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CHARACTERIZATIONS OF m-CONNECTED GRAPHS

Abstract. A generalization of Frank's conjecture [2] as well as its short proof are given in this paper. A counterexample to Győri's conjecture formulated in his paper [4] on p. 272 is also presented.

1. Introduction. The Menger theorem (see [1], p. 167) is one of the most well-known characterizations of m-connected graphs. A few years ago, Győri [3], [5] and Lovász [6] independently solved Frank's conjecture [2] (its weaker version was formulated also by Maurer [7]) and gave a very interesting characterization of m-connectivity. Some conditions for a graph to be m-connected have been found by Győri in [4], where he formulated a related conjecture. In the last section of our paper, we give a counterexample to his conjecture.

2. A generalization of Frank's conjecture. By a *graph* we mean a finite undirected graph without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of G, respectively. For $W \subset V(G)$, $G(W)$ denotes the subgraph of G induced by W. A graph G is *m-connected* if $|V(G)| > m \geqslant 1$ and, for every $W \subset V(G)$, $1 \leqslant |W| \leqslant m-1$, the graph $G(V \setminus W)$ is connected (G is *1-connected* if it is connected).

Equivalently, a graph G is m-connected ($m \geqslant 1$) if and only if from every subset of vertices $W \subset V$ such that $|V \setminus W| \geqslant m$ there exist in G edges leading to at least m vertices of $V \setminus W$.

We can prove the following theorem:

THEOREM 1. *A graph G with n vertices is m-connected if and only if for k different vertices $v_i \in V(G)$ $(i = 1, \ldots, k;\ k \leqslant m)$, m positive integers n_i $(1 \leqslant n_i \leqslant n-k+1;\ i = 1, \ldots, m)$ such that*

$$\sum_{i=1}^{m} n_i \geqslant n+m-k$$

and for $v_j \in \{v_1, \ldots, v_k\} = V_0$ $(j = k+1, \ldots, m)$ there exist subsets $V_i \subset V(G)$ $(i = 1, \ldots, m)$ such that

$$\bigcup_{i=1}^{m} V_i = V(G), \qquad v_i \in V_i,\ |V_i| = n_i,\ v_i \neq v_j$$

imply

$$v_i \notin V_j,$$

and the graph $G(V_i)$ *is connected for* $i, j = 1, \ldots, m$.

Remark 1. It is sufficient to prove Theorem 1 for the case

$$\sum_{i=1}^{m} n_i = n+m-k$$

only, because any subset of $V(G)$ containing only one vertex from the set V_0 and inducing a connected subgraph can be extended to a subset of $V(G)$ which contains only this vertex from the set V_0 and induces a connected subgraph with an arbitrary number of vertices less than $n-k+2$. Also, we may restrict ourselves to the case $k = m$. This follows from the fact that from every vertex $v_i \in V_0$ there are edges leading to at least m different elements of $V(G) \setminus \{v_i\}$, $i = 1, \ldots, k$. For V_0 and $j = k+1, \ldots, m$ we may choose, step by step, m different vertices

$$V_0' = \{v_1, \ldots, v_k, v_{k+1}', \ldots, v_m'\}$$

in a way such that for $j \in \{k+1, \ldots, m\}$ if $v_j = v_i$ for some $i \in \{1, \ldots, k\}$, then $(v_j, v_j') \in E(G)$. We apply Theorem 1 to V_0' and $\{n_1, \ldots, n_k, n_{k+1}-1, \ldots, n_m-1\}$ (assuming $n_j > 1$ for $j \in \{k+1, \ldots, m\}$) and join chosen vertices by edges (v_j, v_j'), respectively, to receive the connected graphs induced by the sets V_j $(j = k+1, \ldots, m)$.

For the case $k = m$ Theorem 1 becomes Corollary 2.

Definition 1. Let v_i $(i = 1, \ldots, m)$ be m different vertices of a graph G with $|V(G)| = n > m$, and n_i $(i = 1, \ldots, m)$ be positive integers with

$$\sum_{i=1}^{m} n_i = n.$$

We recall (see [4]) that G satisfies the *partition condition*

$$P_m(G; v_1, \ldots, v_m;\ n_1, \ldots, n_m)$$

if there is a partition $\{V_1, \ldots, V_m\}$ of $V(G)$ such that $v_i \in V_i$, $|V_i| = n_i$, and the graph $G(V_i)$ is connected for $i = 1, \ldots, m$.

A graph G is said to satisfy the *partition condition*

$$P_m(G; n_1, \ldots, n_m)$$

if G satisfies the partition condition $P_m(G; v_1, \ldots, v_m; n_1, \ldots, n_m)$ for every choice of $\{v_1, \ldots, v_m\} \subset V(G)$.

Let us observe that

Proposition 1. *A graph* G, $|V(G)| = n$, *is m-connected if and only if it satisfies the partition condition*

$$P_m(G; n-m+1, \underbrace{1, \ldots, 1}_{m-1}).$$

To prove Theorem 1 it is sufficient to show that, for $n_i \geqslant 1$, $i = 1, \ldots, m$, and $\sum_{i=1}^{m} n_i = n$,

$$P_m(G; n-m+1, \underbrace{1, \ldots, 1}_{m-1}) \quad \text{implies} \quad P_m(G; n_1, \ldots, n_m)$$

or, more generally, we have

THEOREM 2. *Let $G = (V, E)$ be an m-connected graph, $\{v_1, \ldots, v_m\} \subset V$, $\{V_1, \ldots, V_m\}$ be a partition of $V(G)$, $v_i \in V_i$, and $G(V_i)$ be connected for $i = 1, \ldots, m$. If $|V_1| > 1$, then there exists a partition $\{V_1', \ldots, V_m'\}$ of $V(G)$ such that $|V_1'| = |V_1|-1$, $|V_i'| = |V_i|$ for $i = 2, \ldots, m-1$, $V_m \subset V_m'$, $v_i \in V_i'$, and the graph $G(V_i')$ is connected for $i = 1, \ldots, m$.*

Proof. Without loss of generality we may assume that $(v_i, v_j) \notin E(G)$ for $i \neq j$ and $i, j = 1, \ldots, m$. For every $i \in \{1, \ldots, m-1\}$, we choose any $c_i \in V_i$ such that there exists $d_i \in V_m$, $(c_i, d_i) \in E(G)$, and the number of vertices in $G(V_i \setminus \{c_i\})$ which do not belong to the component containing v_i (we denote these vertices by W_i) is minimal. It may happen that $c_i = v_i$ or $W_i = \emptyset$. Let us put

$$W = \bigcup_{i=1}^{m-1} W_i$$

and assume $|W| = k$. For $W = \emptyset$, Theorem 2 is obvious. Let us assume that our theorem is true for all m'-connected graphs with $|W| < k$ and $m' \leqslant m$.

Case 1. First, we consider the case $c_1 \neq v_1$. If $W_1 = \emptyset$, then we join c_1 to W_m and we have done. If $W_1 \neq \emptyset$, then we put

$$\bar{v}_i = c_i, \quad \bar{V}_i = W_i \cup \{c_i\} \quad \text{for } i = 1, \ldots, m-1,$$

$$\bar{v}_m = v_m, \quad \bar{V}_m = V_m \cup \bigcup_{i=1}^{m-1} \{V_i \setminus (W_i \cup \{c_i\})\}$$

and apply the induction hypothesis to graphs $G(\bar{V}_i)$ for $i = 1, \ldots, m$. ($\bar{W} \subset W$ and $\{c_1, \ldots, c_{m-1}\}$ do not disconnect G, thus $|\bar{W}| < |W| = k$.) There exist

$$v \in \bar{W}, \quad j \in \{1, \ldots, m-1\}, \quad w_j \in V_j \setminus (W_j \cup \{c_j\})$$

such that $(v, w_j) \in E(G)$, and a partition $\{\bar{V}_1', \ldots, \bar{V}_{m-1}'\}$ of the set

$$\{\bigcup_{i=1}^{m-1} (W_i \cup \{c_i\})\} \setminus \{v\}$$

such that $|\bar{V}_1'| = |W_1|$, $|\bar{V}_i'| = |W_i|+1$ for $i = 2, \ldots, m-1$, $c_i \in \bar{V}_i'$, and the graph $G(\bar{V}_i')$ is connected for $i = 1, \ldots, m-1$. The new graphs $G(\tilde{V}_i)$, where

$$\tilde{V}_j = \{v\} \cup \bar{V}_j' \cup \{V_j \setminus (W_j \cup \{c_j\})\}, \quad \tilde{V}_m = V_m,$$

$$\tilde{V}_i = \bar{V}_i' \cup \{V_i \setminus (W_i \cup \{c_i\})\} \quad \text{for } i = 1, \ldots, m-1 \text{ and } i \neq j,$$

are connected for $i = 1, \ldots, m$. If $j = 1$, then $|\tilde{V}_i| = |V_i|$ for $i = 1, \ldots, m-1$, and if $j \neq 1$, then $|\tilde{V}_1| = |V_1|-1$, $|\tilde{V}_j| = |V_j|+1$, $|\tilde{V}_i| = |V_i|$ for $i = 2, \ldots, m-1$ and $i \neq j$. Furthermore, $|\tilde{W}| \leqslant |W|-1$ and we can apply the induction hypothesis once more to $\{\tilde{v}_1, \ldots, \tilde{v}_m\}$ and the partition $\{\tilde{V}_1', \ldots, \tilde{V}_m'\}$, where $\tilde{v}_1 = v_j$, $\tilde{V}_1' = \tilde{V}_j$, $\tilde{v}_j = v_1$, $\tilde{V}_j' = \tilde{V}_1$, $\tilde{v}_i = v_i$, $\tilde{V}_i' = \tilde{V}_i$ for $i = 1, \ldots, m$ and $1 \neq i \neq j$, to complete the proof.

Case 2. If $c_1 = v_1$, then $V_0 = \{v_1, \ldots, v_{m-1}\}$ does not disconnect G and there is a sequence of different integers r_i, $1 \leqslant r_i \leqslant m$, $r_1 = 1$, $r_k = m$, $k \in \{3, \ldots, m\}$, and vertices $\{a_{r_i}, b_{r_i}\} \in V_{r_i} \setminus \{v_{r_i}\}$ (not necessarily different), $b_m \in V_m$, such that

$$(a_{r_i}, b_{r_{i+1}}) \in E(G(V(G) \setminus V_0)) \quad \text{for } i = 1, \ldots, k-1.$$

We begin with

$$\bar{V}_m = \bigcup_{i=2}^{k} V_{r_i}$$

and apply Case 1 to transfer, in a finite number of steps (less than k), a vertex from V_1 to V_m to fulfil our assertion.

Theorem 2 implies

COROLLARY 1 (Győri's theorem [3], [5]). *Let G be an m-connected graph and $\{v_1, \ldots, v_m\}$ be different vertices of G. Suppose that we have a partition $\{V_1, \ldots, V_m, S\}$ of $V(G)$ such that $S \neq \emptyset$, $v_i \in V_i$, and $G(V_i)$ is connected for $i = 1, \ldots, m$. Then there is another partition $\{V_1', \ldots, V_m', S'\}$ of $V(G)$ such that $v_i \in V_i'$, $G(V_i')$ is connected for $i = 1, \ldots, m$, and $|V_m'| = |V_m|+1$, $|V_j'| = |V_j|$ for $j = 1, \ldots, m-1$.*

Proof. If there are edges leading from S to V_m, then we can join the corresponding vertex from S to the set V_m and we have done. If not, we join the vertices of connected components of S to the vertices V_j for some $j \in \{1, \ldots, m-1\}$, and obtain a partition $\{\bar{V}_1, \ldots, \bar{V}_m\}$ of $V(G)$ such that $V_i \subset \bar{V}_i$ for $i = 1, \ldots, m-1$, $V_m = \bar{V}_m$, $|\bar{V}_j| > |V_j|$ for some $j \in \{1, \ldots, m-1\}$, and the graphs $G(\bar{V}_i)$ $(i = 1, \ldots, m)$ are connected. Then we apply Theorem 2 to transfer a vertex from $\bar{V}_j$ to $\bar{V}_m$. By Theorem 2 we have a partition $\{V_1', \ldots, V_m'\}$ of $V(G)$ with the properties: $|\bar{V}_j'| = |\bar{V}_j|-1$, $|\bar{V}_i'| = |\bar{V}_i|$, $|\bar{V}_m'| = |\bar{V}_m|+1 = |V_m|+1$, $v_i \in \bar{V}_i'$, and $G(\bar{V}_i')$ is connected for $i = 1, \ldots, m$. Having this, we can easily construct the desired partition of $V(G)$.

Theorem 1 for $k = m$ (see Remark 1) follows now from Theorem 2 (as well as from Corollary 1).

COROLLARY 2 (Frank's conjecture [2], Győri [3], [5], Lovász [6]). *Let G be an m-connected graph, $\{v_1, \ldots, v_m\}$ be different vertices of G, and $\{n_1, \ldots, n_m\}$ be positive integers such that*

$$\sum_{i=1}^{m} n_i = |V(G)|.$$

Then there is a partition $\{V_1, \ldots, V_m\}$ of $V(G)$ such that $v_i \in V_i$, $|V_i| = n_i$, and $G(V_i)$ is connected for $i = 1, \ldots, m$.

Proof. We apply Theorem 2 starting from the partition

$$V_1 = V \setminus \{v_2, \ldots, v_m\}, \qquad V_i = \{v_i\} \text{ for } i = 2, \ldots, m.$$

An example presented in the last section of the paper shows that Theorem 2 is not true for connected graphs which are not m-connected. This enables us to formulate the following proposition (its proof follows from Proposition 1):

PROPOSITION 2. *Let G be a connected graph and $|V(G)| > m$. If for every choice $\{v_1, \ldots, v_m\} \subset V(G)$ of different vertices and every partition $\{V_1, \ldots, V_m\}$ of $V(G)$ such that $v_i \in V_i$ and $G(V_i)$ is connected for $1, \ldots, m$ the assertion of Theorem 2 is fulfilled, then the graph G is m-connected.*

Thus, Theorem 2 characterizes m-connected graphs.

3. A counterexample to Györi's conjecture. Now, we give an example to show that the following conjecture is not true.

CONJECTURE 1 (Györi [4], p. 272). *Let G be a graph and $|V(G)| > m$. Furthermore, let $\{n_1, \ldots, n_m\}$ and $\{n'_1, \ldots, n'_m\}$ be nondecreasing sequences of positive integers such that $n_i \leqslant n'_i$ for $i = 1, \ldots, m-1$ and*

$$\sum_{i=1}^{m} n_i = \sum_{i=1}^{m} n'_i = |V(G)|.$$

If G satisfies the partition condition $P_m(G; n_1, \ldots, n_m)$, then it also satisfies the partition condition $P_m(G; n'_1, \ldots, n'_m)$.

EXAMPLE 1. Let a graph G have vertices

$$V(G) = \{x_i, y_j, z_k | \; i, k \in \{1, \ldots, 5\}, j \in \{1, 2\}\}$$

and edges

$$E(G) = \{(x_i, y_j), (z_i, y_j), (x_i, x_k), (z_i, z_k) | \; i \neq k, \; i, k \in \{1, \ldots, 5\}, j \in \{1, 2\}\}.$$

This means that y_j ($j = 1, 2$) is adjacent to each vertex of the complete graphs induced by vertices $\{x_1, \ldots, x_5\}$ and $\{z_1, \ldots, z_5\}$, respectively (see Fig. 1, where some edges of G are drawn).

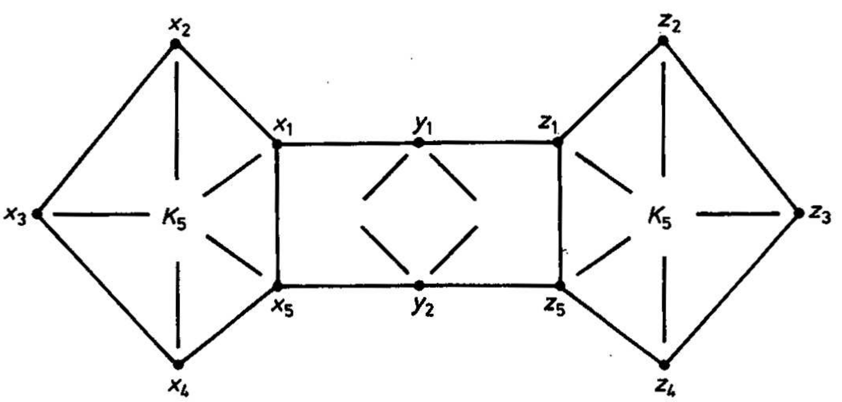

Fig. 1

The graph G is 2-connected. It satisfies the partition condition $P_3(G; 3, 4, 5)$ but does not satisfy the partition condition $P_3(G; x_1, x_2, x_3; 4, 4, 4)$.

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