constant on $[c, d_1]$; therefore φ is constant on $[c, b]$. The proof of the face that φ is constant on $[a, c]$ is similar.

DEFINITION 3. Ψ_c is the class of real-valued continuous functions ψ on $[a, \xi]$ such that

$$g[\psi(\xi)] = \psi(\xi),$$

where $f(\xi) = \xi$, and ψ is constant on $[\underline{x}, \bar{x}]$ whenever $\underline{x} < \bar{x}$.

THEOREM 3. *Suppose that hypotheses* (H_1) *and* (H_2) *are fulfilled. Then formula* (3), *where* ψ *is an arbitrary function from the class* Ψ_c, *determine the general continuous solutions of* (1) *on* $[a, b]$.

The proof of this theorem is similar to the proof of Theorem 1.

References

[1] M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne 46, Warszawa 1968.

[2] A. Smajdor, *On monotonic solutions of some functional equations*, Rozprawy Matematyczne 82 (1971).

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